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# The Effects of Fluency Training on the Acquisition and Retention of Secondary Students' Fraction Skills

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Retention of Secondary Students' Fraction Skills

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# ABSTRACT

The Effects of Fluency Training on the Acquisition and
Retention of Secondary Students' Fraction Skills

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Secondary students, especially those with learning disabilities, often lack an understanding of computations involving fractions. Much of the secondary math core, especially algebra, requires an understanding of fractions to be able to successfully complete core classes. Instruction on fraction concepts in not part of the secondary core standards. These students are expected to already have this knowledge. There is a need for students with learning disabilities who struggle with fraction computations to receive instruction on fraction concepts in addition to their core instruction. This study used direct instruction and fluency practice as an intervention to teach basic fraction skills to two secondary students with learning disabilities. A multiple probe multiple baseline design was used. Results suggest that fluency training has a positive impact on secondary students' acquisition and retention of basic fraction skills. The implications of this study suggest that this intervention is a viable option to help students acquire fraction skills in a minimal amount of time.

Keywords: mathematical instruction, fractions, secondary, disabilities, fluency

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# DESCRIPTION OF THESIS STRUCTURE

This thesis, *The Effects of Fluency on the Acquisition and Retention of Secondary Students' Fraction Skills*, is written in a hybrid format upon the approval of my department. The hybrid format brings together traditional thesis requirements with journal publication formats**.**

The preliminary pages of the thesis reflect requirements for submission to the university. The thesis report is presented as a journal article, and conforms to length and style requirements for submitting research reports to education journals.

The literature review is included in Appendix A. Appendix B contains research approval from the Institutional Review Board, followed by the study's instruments. Appendix C shows the Fluency Probes and Appendix D includes Norm Timings.

This thesis format contains two reference lists. The first reference list contains references included in the journal-ready article. The second list includes all citations used in the Appendix entitled "Review of the Literature."

## Introduction

Students with learning disabilities are expected to meet the same core math objectives as their peers without disabilities while coping with severe deficits in basic math skills. They must not only keep up with new concepts, but also master skills that eluded them the first time they were introduced. To achieve this goal, a student must learn at a rate faster than that of their grade-level peers. However, this is a daunting task. When it comes to mathematics, students with learning disabilities tend to perform around two grade levels below that of their non-disabled peers; and adolescents with learning disabilities perform around a fifth grade mathematics level (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Miller & Hudson, 2007). In addition, Common Core standards for math are aligned with National Council of Teachers of Mathematics (NCTM, 2016) standards that emphasize problem solving and abstract thinking, both of which are difficult for students with disabilities.

According to the National Center for Education Statistics (NCES, 2015), only 6% of 12th grade students with disabilities are proficient in mathematics. For students without disabilities, the percent rises to 25 (National Center for Education Statistics [NCES], 2015). Yet math is an integral aspect of life. People use whole numbers, fractions, decimals, and percent in a wide range of applications. For example, setting an alarm clock, paying for goods and services, estimating time to destinations, calculating gas mileage, tracking weight gain or loss, following recipes, buying paint, and countless other daily applications depend on math skills. The need for higher math skills may not be as apparent. Algebra is not as readily obvious as basic computational skills, but Usiskin (1995) notes that a lack of facility with algebra limits people's opportunities just as low skills in reading or writing can limit them.

Seeing and understanding patterns is key to using algebra. Formulas are patterns that can be applied over and over to solve the same types of problems, such as calculating area, length, or weight (Usiskin, 1995). Beyond basic applications, Bahr and de Garcia (2010) advocate that being mathematically literate is not just entirely about content knowledge, but includes the ability to think in a mathematical way. They define math literacy as "the ability to use resources to find, evaluate, and use information" (pp. 2-3).

Higher levels of mathematics taught in junior high and high school require a broad base of number skills. Algebra, for example, requires prerequisite skills in ratios and proportional thinking, the number system, and expressions and equations (Common Core State Standards Initiative [CCSSI], 2016). Wu (2001) suggests that students struggle with algebra because it is more general and abstract than preceding math concepts. A student must also be fluent in the manipulation of symbolic representations and understand how mathematical properties apply to whole numbers, fractions, and decimals (Wu, 2001).

**Fractions**

Understanding fractions is vital to progression in secondary mathematics; the inability to work fluently with fractions can impede progression in learning new mathematical concepts (Brown & Quinn, 2007). One study investigated the relationship between mathematics achievement and competence with fractions while controlling for intelligence, working memory and a more general competence in mathematics. The cross-lagged analysis found that proficiency in fractions leads to gains in mathematical achievement but math achievement did not lead to fraction competence (Bailey, Hoard, Nugent, & Geary, 2012).

A student who has a good understanding of fractions is likely to be successful in algebra (Wu, 2001). The National Mathematics Advisory Panel (NMAP, 2008) notes that competency in the conceptual understanding of fractions and fluency in solving fraction problems is integral to algebraic understanding. Moreover, the National Mathematics Advisory Panel (2008) reported that U.S. algebra teachers cite lack of fraction understanding as a major stumbling block for their students' ability to learn algebra. Research tends to verify this position. Booth and Newton (2012) studied the influence students' knowledge of fraction and whole number magnitude on algebra readiness tasks, where magnitude is the size of the fraction or whole number as represented on a number line. Results point to knowledge of fraction magnitude as more predictive of algebra readiness than knowledge of whole number magnitude (Booth & Newton, 2012). Similarly, Siegler, Thompson, and Schneider (2011) noted that misunderstanding or not understanding the concept of magnitude in fractions impairs children's transition from whole number thinking to working with parts of a whole. According to Mazzocco, Myers, Lewis, Hanich and Murphy (2013), "elementary school students' knowledge of fractions is a stronger predictor of their overall later high school mathematics achievement than their elementary whole number arithmetic knowledge" (p. 372). When comparing sixth grade students with a math learning disability (MLD) to those considered to be low achievers, Mazzocco et al. found that the students with MLD had "significant misconceptions" (p. 373) regarding fractions. They struggled to order and compare fractions, and to name decimal representations (Mozzocco et al., 2013).

The ability to learn fraction concepts requires more in-depth understanding of numbers than is typically required for proficiency with whole numbers. This is because a reorganization of numerical knowledge is necessary when learning to work with fractions (Siegler, Fasio, Bailey & Zhou, 2013). When introduced to whole numbers, children learn that each number has a logical successor. They learn that whole numbers can be paired with objects to count sets and that sets have cardinality. They learn that adding or taking away members of sets increases or decreases the cardinality (Siegler et al., 2011; Siegler et al., 2013). Moreover, properties of whole numbers such as never decreasing with multiplication or increasing with division do not hold true for fractions. Several theorists have written about these differences. Gelman and Williams (1998) concluded that whole number learning actually interferes with later fraction learning. Geary (2006) wrote that people are preprogrammed to understand whole numbers, but not to understand fractions. Others believe that the conceptual change necessary to move from whole number understanding to fractions is the source of difficulty (Vosniadou, Vamvakoussi, & Skopeiliti, 2008). Regardless of the cognitive demands of learning to calculate and use fractions, successful manipulation of them requires fluency.

**Fluency**

Fluency is generally defined as the ability to respond quickly and accurately to a task (National Council of Teachers of Mathematics [NCTM], 2016). Lack of fluency is characterized as one of the "persistent problems" in learning and using math (Kellman et al., 2008, p. 357). In mathematics, computational fluency is part of the larger concept of procedural fluency, which includes not only speed and accuracy, but understanding of when and how to compute fractions (NCTM, 2014). Lack of computational fluency can greatly hinder students' attainment of higher level mathematical skills, and students with disabilities commonly lack the ability to recall basic number facts or to use them in computation (Farrell & McDougall, 2008). Calhoon, Emerson, Flores, and Houchins (2007) studied which grade levels and in which skill areas high school students with mathematics disabilities were computationally fluent. Results indicated that the computational fluency of these students was at a 3rd grade level, with fractions and decimals being the skills that students struggled with the most.

Researchers have studied the effects of fluency training on the mastery and retention of learning targets (Lee & Singer-Dudek, 2012). A student who is fluent with math facts is able to complete more problems in a given amount of time. This leads to an increased opportunity to respond and more chances for reinforcement of correct responses, both of which are important components of learning (McCallum & Schmitt, 2011). A fluent student is likely to have lower levels of anxiety and avoidance when presented with a math task (McCallum & Schmitt, 2011; Cates & Rymer, 2003). From a cognitive processing perspective, a student solving complex math problems who is fluent in basic math facts and procedures uses less cognitive capacity, leaving more capacity available to solve the problems (McCallum & Schmitt, 2011; Poncy, Skinner, & Jaspers, 2007). In addition, behavior fluency theorists cite skill maintenance, increased capacity to remain on task when solving a problem, and ability to adapt and combine skills in new situations as benefits of fluency (Binder, 1996; Farrell & McDougall, 2008; Singer-Dudek & Greer, 2005).

Much of the literature on fluency refers to an instructional hierarchy developed by Haring and Eaton in *The Fourth R: Research in the Classroom* (Haring, Lovitt, Eaton & Hansen, 1978). This hierarchy outlines four stages of learning: acquisition, fluency, generalization and adaptation. During the acquisition stage, students begin to learn the target skill. The goal is to increase accuracy. Once students are able to complete the skill accurately, they move to the fluency stage. This stage consists of instructional techniques aimed at improving the student's frequency of responding accurately. The authors give three possible definitions of fluency. First, the ability to perform the skill at a level that establishes maintenance of that skill. Second, the ability to perform the skill at a level that establishes success the next time the same or similar task is performed. Third, the ability to perform the skill at a level that is similar to successful peers. Students must become fluent in the target skill in order to move to the next two stages of generalization and adaptation. Students are considered to have mastered the target skill when they have successfully moved through all four stages. According to this model, students do not achieve mastery until they become fluent in the skill (Haring et al., 1978). One research-based approach to teaching for fluency is direct instruction.

**Direct Instruction**

In a direct instruction teaching model, the teacher provides clear instruction, adequate opportunities to respond and ongoing monitoring of student performance (Carnine, Silbert, Kame'enui, & Tarver, 2010). After researching what components of teaching determine student academic success, Rosenshine (1986) summarized his findings and called the collection direct instruction. High levels of student engagement, teacher-led sequenced and structured instruction, monitoring of student performance, and immediate feedback to students were all elements of teaching that were included in his description (Rosenshine, 1986).

Misquitta (2011) reviewed literature that was published between 1990 and 2008 to find studies that involved teaching fraction skills to students who had difficulty with mathematics. In his review, he found that there were three interventions that were effective for the instruction of fractions. One of these three interventions was direct instruction (Misquitta, 2011). The effectiveness of direct instruction has been shown in studies with both elementary and secondary students. In one such study on three elementary aged students, an instructional program based on direct instruction principles was implemented. The results showed improvements on students' fraction performance as well as an ability to maintain these improvements over a short follow-up period (Perkins & Cullinan, 1985). In a similar study directed at secondary students, four seventh graders with learning disabilities were taught fraction and decimal skills using the direct instruction method. These students were able to perform at a level above that of their peers on standardized and informal assessments (Scarlato & Burr, 2002).

Direct instruction can use scripted teaching materials that instructors follow word-for-word, or can be unscripted and require teachers to use their own language to follow the steps for modeling, prompting responses, feedback, guided practice, and independent practice (Kim & Axelrod, 2005). Direct instruction monitors response accuracy to determine whether to continue teaching or to reteach what has not been mastered. A formal system for using student response data to make instruction decisions is exemplified by precision teaching.

**Precision Teaching**

Precision teaching was developed by Ogden R. Lindsley, a behavioral scientist who worked under B. F. Skinner at Harvard University in the 1950's. According to Lindsley (1992) "Precision teaching is basing educational decisions on changes in continuous self-monitored performance frequencies displayed on standard celeration charts" (p. 51); therefore, it is a decision-making system rather than a set of instructional behaviors. Lindsley left his behavioral research at Harvard when he became a professor at the University of Kansas in 1965. In his model for precision teaching, Lindsley used six principles of Skinner's work in behavior analysis: consequences for behavior, "the learner knows best" (Lindsley, 1972), observable behavior, daily monitoring of frequency, measuring behavior using frequency and using a standard display for data (Potts, Eshleman, & Cooper, 1993). Precision teaching emphasizes both speed and accuracy. Through the use of multiple opportunities to respond in the form of fluency practice and the daily charting of student performance, precision teaching provides constant feedback on student behavior. This feedback enables teachers to differentiate and modify instruction to fit the needs of each student. In this way, the student tells the teacher if instruction is effective (West & Young, 1992).

Precision teaching has extended the definition of fluency to include the attainment of critical learning outcomes in addition to ease and quickness of responding to an accuracy requirement. This is referred to as behavioral fluency and is produced through frequency building. Kubina and Yurich (2012) define frequency building as “the timed repetition of the behavior with performance feedback” (p. 324). One of the learning outcomes associated with behavioral fluency is long-term retention. Retention is the ability to accurately perform a behavior when it has not been practiced for an extended period of time (Kubina & Yurich, 2012). In research on the effects of fluency training on retention, Bucklin, Dickinson and Brethower (2000) found that fluency training led to gains in retention rates for college students. The authors also cite three controlled studies conducted within educational settings that looked at the effects of fluency on retention. Ivarie (1986) found that for students who were considered average or below average, fluency training resulted in higher retention rates. Similarly, Berquam (1981) observed that, given a retention assessment, students who were trained using timed fluency practice were faster and more accurate than their peers who were trained using untimed fluency practice. In the third study, Shirley and Pennypacker (1994) looked at three conditions for learning spelling words: no specified criterion, an accuracy criterion, and a fluency criterion. Participants achieved higher retention rates on the fluency condition when compared to the no specified criterion condition. For one of the two participants, fluency training also resulted in better retention rates compared to the accuracy criterion condition (as cited by Bucklin et al, 2000).

In addition to increased accuracy and retention, research has shown that when a student is fluent, his abilities to master more advanced math skills is enhanced (Poncy, Skinner, & Jaspers, 2007; Poncy, 2010). Precision teachers refer to this as steeper slopes and rising bottoms in reference to graphs of student performance. Steeper slopes refer to faster acquisition of skills resulting in fewer data points and a steeper line on a data graph. Rising bottoms comes from the observance of higher scores with initial trials that, when graphed, create higher beginning data points (West & Young, 1992). With outcomes such as increased accuracy rates, better retention and an increased ability to master subsequent tasks, fluency training is a viable option for high school students who have not become proficient with math skills taught in the lower grades.

**Problem Statement**

Many students with disabilities struggle to understand fractions which leads to difficulty learning new math concepts in algebra, as solving algebraic equations often requires the manipulation of fractions. In order to help these students achieve at the rate of their peers, it is important to help them become both fluent and accurate with fractions. In addition, students must retain these skills throughout their high school careers as they are repeatedly presented with tasks that require them to apply the skills. Unfortunately, math instruction for high school students does not typically include basic fraction skills. Educators in Utah use the Utah Core Standards to guide instruction. The Utah standards are based on the Common Core State Standards, developed by the National Governors Association Center for Best Practices and the Council of Chief State School Officers (Common Core State Standards Initiative, 2016). These standards outline what students are expected to learn in each content area across all grade levels. The Utah core standards for mathematics introduce fractions in the third grade and by seventh grade students are expected to “apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers" (Common Core State Standards for Mathematics, 2016, p. 47). High school students who have not gained proficiency in fractions need to be re-taught these basic fraction skills. This is a difficult task for both the student and the teacher. It is therefore important to discover ways to effectively teach these skills to students when there is not typically a time set aside in the school day to do so. To address the issue, this study looked at the effects of direct instruction combined with fluency training on the acquisition and retention of fraction skills in secondary students.

**Research Questions**

1. Can secondary students with learning disabilities master learning targets for identifying fractions, writing equivalent fractions, and reducing fractions with both fluency and accuracy using direct instruction and elements of precision teaching to increase students' opportunities to respond?
2. Will these secondary students retain their rate of fluency and accuracy for identifying fractions, writing equivalent fractions, and reducing fractions on retention tests given at two and four week intervals following initial mastery?

## Method

**Setting**

This study took place at an urban charter school in northern Utah. The school services 1100 students in grades K-12. Waiver of participant consent was authorized by the BYU Institutional Review Board because the data used in this study was that of student performance on an instructional sequence that was implemented in a secondary setting. This instruction was part of the normal special education program curriculum. Participants were enrolled in a math lab class in which students received individualized instruction on basic math skills. Instruction was provided by a licensed special education teacher. Students in these classes also attended a core math class, in which instruction was based on the Utah Core Standards for high school mathematics and taught by a special education teacher, not the same teacher who provided instruction during intervention. Students attended both the math lab and the core math class for 50 minutes three times a week and 90 minutes one day a week each class.

During the intervention phase, participants were pulled individually from their math lab class for 15-20 minute sessions, 2-3 times a week. Intervention occurred in a classroom used for small group and one-on-one instruction with the student and teacher both sitting at the same table. A small whiteboard, fraction strips, fraction tiles, and Cuisenaire rods were used for the modeling and practice of skills being taught. Intervention instruction was delivered by a licensed special education teacher with a bachelors in special education and an emphasis in math instruction. She had taken and passed three PRAXIS exams- Elementary Education Content Knowledge, Middle School Math and Principles for Learning and Teaching Grades 7-12. She had six years of experience teaching special education in a secondary school. She taught language arts and science during three of the six years and then exclusively math for the three years leading up to this study. At the time of intervention, she was working as the Special Education Program Director and was not teaching classes.

**Participants**

Participants were selected based on four criteria: (a) students had a learning disability, based on the discrepancy model, which compares a student's cognitive abilities with current achievement. Cognitive abilities had been assessed within the past 3 years by a school psychologist using the WISC-IV and achievement had been assessed within the last year with the Woodcock Johnson III Tests of Achievement, (b) students were in grades 7 to 12, (c) students had a current IEP with specified services in math instruction, and (d) students were below the proficiency level in the area of fractions. Four potential participants were initially recommended for intervention by the math lab teacher, based on their performance on a recent fraction unit. They were each screened using the *Key to Fractions* (Rasmussen, 1980) diagnostic assessments. These assessments are broken into four parts: fraction concepts, multiplying and dividing fractions, adding and subtracting fractions and mixed numbers. Students who scored below 60% on at least one of the four screening parts were chosen to participate in the intervention. All four qualified as participants in the study, but two were dropped from the study due to excessive absences. Final participants were a 7$^{th}$ grade female that we will call Lisa and an 11$^{th}$ grade female that we will call Mary. As stated above, these student were found to have a learning disability with a discrepancy in math achievement. This means that they had cognitive scores in the average or above range and math achievement scores in the below to well-below range.

**Materials**

Materials included eight lesson plans pertaining to a specific fraction skill, *Key to Fractions* student workbooks, a one-minute timed assessment fluency sheet corresponding to each fraction skill, a computer for data management, and manipulatives in the form of fraction strips, fraction circles and Cuisenaire rods.

Instructional plans were created using the *Key to Fractions* workbook series. This series consists of four workbooks: *Fraction Concepts*; *Multiplying and Dividing*; *Adding and Subtracting*; and *Mixed Numbers* (Rasmussen, 1980). The content in these workbooks was broken down into eight fraction instruction lesson plans: identifying fractions; equivalent fractions, reducing, comparing and ordering fractions; addition/subtraction of fractions; multiplication of fractions; division of fractions; and changing between mixed numbers and improper fractions. An instructional sequence was written for each lesson and the workbook pages were used during these teacher-directed lessons.

**Measures**

For each lesson plan/fraction skill, one fluency sheet was created or found from existing resources. This fluency sheet was used as the practice worksheet as well as the baseline/retention probe. For fluency on equivalent fractions, reducing fractions and changing mixed numbers to improper fractions, the See to Write Skill Builder fluency sheets were used (Beck, 1995). Math-Drills.com was used to make the fluency sheet on comparing fractions. Fluency sheets on identifying, adding/subtracting, multiplying and dividing were developed by the researcher. A copy of each fluency sheet can be found in the appendix.

Participants' acquisition of each fraction skill was measured by fluency worksheets specific to the fraction skill. Fluency was measured as correct digits per minute. A standard rate of proficiency was established for each fraction skill using the average scores of fifty-two peers without disabilities from $8^{th}$ – $11^{th}$ grades. Peers were chosen at random from advisory classes in which students are grouped according to grade level. During this advisory time, a special education teacher administered the timings with the assistance of a paraeducator. All students were given a stapled packet containing two copies of each fluency worksheet. Before each timing, the special education teacher gave a short description of the fraction skill contained on the fluency sheet. For example, before starting the fluency sheet on equivalent fractions, the teacher would say "there is more than one way to write a fraction. One-half can be written as two-fourths." The teacher then instructed the students to get ready, then to start. After one minute, students were told to stop and turn to the next page in the packet which was identical to the previous page. Students were timed a second time on that particular fraction skill.

Once both fluency sheets were completed, students were instructed on the next fraction skill and given two 1-minute timings. This continued until students completed two timings of each fraction skill. To account for outliers, the median score was taken for each of the two trials then the average of the two median scores was found and rounded up to the nearest whole number. This score was used as the proficiency criteria. The results of these norm timings are depicted in Table 1.

**Research Design**

A multiple probe multiple baseline design was used in this study. The multiple probe multiple baseline design is a variant of the multiple baseline design. First made known by Horner and Baer in 1978, the multiple probe multiple baseline design was created as a more efficient way to collect data. This is done by collecting data intermittently at strategic times, rather than during every intervention session. In this way, trends and patterns can be identified while saving time and effort (Kennedy, 2005). Another benefit of this design is that it does not require withdrawal to establish a functional relation. The current study employed a multiple baseline across fraction skills with retention probes at specific intervals.

**Procedure**

Intervention was based on the main elements of precision teaching: opportunities to respond, an emphasis on speed and accuracy and daily charting. In precision teaching, celeration charts are the standard for recording performance and students are taught to chart their own data. In this study, student data was entered into an Excel spreadsheet and set up to graph each data point as it was entered into the spreadsheet. Students did not chart their own performance but were shown their graph at the end of each session. This graph included their baseline data so students were able to see their progress over time.

Intervention took place 15-20 minutes a day, 2-3 times a week and consisted of a four part teaching model: direct instruction, fluency timings, daily charting and baseline/retention probes.

**Placement**. Intervention placement was established using one-minute timed fluency probes, one probe for each fraction skill to be taught. Beginning with the first probe, participants were given one minute to complete as many problems as they could. If they met requirements for mastery, they were given the next probe. This continued until participants reached a fluency probe that they were not able to complete at the mastery level.

**Instruction.** Instruction was given by a licensed Level 2 special education teacher, as defined by Utah licensure rules. The instruction in this intervention used components of direct instruction.  It was teacher led, contained structured and sequenced teacher-student interactions, monitoring of student progress and the provision of immediate feedback to participants.

**Opportunities to respond.** Following instruction on each fraction skill, participants practiced the skill using one-minute timed fluency probes until they reached mastery on three (it was not required that this be on consecutive probes), at which point they were given the next lesson. Proficiency criteria were set at a specific rate of correct digits per minute, depending on the fraction skill (Table 1). Participants were given an average of three fluency probes per intervention session. At the end of each session, digits correct were calculated, entered into Excell and the resulting graph shown to the student.

**Baseline.** Baseline was established using the same timed fluency probes as those used for curriculum placement. The data points from each of the curriculum placement probes served as the first baseline data point for each skill. Repeated fluency probes were given on the first fraction skill to be taught until a stable baseline was achieved. At this point, instruction on that skill began. Baseline probes on remaining fraction skills to be taught were given intermittently throughout the intervention.

**Retention probes.** Once a student reached mastery on a specific fraction skill, she was not given the related fluency probe until she was tested for retention. Retention probes were given at 2 week intervals following initial mastery of a fraction skill. Due to school ending, there was a 5 week gap between the 2nd and 3rd retention probes on Identifying Fractions for both participants and a 3 week gap between the 1st and 2nd retention probe on Equivalent Fractions for student 1.

**Interobserver agreement.** Interobserver agreement was conducted for both students across all phases of intervention. This was done on 33% of the probes given to each student. A second party not associated with the study checked the scores of the probes by a) looking to see whether each answer was correct or not and b) checking that number of correct digits per minute was calculated correctly. Agreement was at 99%.

## Results

Both participants were placed at Lesson 1 in the instructional program and completed the units for identifying fractions, equivalent fractions, and reducing fractions. Although more fluency probes were given, only the highest score for each session was graphed. According to Cooper, Heron and Heward (2007), a data point represents two things: "(a) a quantifiable measure of the target behavior recorded during a given observation period and (b) the time and or experimental conditions under which that particular measure was conducted" (p. 130). Graphing all probes within a session misconstrues part (a) mentioned above and makes it difficult to compare both the timings within a session and the timings from the next session. The researchers felt that one data point per session presented a more consistent data display.

**Lisa**

Results of the intervention with Lisa are displayed in Figure 1. For identifying fractions, Lisa met the criterion of 53 digits per minute during the 5th session of intervention. She scored at or above criterion on 2 of the 5 retention probes. Lisa reached the criterion of 22 digits per minute for the skill of equivalent fractions on the 6th session but only retained this level for 1 of the 5 retention probes. She met the criterion of 24 digits per minute for reducing fractions during the 7th session and maintained criterion for 1 of the 3 retention probes. Although Lisa did not consistently retain at criterion levels, she completed the fluency probes at a much higher rate than she achieved on baseline probes.

**Mary**

Figure 2 shows the results of the intervention with Mary. She met the criterion of 53 digits per minute for identifying fractions during the 4th intervention session and held it for 5 of 5 retention probes. For equivalent fractions she met the criterion of 22 digits per minute during the 2nd session and continued to meet criterion on 4 of the 5 retention probes. Mary met the criterion of 24 digits per minute for reducing fractions during the 6th session and maintained criterion for 1 of the 3 retention probes. As with Lisa, Mary completed the probes at a much higher rate than on baseline probes, but did not consistently retain at criterion levels.

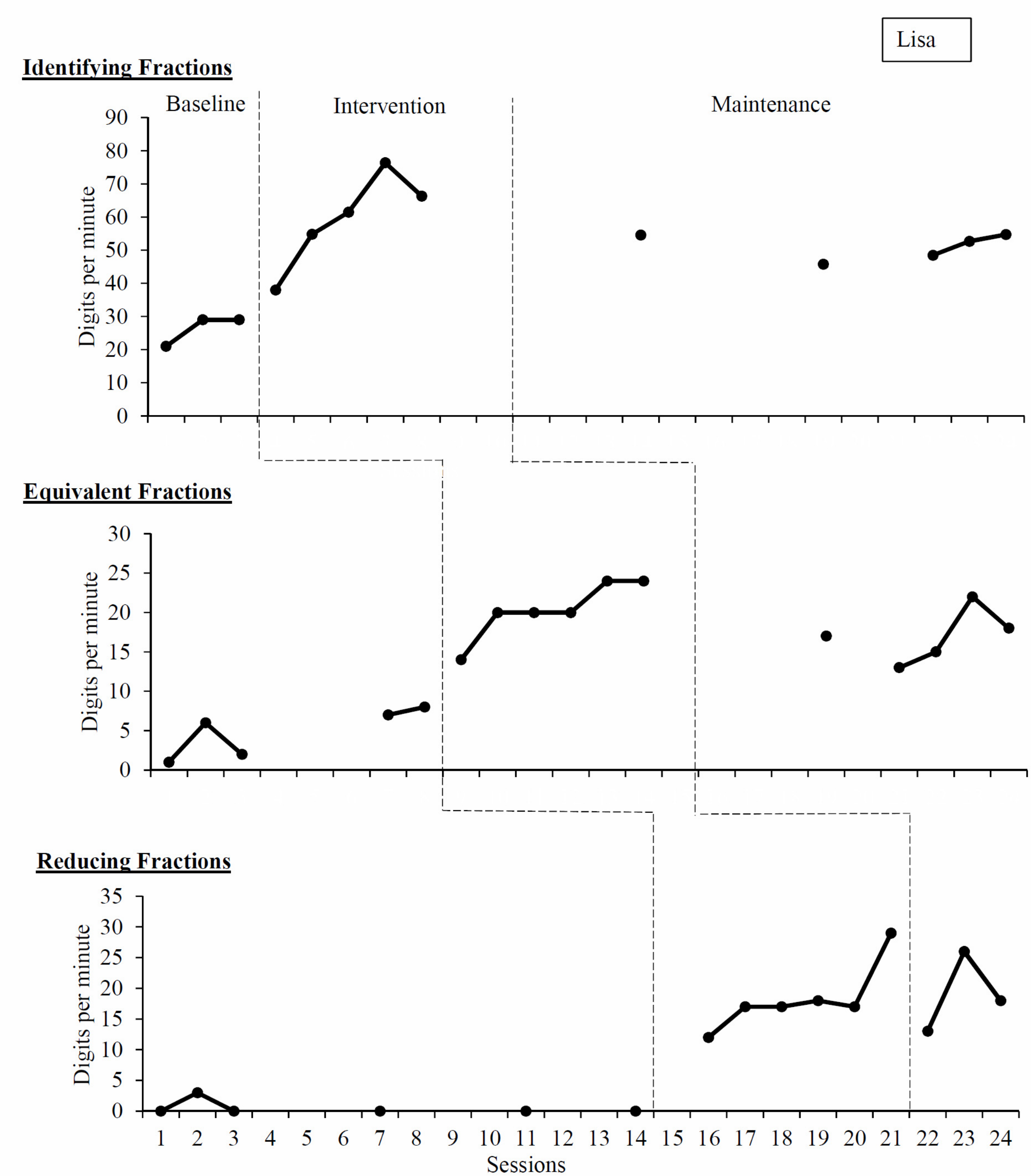


*Figure 1*. Lisa’s data across baseline, intervention, and maintenance.

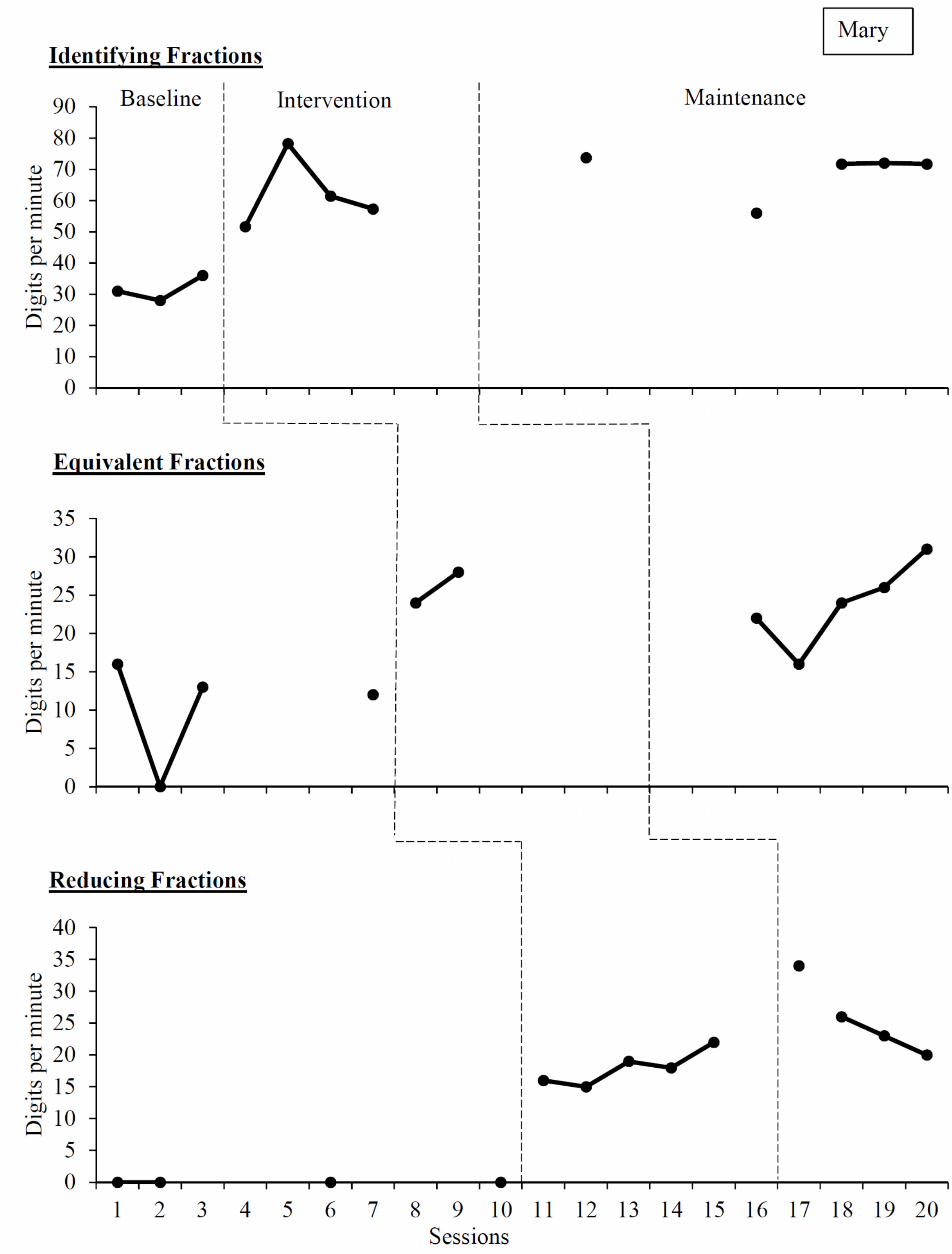


*Figure 2*. Mary's data across baseline, intervention, and maintenance.

## Discussion

The purpose of this study was to determine if students with learning disabilities in mathematics could master learning targets for identifying fractions, writing equivalent fractions, and reducing fractions with both fluency and accuracy. A secondary purpose was to measure maintenance of fluency and accuracy on retention probes given at two and four week intervals following initial mastery.

The study began with four secondary students but data were not collected from one student because he ran away from home and did not come to school for two weeks. A second student was absent 80% of the time and did not participate in the intervention, which is not surprising. A student who lacks fluency is likely to have higher levels of anxiety and avoidance, which may have contributed to these two students' excessive absences (McCallum & Schmitt, 2011). The remaining two students attended intervention regularly, providing data for the study.

In relation to fractions, Calhoon, et al. (2007) found that high school students who lacked fluency were generally functioning at an elementary level. Such was the case with the two participants in this study. They lacked skills in identifying fractions, reducing fractions and finding equivalent fractions. The Core Curriculum Standards classify these skills as $3^{rd}$ and $4^{th}$ grade. Moreover, the NCTM (2014) identifies computational fluency as necessary for students to obtain higher level math skills such as the understanding of when and how to compute fractions.

The lessons in this study were designed and delivered using structured and sequenced student-teacher interactions, opportunities to respond, monitoring, and feedback to the student in accordance with the noted research. Misquitta (2011); Perkins & Cullinan (1985); Rosenshine (1986); and Scarlato and Burr (2002) found that the direct instruction teaching model was effective in creating successful outcomes in student learning. Explicit instruction, multiple opportunities to respond and continuous monitoring are components of both direct instruction and precision teaching. Furthermore, precision teaching emphasizes both speed and accuracy, which are observable and measurable (Lindsley, 1992; Potts et al., 1993). Based on research by Lindsley (1992) and Carnine et al. (2010) regarding the utilization of the components of effective instruction, it is not surprising that the students in this study improved in fraction fluency and accuracy. In addition, the students showed maintenance of the fraction skills at levels much higher than baseline performance.

**Implications for Further Research**

These finding suggest a need for further research on the effects of fluency training on addition, subtraction, multiplication and division of fractions as well as other higher level math skills as required in high school. Another possible study would be to see if a student's increased abilities in fluency and accuracy of fraction skills leads to improved performance on algebra tasks. Maintenance of fraction skills was another aspect of research extending this study that could be conducted. Eight weeks following initial mastery, students in this study maintained skill levels well above baseline, although not at criterion levels. Once students reached mastery, they did not practice the skill until the next maintenance probe. It would be interesting to study whether students could maintain skills at higher levels and over longer periods of time with minimal practice by using activities such as completing a fluency sheet once a week between maintenance probes.

Practitioners should consider providing a small amount of time during the week to provide basic fraction skill instruction as well as time for students to practice these skills in the form of fluency sheets. This intervention is simple to implement and is not staff intensive. It could also be applied to other basic math skills in which the students may have deficits.

**Limitations**

In a multiple probe multiple baseline design, probes are given strategically throughout the intervention. Each skill is probed before any intervention occurs, after criterion for a skill has been met and each time a new phase is introduced. This ensures that there is no carry over from one phase to the next. In this study, there were three phases of intervention- one for each skill taught (identifying fractions, equivalent fractions and reducing fractions). In following the multiple probe multiple baseline design, probes on each of the three skills would be given to establish a baseline. Phase one of the intervention (instruction on identifying fractions) would then begin. Once a student reached criteria on phase one, a probe on the remaining two fraction skills (equivalent and reducing) would be given and phase two of the intervention (instruction on equivalent fractions) would begin. In this way, the researcher could assess whether instruction in phase one had an effect on the student's ability to perform the fraction skills yet to be taught in phases two and three. While students in this study did receive probes prior to the introduction of each new phase of intervention, the probes were not given immediately prior to intervention due to the challenges of school schedules such as school breaks, student absences and in-school activities which interfere with scheduled intervention times. As a result, these finding should be interpreted with caution and further research is required.

Another aspect of this design that can be limiting is the repeated probes prior to instruction. Although this can be frustrating for students as they are required to repeatedly complete tasks that they have not been taught how to do, it was not an apparent problem for the students in this study. Future studies should take this into consideration.

**Conclusion**

Results from this study suggest that a $7^{th}$ and $9^{th}$ grader with learning disabilities in math can perform basic fraction skills at the level of their peers as a result of minimal intensive instruction and practice time. Study participants spent an average of 45 minutes weekly in instruction and practice with fluency sheets, making the procedures reasonable if teachers carefully schedule the time. It is also reasonable that a larger group of students could be accommodated with little difficulty. If teachers identify students, match instruction to their specific needs, and monitor progress to make instructional decisions, then explicit instruction for mastering fractions can effectively increase student achievement.

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# APPENDIX A: EXTENDED REVIEW OF LITERATURE

The Individuals with Disabilities Education Act (IDEA) as well as recent legislation in the form of the Every Student Succeeds Act (ESSA) stress increased standards for students with disabilities as well as access to the same core instruction as that of their non-disabled peers. In addition, there has been an increase in accountability for teachers specific to learning outcomes of all students, including those with disabilities. Students with disabilities are required to take the same state assessments as their non-disabled peers. Therefore, students with learning disabilities are expected to meet the same core objectives as their non-disabled peers while coping with severe skill deficits. They must not only keep up with new concepts, but also master skills that eluded them the first time they were introduced. To achieve this goal, a student must learn at a rate faster than that of their grade-level peers. However, this is a daunting task. When it comes to mathematics, students with learning disabilities tend to perform around two grade levels below that of their non-disabled peers; and adolescents with learning disabilities perform around a fifth grade mathematics level. These students make limited progress in computation from year to year and struggle to attain a conceptual understanding of core concepts that are necessary in order to use algorithms to solve problems involving whole and rational numbers (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Miller & Hudson, 2007). In addition, Common Core standards for math are aligned with National Council of Teachers of Mathematics (NCTM, 2016) standards that emphasize problem solving and abstract thinking, both of which are difficult for students with disabilities who often have deficits in memory and task completion skills (Miller & Hudson, 2007).

According to the National Center for Education Statistics (NCES, 2015), only 6 % of $12^{th}$ grade students with disabilities are proficient in mathematics. For students without disabilities, the percent rises to 25 (National Center for Education Statistics [NCES], 2015). Yet math is an integral aspect of life. People use whole numbers, fractions, decimals, and percent in a wide range of applications. For example, setting an alarm clock, paying for goods and services, estimating time to destinations, calculating gas mileage, tracking weight gain or loss, following recipes, buying paint, and countless other daily applications depend on math skills. The need for higher math skills may not be as apparent. Algebra is not as readily obvious as basic computational skills, but Usiskin (1995) notes that a lack of facility with algebra limits people's opportunities just as low skills in reading or writing can limit them.

Seeing and understanding patterns is key to using algebra. Formulas are patterns that can be applied over and over to solve the same types of problems, such as calculating area, length, or weight (Usiskin, 1995). Beyond basic applications, Bahr and de Garcia (2010) advocate that being mathematically literate is not just entirely about content knowledge, but includes the ability to think in a mathematical way. They define math literacy as "the ability to use resources to find, evaluate, and use information" (p. 2-3).

Higher levels of mathematics taught in junior high and high school require a broad base of number skills. Algebra, for example, requires prerequisite skills in ratios and proportional thinking, the number system, and expressions and equations (Common Core State Standards Initiative [CCSSI], 2016). Wu (2001) suggests that students struggle with algebra because it is more general and abstract than preceding math concepts. A student must also be fluent in the manipulation of symbolic representations and understand how mathematical properties apply to whole numbers, fractions, and decimals (Wu, 2001). In order to do this, a student must have skills in both conceptual and procedural knowledge.

Conceptual knowledge is the ability to make connections. These connections may be between two previously learned concepts or between a previously learned and a newly acquired concept. Students with conceptual knowledge have the ability to identify shared characteristics of different concepts and use previously acquired knowledge in a new situation (Miller & Hudson, 2007). According to Douglas Carnine (1997), a student with conceptual knowledge "not only understand[s] what the concept means but also know[s] how to apply and when to apply it" (p. 131).

Procedural knowledge is just as it sounds, knowing the specific steps, or procedure, to solve a problem. It includes algorithms and memory strategies. Unlike conceptual knowledge, a person does not need to understand what each of the steps in a procedure means in order to apply that procedural knowledge (Hallett, Nunes, & Bryant, 2010).

An ongoing subject of research in mathematical cognition is that of the relationship between computational and procedural knowledge (Hallett et al, 2010). In their article, Rittle-Johnson, Schneider & Star (2015) quoted early research by Resnick and Ford (1981), "the relationship between computational skill and conceptual understanding is one of the oldest concerns in the psychology of mathematics" (p. 587). The question of whether conceptual knowledge or procedural knowledge should be taught first or if they should be taught concurrently has been debated for years. Hallett et al. (2010) argue that while conceptual and procedural knowledge are two different methods to problem solving, a student can use both methods, as well as other tools, to solve mathematical problems. Despite any disagreement on the relationship between these two types of knowledge, it is understood that it is necessary for mathematical learners to develop skills in both areas (Rittle-Johnson et al, 2015). Research concerning computational and procedural knowledge as related to understanding fractions support the need for both types of knowledge. Some studies have found that students required a conceptual understanding prior to implementing procedural knowledge in fraction computation while others reported cases when students were able to solve fraction problems by applying procedural knowledge without the understanding of the fraction concept (Hallett et al, 2010).

Researchers found that students with math learning disabilities struggled the most in six areas of mathematics. Two of these six areas were basic operations involving fractions and fraction terminology (Calhoun, Emerson, Flores & Houchins, 2007).

**Fractions**

Understanding fractions is vital to progression in secondary mathematics; the inability to work fluently with fractions can impede progression in learning new mathematical concepts (Brown & Quinn, 2007). One study investigated the relationship between mathematics achievement and competence with fractions while controlling for intelligence, working memory and a more general competence in mathematics. The cross-lagged analysis found that proficiency in fractions leads to gains in mathematical achievement but math achievement did not lead to fraction competence (Bailey, Hoard, Nugent, & Geary, 2012).

A student who has a good understanding of fractions is likely to be successful in algebra (Wu, 2001). The National Mathematics Advisory Panel (NMAP, 2008) notes that competency in the conceptual understanding of fractions and fluency in solving fraction problems is integral to algebraic understanding. Moreover, the National Mathematics Advisory Panel (2008) reported that U.S. algebra teachers cite lack of fraction understanding as a major stumbling block for their students' ability to learn algebra. Research tends to verify this position. Booth and Newton (2012) studied the influence students' knowledge of fraction and whole number magnitude on algebra readiness tasks, where magnitude is the size of the fraction or whole number as represented on a number line. Results point to knowledge of fraction magnitude as more predictive of algebra readiness than knowledge of whole number magnitude (Booth & Newton, 2012). Similarly, Siegler, Thompson, and Schneider (2011) noted that misunderstanding or not understanding the concept of magnitude in fractions impairs children's transition from whole number thinking to working with parts of a whole. According to Mazzocco, Myers, Lewis, Hanich and Murphy (2013), "elementary school students' knowledge of fractions is a stronger predictor of their overall later high school mathematics achievement than their elementary whole number arithmetic knowledge" (p. 372). When comparing sixth grade students with a math learning disability (MLD) to those considered to be low achievers, Mazzocco et al. found that the students with MLD had "significant misconceptions" (p.373) regarding fractions. They struggled to order and compare fractions, and to name decimal representations (Mozzocco et al., 2013).

The ability to learn fraction concepts requires more in-depth understanding of numbers than is typically required for proficiency with whole numbers. This is because a reorganization of numerical knowledge is necessary when learning to work with fractions (Siegler, Fasio, Bailey & Zhou, 2013). When introduced to whole numbers, children learn that each number has a logical successor. They learn that whole numbers can be paired with objects to count sets and that sets have cardinality. They learn that adding or taking away members of sets increases or decreases the cardinality (Siegler et al., 2011; Sieglar et al., 2013). Moreover, properties of whole numbers such as never decreasing with multiplication or increasing with division do not hold true for fractions. Several theorists have written about these differences. Gelman and Williams (1998) concluded that whole number learning actually interferes with later fraction learning. Geary (2006) wrote that people are preprogrammed to understand whole numbers, but not to understand fractions. Others believe that the conceptual change necessary to move from whole number understanding to fractions is the source of difficulty (Vosniadou, Vamvakoussi, & Skopeiliti, 2008). Regardless of the cognitive demands of learning to calculate and use fractions, successful manipulation of them requires fluency.

**Fluency.** Fluency is generally defined as the ability to respond quickly and accurately to a task (National Council of Teachers of Mathematics [NCTM], 2016.). Lack of fluency is characterized as one of the "persistent problems" in learning and using math (Kellman et al., 2008, p. 357). In mathematics, computational fluency is part of the larger concept of procedural fluency, which includes not only speed and accuracy, but understanding of when and how to compute fractions (NCTM, 2014). A deficit in computational fluency can inhibit mathematical comprehension just as deficits in decoding can inhibit reading comprehension (Calhoon, Emerson, Flores and Houchins, 2007). Lack of computational fluency can greatly hinder students' attainment of higher level mathematical skills, and students with disabilities commonly lack the ability to recall basic number facts or to use them in computation (Farrell & McDougall, 2008). Calhoon et al. (2007) studied which grade levels and in which skill areas high school students with mathematics disabilities were computationally fluent. Results indicated that the computational fluency of these students was at a 3rd grade level, with fractions and decimals being the skills that students struggled with the most.

Researchers have studied the effects of fluency training on the mastery and retention of learning targets (Lee & Singer-Dudek, 2012). A student who is fluent with math facts is able to complete more problems in a given amount of time. This leads to an increased opportunity to respond and more chances for reinforcement of correct responses, both of which are important components of learning (McCallum & Schmitt, 2011). A fluent student is likely to have lower levels of anxiety and avoidance when presented with a math task (McCallum & Schmitt, 2011;

Cates & Rymer, 2003). From a cognitive processing perspective, a student solving complex math problems who is fluent in basic math facts and procedures uses less cognitive capacity, leaving more capacity available to solve the problems (McCallum & Schmitt, 2011; Poncy, Skinner, & Jaspers, 2007). In addition, behavior fluency theorists cite skill maintenance, increased capacity to remain on task when solving a problem, and ability to adapt and combine skills in new situations as benefits of fluency (Binder, 1996; Singer-Dudek & Greer, 2005; Farrell & McDougall, 2008).

Much of the literature on fluency refers to an instructional hierarchy developed by Haring and Eaton in *The Fourth R: Research in the classroom* (Haring, Lovitt, Eaton & Hansen, 1978). This hierarchy outlines four stages of learning: acquisition, fluency, generalization and adaptation. During the acquisition stage, students begin to learn the target skill. The goal is to increase accuracy. Once students are able to complete the skill accurately, they move to the fluency stage. This stage consists of instructional techniques aimed at improving the student's frequency of responding accurately. The authors give three possible definitions of fluency. First, the ability to perform the skill at a level that establishes maintenance of that skill. Second, the ability to perform the skill at a level that establishes success the next time the same or similar task is performed. Third, the ability to perform the skill at a level that is similar to successful peers. Students must become fluent in the target skill in order to move to the next two stages of generalization and adaptation. Students are considered to have mastered the target skill when they have successfully moved through all four stages. According to this model, students do not achieve mastery until they become fluent in the skill (Haring et al., 1978). One research-based approach to teaching for fluency, as well as accuracy, is direct instruction.

**Direct Instruction**

In a direct instruction teaching model, the teacher provides clear instruction, adequate opportunities to respond and ongoing monitoring of student performance (Carnine, Silbert, Kame'enui, & Tarver, 2010). After researching what components of teaching determine student academic success, Rosenshine (1986) summarized his findings and called the collection direct instruction. High levels of student engagement, teacher led sequenced and structured instruction, monitoring of student performance, and immediate feedback to students were all elements of teaching that were included in his description (Rosenshine, 1986).

Misquitta (2011) reviewed literature that was published between 1990 and 2008 to find studies that involved teaching fraction skills to students who had difficulty with mathematics. In his review, he found that there were three interventions that were effective for the instruction of fractions. One of these three interventions was direct instruction (Misquitta, 2011). The effectiveness of direct instruction has been shown in studies with both elementary and secondary students. In one such study on three elementary aged students, an instructional program based on direct instruction principles was implemented. The results showed improvements on students' fraction performance as well as an ability to maintain these improvements over a short follow-up period (Perkins & Cullinan, 1985). In a similar study directed at secondary students, four seventh graders with learning disabilities were taught fraction and decimal skills using the direct instruction method. These students were able to perform at a level above that of their peers on standardized and informal assessments (Scarlato & Burr, 2002).

Direct instruction can use scripted teaching materials that instructors follow word-for-word, or can be unscripted and require teachers to use their own language to follow the steps for modeling, prompting responses, feedback, guided practice, and independent practice (Kim &

Axelrod, 2005). Direct instruction monitors response accuracy to determine whether to continue teaching or to reteach what has not been mastered. A formal system for using student response data to make instruction decisions is exemplified by precision teaching.

**Precision teaching.** Precision teaching was developed by Ogden R. Lindsley, a behavioral scientist who worked under B. F. Skinner at Harvard University in the 1950's. According to Lindsley (1992) "Precision teaching is basing educational decisions on changes in continuous self-monitored performance frequencies displayed on standard celeration charts" (p. 51); therefore, it is a decision-making system rather than a set of instructional behaviors. Lindsley left his behavioral research at Harvard when he became a professor at the University of Kansas in 1965. In his model for precision teaching, Lindsley used six principles of Skinner's work in behavior analysis: consequences for behavior, "the learner knows best" (Lindsley, 1972), observable behavior, daily monitoring of frequency, measuring behavior using frequency and using a standard display for data (Potts, Eshleman & Cooper, 1993). Precision teaching emphasizes both speed and accuracy. Through the use of multiple opportunities to respond in the form of fluency practice and the daily charting of student performance, precision teaching provides constant feedback on student behavior. This feedback enables teachers to differentiate and modify instruction to fit the needs of each student. In this way, the student tells the teacher if instruction is effective (West & Young, 1992).

Precision teaching has extended the definition of fluency to include the attainment of critical learning outcomes in addition to ease and quickness of responding to an accuracy requirement. This is referred to as behavioral fluency and is produced through frequency building. Kubina and Yurich (2012) define frequency building as "the timed repetition of the behavior with performance feedback" (p. 324). One of the learning outcomes associated with behavioral fluency is long-term retention. Retention is the ability to accurately perform a behavior when it has not been practiced for an extended period of time (Kubina & Yurich, 2012). In research on the effects of fluency training on retention, Bucklin, Dickinson and Brethower (2000) found that fluency training led to gains in retention rates for college students. The authors also cite three controlled studies conducted within educational settings that looked at the effects of fluency on retention. Ivarie (1986) found that for students who were considered average or below average, fluency training resulted in higher retention rates. Similarly, Berquam (1981) observed that, given a retention assessment, students who were trained using timed fluency practice were faster and more accurate than their peers who were trained using untimed fluency practice. In the third study, Shirley and Pennypacker (1994) looked at three conditions for learning spelling words: no specified criterion, an accuracy criterion, and a fluency criterion. Participants achieved higher retention rates on the fluency condition when compared to the no specified criterion condition. For one of the two participants, fluency training also resulted in better retention rates compared to the accuracy criterion condition (as cited by Bucklin et al, 2000).

In addition to increased accuracy and retention, research has shown that when a student is fluent, his abilities to master more advanced math skills is enhanced (Poncy et al, 2007; Poncy, 2010). Precision teachers refer to this as steeper slopes and rising bottoms in reference to graphs of student performance. Steeper slopes refer to faster acquisition of skills resulting in fewer data points and a steeper line on a data graph. Rising bottoms comes from the observance of higher scores with initial trials that, when graphed, create higher beginning data points (West & Young, 1992). With outcomes such as increased accuracy rates, better retention and an increased ability to master subsequent tasks, fluency training is a viable option for high school students who have not become proficient with math skills taught in the lower grades.

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# APPENDIX B: CONSENT TO CONDUCT RESEARCH

Institutional Review Board
for Human Subjects

Brigham Young University
A-285 ASB Provo, Utah 84602
(801) 422-3841 / Fax: (801) 422-0620

December 8, 2016

Jani Ashbaker
722 W 4050 N
Pleasant View, UT 84414

Re: The Effects of Fluency Training on the Acquisition and Retention of Secondary Students' Fraction Skills

Dear Jani Ashbaker

This is to inform you that Brigham Young University's IRB has approved the above research study.

The approval period is from 12-8-2016 to 12-7-2017. Your study number is X16435. Please be sure to reference this number in any correspondence with the IRB.

Continued approval is conditional upon your compliance with the following requirements.

All protocol amendments and changes to approved research must be submitted to the IRB and not be implemented until approved by the IRB.

A few months before this date we will send out a continuing review form. There will only be two reminders. Please fill this form out in a timely manner to ensure that there is not a lapse in your approval.

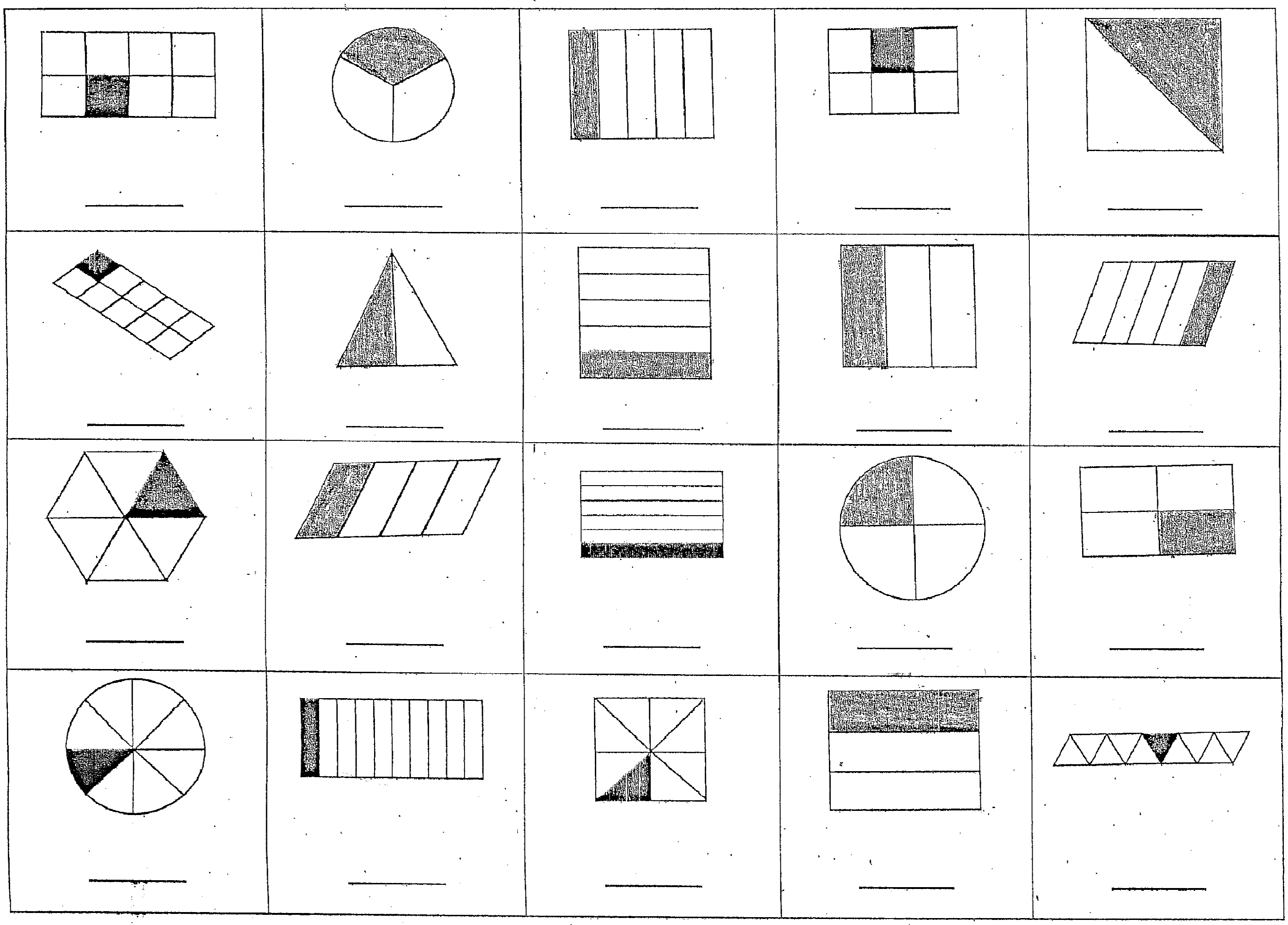
If you have any questions, please do not hesitate to call me.

Sincerely,

Robert Ridge, PhD, Chair

# APPENDIX C: FLUENCY PROBES

Fractions- Identifying Halves, Thirds, Fourths, Fifths, Sixths, Eighths, and Tenths of a Region



BASIC Skill Builders

SEE TO WRITE

| | Correct | Error |
|---|---|---|
| First Try | | |
| Second Try | | |

**Equivalent Fraction Combinations**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{3}{4} = \frac{\_\_}{8}$ | $\frac{2}{3} = \frac{\_\_}{12}$ | $\frac{4}{5} = \frac{\_\_}{20}$ | $\frac{2}{5} = \frac{\_\_}{10}$ | $\frac{1}{3} = \frac{\_\_}{6}$ | $\frac{2}{3} = \frac{\_\_}{15}$ | $\frac{7}{8} = \frac{\_\_}{32}$ | $\frac{5}{6} = \frac{\_\_}{18}$ | $\frac{9}{10} = \frac{\_\_}{50}$ | $\frac{3}{4} = \frac{\_\_}{12}$ | (15) |
| $\frac{1}{5} = \frac{\_\_}{20}$ | $\frac{5}{9} = \frac{\_\_}{27}$ | $\frac{3}{11} = \frac{\_\_}{22}$ | $\frac{1}{4} = \frac{\_\_}{24}$ | $\frac{3}{8} = \frac{\_\_}{32}$ | $\frac{11}{16} = \frac{\_\_}{48}$ | $\frac{1}{2} = \frac{\_\_}{10}$ | $\frac{3}{8} = \frac{\_\_}{24}$ | $\frac{5}{6} = \frac{\_\_}{18}$ | $\frac{5}{8} = \frac{\_\_}{40}$ | (30) |
| $\frac{3}{7} = \frac{\_\_}{14}$ | $\frac{1}{5} = \frac{\_\_}{20}$ | $\frac{3}{4} = \frac{\_\_}{16}$ | $\frac{1}{3} = \frac{\_\_}{15}$ | $\frac{5}{8} = \frac{\_\_}{24}$ | $\frac{6}{7} = \frac{\_\_}{35}$ | $\frac{5}{14} = \frac{\_\_}{28}$ | $\frac{1}{6} = \frac{\_\_}{12}$ | $\frac{1}{3} = \frac{\_\_}{6}$ | $\frac{4}{15} = \frac{\_\_}{30}$ | (44) |
| $\frac{3}{5} = \frac{\_\_}{20}$ | $\frac{11}{24} = \frac{\_\_}{48}$ | $\frac{2}{3} = \frac{\_\_}{9}$ | $\frac{2}{9} = \frac{\_\_}{18}$ | $\frac{7}{10} = \frac{\_\_}{20}$ | $\frac{21}{50} = \frac{\_\_}{100}$ | $\frac{5}{8} = \frac{\_\_}{16}$ | $\frac{1}{4} = \frac{\_\_}{16}$ | $\frac{1}{6} = \frac{\_\_}{24}$ | $\frac{1}{2} = \frac{\_\_}{12}$ | (59) |
| $\frac{1}{2} = \frac{\_\_}{12}$ | $\frac{5}{7} = \frac{\_\_}{21}$ | $\frac{7}{8} = \frac{\_\_}{80}$ | $\frac{2}{5} = \frac{\_\_}{10}$ | $\frac{3}{4} = \frac{\_\_}{8}$ | $\frac{4}{5} = \frac{\_\_}{10}$ | $\frac{2}{3} = \frac{\_\_}{27}$ | $\frac{3}{4} = \frac{\_\_}{36}$ | $\frac{1}{3} = \frac{\_\_}{12}$ | $\frac{4}{15} = \frac{\_\_}{60}$ | (74) |

BASIC Skill Builders

SEE TO WRITE

|  | Correct | Error |
|---|---|---|
| First Try |  |  |
| Second Try |  |  |

**Reducing Fractions—Random**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{2}{4}$ = ___ | $\frac{2}{10}$ = ___ | $\frac{4}{12}$ = ___ | $\frac{4}{6}$ = ___ | $\frac{4}{16}$ = ___ | $\frac{3}{18}$ = ___ | $\frac{6}{15}$ = ___ | $\frac{5}{20}$ = ___ | $\frac{9}{12}$ = ___ | $\frac{6}{14}$ = ___ | (20) |
| $\frac{2}{12}$ = ___ | $\frac{14}{16}$ = ___ | $\frac{12}{20}$ = ___ | $\frac{2}{6}$ = ___ | $\frac{8}{18}$ = ___ | $\frac{10}{20}$ = ___ | $\frac{12}{16}$ = ___ | $\frac{6}{20}$ = ___ | $\frac{6}{8}$ = ___ | $\frac{12}{18}$ = ___ | (40) |
| $\frac{4}{14}$ = ___ | $\frac{16}{20}$ = ___ | $\frac{4}{8}$ = ___ | $\frac{10}{18}$ = ___ | $\frac{3}{15}$ = ___ | $\frac{10}{15}$ = ___ | $\frac{18}{20}$ = ___ | $\frac{5}{10}$ = ___ | $\frac{8}{12}$ = ___ | $\frac{6}{16}$ = ___ | (60) |
| $\frac{9}{15}$ = ___ | $\frac{2}{8}$ = ___ | $\frac{6}{10}$ = ___ | $\frac{8}{16}$ = ___ | $\frac{12}{15}$ = ___ | $\frac{5}{15}$ = ___ | $\frac{2}{20}$ = ___ | $\frac{4}{10}$ = ___ | $\frac{3}{12}$ = ___ | $\frac{10}{16}$ = ___ | (80) |
| $\frac{14}{20}$ = ___ | $\frac{16}{18}$ = ___ | $\frac{2}{16}$ = ___ | $\frac{8}{20}$ = ___ | $\frac{6}{12}$ = ___ | $\frac{10}{14}$ = ___ | $\frac{14}{18}$ = ___ | $\frac{4}{20}$ = ___ | $\frac{8}{12}$ = ___ | $\frac{2}{14}$ = ___ | (100) |
| $\frac{4}{18}$ = ___ | $\frac{10}{12}$ = ___ | $\frac{8}{14}$ = ___ | $\frac{2}{18}$ = ___ | $\frac{6}{16}$ = ___ | $\frac{12}{14}$ = ___ | $\frac{4}{10}$ = ___ | $\frac{9}{18}$ = ___ | $\frac{2}{6}$ = ___ | $\frac{3}{15}$ = ___ | (120) |
| $\frac{8}{14}$ = ___ | $\frac{9}{15}$ = ___ | $\frac{2}{4}$ = ___ | $\frac{6}{8}$ = ___ | $\frac{14}{20}$ = ___ | $\frac{6}{16}$ = ___ | $\frac{16}{18}$ = ___ | $\frac{6}{12}$ = ___ | $\frac{14}{16}$ = ___ | $\frac{4}{6}$ = ___ | (140) |
| $\frac{5}{15}$ = ___ | $\frac{8}{20}$ = ___ | $\frac{4}{8}$ = ___ | $\frac{2}{20}$ = ___ | $\frac{10}{16}$ = ___ | $\frac{10}{12}$ = ___ | $\frac{12}{18}$ = ___ | $\frac{5}{20}$ = ___ | $\frac{3}{6}$ = ___ | $\frac{12}{15}$ = ___ | (160) |
| $\frac{2}{8}$ = ___ | $\frac{6}{14}$ = ___ | $\frac{2}{6}$ = ___ | $\frac{9}{18}$ = ___ | $\frac{8}{10}$ = ___ | $\frac{18}{20}$ = ___ | $\frac{12}{16}$ = ___ | $\frac{6}{10}$ = ___ | $\frac{6}{18}$ = ___ | $\frac{12}{24}$ = ___ | (180) |
| $\frac{16}{20}$ = ___ | $\frac{2}{14}$ = ___ | $\frac{6}{15}$ = ___ | $\frac{12}{20}$ = ___ | $\frac{9}{12}$ = ___ | $\frac{3}{12}$ = ___ | $\frac{10}{15}$ = ___ | $\frac{4}{20}$ = ___ | $\frac{8}{12}$ = ___ | $\frac{10}{14}$ = ___ | (200) |

# Comparing Fractions (A)

Compare each pair of fractions using a <, > or = sign.

| | | | |
|---|---|---|---|
| $\frac{3}{4}$ □ $\frac{1}{3}$ | $\frac{2}{9}$ □ $\frac{6}{9}$ | $\frac{2}{3}$ □ $\frac{3}{10}$ | $\frac{3}{5}$ □ $\frac{3}{11}$ |
| $\frac{4}{6}$ □ $\frac{7}{8}$ | $\frac{6}{8}$ □ $\frac{1}{2}$ | $\frac{6}{12}$ □ $\frac{4}{5}$ | $\frac{3}{6}$ □ $\frac{2}{7}$ |
| $\frac{7}{8}$ □ $\frac{8}{11}$ | $\frac{7}{9}$ □ $\frac{5}{8}$ | $\frac{3}{8}$ □ $\frac{1}{5}$ | $\frac{3}{4}$ □ $\frac{4}{6}$ |
| $\frac{1}{8}$ □ $\frac{3}{6}$ | $\frac{1}{11}$ □ $\frac{1}{2}$ | $\frac{1}{3}$ □ $\frac{1}{12}$ | $\frac{2}{3}$ □ $\frac{2}{5}$ |
| $\frac{3}{7}$ □ $\frac{1}{11}$ | $\frac{2}{3}$ □ $\frac{5}{10}$ | $\frac{8}{12}$ □ $\frac{2}{5}$ | $\frac{2}{3}$ □ $\frac{2}{12}$ |
| $\frac{2}{3}$ □ $\frac{5}{7}$ | $\frac{3}{10}$ □ $\frac{1}{10}$ | $\frac{4}{5}$ □ $\frac{1}{8}$ | $\frac{1}{2}$ □ $\frac{3}{12}$ |
| $\frac{5}{7}$ □ $\frac{1}{2}$ | $\frac{2}{4}$ □ $\frac{2}{6}$ | $\frac{5}{6}$ □ $\frac{1}{6}$ | $\frac{1}{2}$ □ $\frac{5}{6}$ |
| $\frac{4}{7}$ □ $\frac{5}{8}$ | $\frac{5}{9}$ □ $\frac{3}{12}$ | $\frac{1}{2}$ □ $\frac{4}{12}$ | $\frac{4}{8}$ □ $\frac{5}{6}$ |
| $\frac{6}{11}$ □ $\frac{6}{7}$ | $\frac{4}{9}$ □ $\frac{3}{6}$ | $\frac{2}{4}$ □ $\frac{2}{11}$ | $\frac{2}{6}$ □ $\frac{3}{5}$ |
| $\frac{1}{2}$ □ $\frac{5}{10}$ | $\frac{4}{12}$ □ $\frac{5}{12}$ | $\frac{8}{9}$ □ $\frac{3}{7}$ | $\frac{5}{9}$ □ $\frac{6}{10}$ |

Adding and Subtracting Fractions

| | | | | |
|---|---|---|---|---|
| $\frac{12}{17}-\frac{3}{17}$ ______ | $\frac{2}{3}+\frac{1}{9}$ ______ | $\frac{8}{3}-\frac{3}{2}$ ______ | $\frac{3}{5}-\frac{1}{2}$ ______ | $\frac{1}{5}+\frac{11}{5}$ ______ |
| $\frac{7}{10}+\frac{2}{5}$ ______ | $\frac{1}{2}-\frac{1}{20}$ ______ | $\frac{11}{12}-\frac{5}{12}$ ______ | $\frac{3}{4}+\frac{1}{16}$ ______ | $\frac{5}{2}+\frac{2}{3}$ ______ |
| $\frac{4}{3}-\frac{2}{5}$ ______ | $\frac{1}{4}+\frac{1}{2}$ ______ | $\frac{1}{4}+\frac{1}{2}$ ______ | $\frac{9}{11}+\frac{1}{11}$ ______ | $\frac{17}{20}-\frac{4}{5}$ ______ |
| $\frac{1}{8}+\frac{5}{8}$ ______ | $\frac{1}{2}+\frac{3}{16}$ ______ | $\frac{1}{2}-\frac{1}{8}$ ______ | $\frac{2}{3}-\frac{1}{3}$ ______ | $\frac{3}{2}-\frac{9}{7}$ ______ |

Multiplying Proper Fractions

| | | | | |
|---|---|---|---|---|
| $\frac{2}{3} \times \frac{8}{11}$ ______ | $\frac{1}{2} \times \frac{3}{4}$ ______ | $\frac{1}{3} \times \frac{2}{3}$ ______ | $\frac{1}{5} \times \frac{4}{5}$ ______ | $\frac{1}{2} \times \frac{1}{2}$ ______ |
| $\frac{1}{3} \times \frac{4}{5}$ ______ | $\frac{1}{2} \times \frac{7}{11}$ ______ | $\frac{3}{5} \times \frac{3}{4}$ ______ | $\frac{7}{9} \times \frac{1}{2}$ ______ | $\frac{1}{2} \times \frac{1}{6}$ ______ |
| $\frac{1}{2} \times \frac{3}{4}$ ______ | $\frac{5}{7} \times \frac{1}{2}$ ______ | $\frac{1}{8} \times \frac{1}{4}$ ______ | $\frac{1}{2} \times \frac{5}{6}$ ______ | $\frac{5}{6} \times \frac{1}{3}$ ______ |
| $\frac{4}{9} \times \frac{2}{3}$ ______ | $\frac{1}{4} \times \frac{3}{4}$ ______ | $\frac{8}{9} \times \frac{2}{3}$ ______ | $\frac{9}{11} \times \frac{1}{2}$ ______ | $\frac{1}{3} \times \frac{7}{10}$ ______ |

Dividing Proper Fractions

| | | | | |
|---|---|---|---|---|
| $\frac{1}{3} \div \frac{3}{4}$ ______ | $\frac{1}{5} \div \frac{2}{3}$ ______ | $\frac{1}{3} \div \frac{7}{10}$ ______ | $\frac{1}{2} \div \frac{2}{3}$ ______ | $\frac{1}{5} \div \frac{2}{7}$ ______ |
| $\frac{1}{7} \div \frac{1}{5}$ ______ | $\frac{2}{9} \div \frac{3}{4}$ ______ | $\frac{1}{7} \div \frac{1}{2}$ ______ | $\frac{3}{7} \div \frac{5}{9}$ ______ | $\frac{1}{4} \div \frac{8}{9}$ ______ |
| $\frac{3}{4} \div \frac{8}{9}$ ______ | $\frac{1}{4} \div \frac{7}{9}$ ______ | $\frac{1}{5} \div \frac{1}{2}$ ______ | $\frac{1}{8} \div \frac{6}{7}$ ______ | $\frac{3}{8} \div \frac{4}{5}$ ______ |
| $\frac{1}{4} \div \frac{4}{5}$ ______ | $\frac{5}{7} \div \frac{4}{5}$ ______ | $\frac{2}{9} \div \frac{1}{2}$ ______ | $\frac{1}{5} \div \frac{2}{7}$ ______ | $\frac{4}{9} \div \frac{1}{2}$ ______ |

BASIC Skill Builders

SEE TO WRITE

| | Correct | Error |
|---|---|---|
| First Try | | |
| Second Try | | |

**Changing Mixed Numbers to Improper Fractions**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $5\frac{1}{3}$ | $2\frac{3}{4}$ | $1\frac{5}{9}$ | $2\frac{1}{6}$ | $8\frac{1}{3}$ | $2\frac{3}{8}$ | $3\frac{3}{5}$ | $6\frac{1}{6}$ | $2\frac{2}{3}$ | $5\frac{1}{4}$ | $7\frac{1}{5}$ | $1\frac{2}{5}$ | (24) |
| $6\frac{1}{4}$ | $5\frac{5}{7}$ | $9\frac{1}{2}$ | $4\frac{1}{4}$ | $3\frac{1}{3}$ | $4\frac{5}{8}$ | $7\frac{1}{3}$ | $5\frac{3}{7}$ | $4\frac{1}{4}$ | $4\frac{1}{3}$ | $1\frac{7}{8}$ | $2\frac{1}{14}$ | (48) |
| $1\frac{3}{4}$ | $3\frac{3}{4}$ | $2\frac{3}{5}$ | $5\frac{1}{6}$ | $3\frac{1}{5}$ | $2\frac{7}{8}$ | $5\frac{2}{3}$ | $4\frac{5}{6}$ | $6\frac{1}{5}$ | $5\frac{1}{5}$ | $4\frac{2}{5}$ | $3\frac{8}{9}$ | (72) |
| $1\frac{1}{2}$ | $2\frac{1}{2}$ | $3\frac{1}{4}$ | $6\frac{1}{2}$ | $4\frac{1}{3}$ | $6\frac{5}{8}$ | $4\frac{3}{4}$ | $2\frac{9}{10}$ | $9\frac{1}{2}$ | $4\frac{1}{3}$ | $2\frac{2}{11}$ | $2\frac{9}{10}$ | (96) |

# APPENDIX D: NORM TIMINGS

Table 1

*Norm Timings by Fraction Skill*

| Fraction Skill | Digits per Minute |
|---|---|
| Identify | 53 |
| Equivalent | 22 |
| Reduce | 24 |
| Compare | 13 |
| Add/Subtract | 20 |
| Multiply | 62 |
| Divide | 41 |
| Mixed to Improper | 38 |