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Characterization of classes of graphs with large general position number

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ABSTRACT

Getting inspired by the famous no-three-in-line problem and by the general position subset selection problem from discrete geometry, the same is introduced into graph theory as follows. A set *S* of vertices in a graph *G* is a general position set if no element of *S* lies on a geodesic between any two other elements of *S*. The cardinality of a largest general position set is the general position number gp(G) of *G*. The graphs *G* of order *n* with $gp(G) \in \{2, n, n - 1\}$ were already characterized. In this paper, we characterize the classes of all connected graphs of order $n \ge 4$ with the general position number n - 2.

KEYWORDS

Diameter; girth; general position set; general position number

AMS SUBJECT CLASSIFICATION 05C12; 05C69

1. Introduction

The general position problem in graphs was introduced by P. Manuel and S. Klavžar [6] as a natural extension of the well-known century old Dudeney's no-three-in-line problem and the general position subset selection problem from discrete geometry [3, 4, 9]. The general position problem in graph theory was introduced in [6] as follows. A set S of vertices in a graph G is a general position set if no element of S lies on a geodesic between any two other elements of S. A largest general position set is called a *gp-set* and its size is the *general position number* (*gp*-number, in short), gp(G), of G.

The same concept was in use two years earlier in [2] under the name geodetic irredundant sets. The concept was defined in a different method, see the preliminaries below. In [2] it is proved that for a connected graph of order *n*, the complete graph of order n is the only graph with the largest general position number *n*; and gp(G) = n - 1 if and only if $G = K_1 + \bigcup_i m_i K_i$ with $\sum m_i \ge 2$ or $G = K_n - \{e_1, e_2, ..., e_k\}$ with $1 \le k \le n-2$, where e_i 's all are edges in K_n which are incident to a common vertex v. In [6], certain general upper and lower bounds on the gp-number are proved. In the same paper it is proved that the general position problem is NP-complete for arbitrary graphs. The gp-number for a large class of subgraphs of the infinite grid graph, for the infinite diagonal grid, and for Beneš networks were obtained in the subsequent paper [7]. Anand et al. [1] gives a characterization of general position sets in arbitrary graphs. As a consequence, the gp-number of graphs of diameter 2, cographs, graphs with at least one universal vertex, bipartite graphs and their complements were obtained. Subsequently, gp-number for the complements of trees, of grids, and of hypercubes were deduced in [1]. Recently, in [5] a sharp lower bound on the *gp*-number is proved for Cartesian products of graphs. In the same paper the *gp*-number for joins of graphs, coronas over graphs, and line graphs of complete graphs are determined. Recent developments on general position number can be seen in [8].

2. Preliminaries

Graphs used in this paper are finite, simple and undirected. The distance $d_G(u, v)$ between u and v is the minimum length of an u, v-path. An u, v-path of minimum length is also called an u, v-geodesic. The maximum distance between all pairs of vertices of G is the diameter, diam(G), of G. A subgraph H of a graph G is isometric subgraph if $d_H(u, v) =$ $d_G(u, v)$ for all $u, v \in V(H)$. The interval $I_G[u, v]$ between vertices u and v of a graph G is the set of vertices that lie on some u, v-geodesic of G. For $S \subseteq V(G)$ we set $I_G[S] =$ $\cup_{u,v \in S} I_G[u, v]$. We may simplify the above notation by omitting the index G whenever G is clear from the context.

A set of vertices $S \subseteq V(G)$ is a general position set of G if no three vertices of S lie on a common geodesic in G. A gp-set is thus a largest general position set. Call a vertex $v \in T \subseteq$ V(G) to be an *interior vertex* of T, if $v \in I[T - \{v\}]$. Now, T is a general position set if and only if T contains no interior vertices. In this way general position sets were introduced in [2] under the name geodetic irredundant sets. The maximum order of a complete subgraph of a graph G is denoted by $\omega(G)$. Let $\eta(G)$ be the maximum order of an induced complete multipartite subgraph of the complement of G. Finally, for $n \in \mathbb{N}$ we will use the notation $[n] = \{1, ..., n\}$.

In this paper, we make use of the following results.

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Theorem 2.1. [2] Let G be a connected graph of order n and diameter d. Then $gp(G) \le n - d + 1$.

Theorem 2.2. [2] *For any cycle* C_n ($n \ge 5$), $gp(C_n) = 3$.

We recall the characterization of general position sets from [1], for which we need some additional information. Let G be a connected graph, $S \subseteq V(G)$, and $\mathcal{P} = \{S_1, ..., S_p\}$ a partition of S. Then \mathcal{P} is *distance-constant* if for any $i, j \in$ $[p], i \neq j$, the distance d(u, v), where $u \in S_i$ and $v \in S_j$ is independent of the selection of u and v. If \mathcal{P} is a distanceconstant partition, and $i, j \in [p], i \neq j$, then let $d(S_i, S_j)$ be the distance between a vertex from S_i and a vertex from S_j . Finally, we say that a distance-constant partition \mathcal{P} is *intransitive* if $d(S_i, S_k) \neq d(S_i, S_j) + d(S_j, S_k)$ holds for arbitrary pairwise different $i, j, k \in [p]$.

Theorem 2.3. [1] Let G be a connected graph. Then $S \subseteq V(G)$ is a general position set if and only if the components of G[S] are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of S.

Theorem 2.4. [1] If diam(G) = 2, then $gp(G) = max\{\omega(G), \eta(G)\}$.

v_2 v_4 v_4

3. The characterization

In the following, we characterize all connected graphs *G* of order $n \ge 4$ with the *gp*- number n - 2. Since the complete graph K_n is the only connected graph of order *n* with the *gp*-number *n*, by Theorem 2.1, we need to consider only graphs with diameter 2 or 3. First, we introduce four families of graphs with the diameter 3; and four families of graphs with the diameter 2.

Let \mathcal{F}_1 be the collection of all graphs obtained from the cycle $C: u_1, u_2, u_3, u_4, u_1$ by adding k new vertices $v_1, v_2, ..., v_k (k \ge 1)$ and joining each $v_i, i \in [k]$ to the vertex u_1 . Graphs from the family \mathcal{F}_1 are presented in Figure 1.

Let \mathcal{F}_2 be the collection of all graphs obtained from the path $P_2: x, y$ and complete graphs $K_{n_1}, K_{n_2}, ..., K_{n_r} (r \ge 1)$, $K_{m_1}, K_{m_2}, ..., K_{m_s} (s \ge 1)$ and $K_{l_1}, K_{l_2}, ..., K_{l_t}$ (possibly complete graphs of this kind may be empty), by joining both x and y to all vertices of $K_{l_1}, K_{l_2}, ..., K_{l_t}$; joining x to all vertices of $K_{n_1}, K_{n_2}, ..., K_{n_r}$; and joining y to all vertices of $K_{m_1}, K_{m_2}, ..., K_{m_s}$. Graphs from the family \mathcal{F}_2 are presented in Figure 2. Trees with diameter 3 are called double stars and they belong to the class \mathcal{F}_2 .

Let \mathcal{F}_3 be the collection of all graphs obtained from the path $P_4 : u, x, y, v$ and a complete graph $K_r (r \ge 1)$ by joining both u and x to all vertices of K_r and joining y to a subset S of vertices of $V(K_r)$ (possibly S may be empty or $S = V(K_r)$). Graphs from the family \mathcal{F}_3 are presented in Figure 3.

Let \mathcal{F}_4 be the collection of all graphs obtained from the path $P_3: x, y, v$ and complete graphs $K_q, K_{n_1}, K_{n_2}, ..., K_{n_r} (r \ge 1)$, $K_{m_1}, K_{m_2}, ..., K_{m_s} (s \ge 1)$ by joining x to all vertices of $K_{n_1}, K_{n_2}, ..., K_{n_r}$; joining x and v to all vertices of $K_{m_1}, K_{m_2}, ..., K_{m_s}$; joining x and y to all vertices of $K_{m_1}, K_{m_2}, ..., K_{m_s}$; joining x and y to all vertices of K_q . Graphs from the family \mathcal{F}_4 are presented in Figure 4.

Figure 1. Family \mathcal{F}_1 .



Figure 2. Family \mathcal{F}_2 .







Figure 4. Family \mathcal{F}_4 .



Figure 5. Family \mathcal{F}_5 .

Next, we introduce four families of graphs with diameter 2.

Let \mathcal{F}_5 be the collection of all graphs obtained from the complete graph K_{n-2} ($n \ge 5$) by adding two new vertices u and v, joining u to all vertices of non-empty subset S of $V(K_{n-2})$ of size at most n - 3; and joining v to all vertices of non-empty subset T of $V(K_{n-2})$ of size at most n - 3. The set S must intersect with the set T so that, the diameter of each graph from the family \mathcal{F}_5 is 2. Graphs from the family \mathcal{F}_5 are presented in Figure 5.

Let \mathcal{F}_6 be the collection of all graphs obtained from the family \mathcal{F}_5 by adding the edge *uv*. Moreover; in this case, the set *S* may be disjoint with the set *T*. Graphs from the family \mathcal{F}_6 are presented in Figure 6.

Let \mathcal{F}_7 be the collection of all graphs obtained from the complete graphs $K_{n_1}, K_{n_2}, ..., K_{n_r} (r \ge 2)$ by adding two new vertices x and y, joining x to a non-empty subset S_i of $V(K_{n_i})$ for all $i \in [r]$; and y to a non-empty subset T_i of $V(K_{n_i})$ for all $i \in [r]$ (the edges are in a way that for any $u \in V(K_{n_i})$ and $v \in V(K_{n_j})$ with $i \neq j$ must have a common neighbor). Moreover, for some $i \in [r]$; the set S_i must



Figure 6. Family \mathcal{F}_6 .



Figure 7. Family \mathcal{F}_7 .

intersect with the set T_i so that, the diameter of each graph from the family \mathcal{F}_7 is 2. Graphs from the family \mathcal{F}_7 are presented in Figure 7. It is clear that both C_4 and C_5 belong to class \mathcal{F}_7 .

Let \mathcal{F}_8 the collection of all graphs obtained from the family \mathcal{F}_7 by adding the edge *xy*. In this case, the set S_i may be disjoint with the set T_i for all $i \in [r]$. Graphs from the family \mathcal{F}_8 are presented in Figure 8.

Theorem 3.1. Let G be a connected graph of order $n \ge 4$, then gp(G) = n - 2 if and only if G belongs to the family $\bigcup_{i=1}^{8} \mathcal{F}_{i}$.

Proof. First, suppose that *G* is a connected graph of order *n* with gp(G) = n - 2. Then it follows from Theorem 2.1 that diam(G) is either 2 or 3. We consider the following two cases.

Case 1: diam(G) = 3. If G is a tree, then G is a double star and hence it belongs to \mathcal{F}_2 . So, assume that G has cycles. Let girth(G) denotes the length of a shortest cycle in G.

Let *C* be any shortest cycle in *G*. Then it is clear that *C* is an isometric subgraph of *G*. This shows that if *S* is a general position set in *G*, then $S \cap V(C)$ is a general position set in *C*. Hence it follows from Theorem 2.2 that any general position



Figure 8. Family \mathcal{F}_8 .

set of *G* contains at most three vertices from the cycle *C*. Now, since gp(G) = n - 2, we have that the length of *C* is at most 5 and so girth(*G*) \leq 5.

Next, we claim that there is no connected graph of order *n* with girth(G) = 5 and gp(G) = n - 2. For, assume the contrary that there is a connected graph of order n with girth(G) = 5 and gp(G) = n - 2. Let $C : u_1, u_2, u_3, u_4, u_5, u_1$ be a shortest cycle of length 5 in G. Since girth(G) = 5, it follows that the vertices from $N(u_i)$ are independent for all $i \in [5]$. Also, as above we have that any general position set of G has at most three vertices from the cycle C. Let S be a general position set in G. Since gp(G) = n - 2, we have that $S = V(G) \setminus \{u_i, u_i\}$. If u_i and u_i are successive vertices in C, then it follows that the induced subgraph of S has a P_3 , which is impossible. Hence without loss of generality, we may assume that i=1 and j=3. So $S = V(G) \setminus \{u_1, u_3\}$. Now, since $u_2, u_4, u_5 \in S$ and $N(u_i)$ is independent, by Theorem 2.3, it follows that $deg(u_i) \leq 3$ for i = 2, 4, 5. Now we claim that $deg(u_2) = deg(u_4) = deg(u_5) = 2$. Otherwise, we may assume that $deg(u_2) = 3$ and let x be the neighbour of u_2 different from u_1 and u_3 . Since girth(G) = 5, it follows that x is not adjacent with the remaining vertices of C. Now, since $u_2, u_5, x \in S$, by Theorem 2.3, $d(u_5, x) = d(u_5, u_2) = 2$. Let $P: u_5, y, x$ be a u_5, x -geodesic of length 2. Then it is clear that $y \notin V(C)$ and so $y \in S$. This leads to the fact that induced subgraph of S has a P_3 , impossible in a general pos-Hence $\deg(u_2) = 2$. ition set. Similarly $deg(u_4) =$ $\deg(u_5) = 2.$

Now, if $N(u_1) \neq \emptyset$, then $u_5 \in I[x, u_4]$ for all $x \in N(u_1)$ (otherwise *S* contains an induced P_3), impossible. Hence $N(u_1) = \emptyset$. Similarly, $N(u_2) = \emptyset$. Hence $G \cong C_5$. But $gp(C_5) = 3 = n - 2$ and $diam(G) = diam(C_5) = 2$. Hence there is no connected graph of order *n* with diam(G) = 3, girth(G) = 5 and gp(G) = n - 2. Hence girth(G) is at most 4.

Now, assume that girth(G) = 4and let C: u_1, u_2, u_3, u_4, u_1 be a shortest cycle of length 4 in G. Since diam(G) = 3, we have that $G \not\cong C_4$. Now, we may assume that $u_1 \in V(C)$ be a vertex such that $\deg(u_1) \ge 3$ and let x be a neighbour of u_1 such that $x \notin V(C)$. Since S is a general position set and |S| = n - 2, we have that S contains exactly 2 vertices from C. We claim that $u_1 \notin S$. For otherwise assume that $u_1 \in S$. Since |S| = n - 2 and $x, u_1 \in S$, it follows from Theorem 2.3 that $u_2, u_4 \notin S$ and $u_3 \in S$. This shows that the path x, u_1, u_2, u_3 must be a x, u_3 -geodesic (otherwise, since |S| = n - 2, S contains an induced P_3 . Hence $d(x, u_3) \neq d(u_1, u_3)$, which is impossible in a general position set. Hence $u_1 \notin S$.

Now, we claim that u_1 is the unique vertex in C with degree at least 3. Assume the contrary that there exists $u_i \in$ C with $j \neq 1$ and deg $(u_i) \geq 3$. Then as above we have that $u_i \notin S$. Now, if u_i and u_i are adjacent vertices in C, then we can assume that j = 2. It follows from the fact that S is a general position set of size n-2, $d(u_3, x) = 3$ and u_3, u_4, u_1, x is a geodesic in G, where x is a neighbour of u_1 such that $x \notin V(C)$. This shows that the vertices x, u_4, u_3, x lie on a common geodesic, a contradiction. Similarly if u_1 and u_i are non adjacent vertices in C then $u_i = u_3$ and u_2 , u_4 belong to S. Moreover, as above S is a general position set of size n-2, we have that $x, y \in S$ and d(x, y) = 4, where $x \in N(u_1) \setminus V(C)$ and $y \in N(u_3) \setminus V(C)$, which is impossible. Thus u_1 is the unique vertex in C with $deg(u_1) \ge 3$. Also, since girth(G) = 4, we have that $N(u_1)$ induces an independent set. Hence the graph belongs to \mathcal{F}_1 .

Now, consider girth(G) = 3 and diam(G) = 3. Let P: u, x, y, v be a u, v- shortest path in G of length 3. Then Scontains atmost 2 vertices from V(P). Since |S| = n - 2, we have that S contains exactly two vertices from V(P). We consider the following four cases.

Subcase 1.1: $u, v \in S$. Then $x, y \notin S$. Moreover, S = $V(G) \setminus \{x, y\}$. Now, let z be any neighbour of u. Since S is a general position set of size n-2, it follows that $I[z, v] \subseteq$ V(P). This shows that $d(z, v) \leq 3$. If d(z, v) = 2, then z must be adjacent with y and so u, z, y, v is a u, v^{-} geodesic, which contradicts the fact that S is a general position set. Hence d(z, v) = 3 and since $I[z, v] \subseteq V(P)$, we have that z is adjacent with x but it is not adjacent with y. Similarly, we have that any neighbour of v is adjacent with y but nonadjacent with x. Now, assume that z be any vertex in G such that $z \notin V(P)$ and z is non-adjacent with both u and v. Then as in the previous case, we have that $I[z, v] \subseteq V(P)$. Also, we have $d(z, v) \in \{2, 3\}$ and $d(z, u) \in \{2, 3\}$. Hence it follows that z is adjacent to x or y or both. Also, by Theorem 2.3, we have that the components of S are in-transitive distance-constant cliques. Hence the graph reduces to the class \mathcal{F}_2 .

Subcase 1.2: $u, x \in S$. Then $y, v \notin S$ and $S = V(G) \setminus \{y, v\}$. Now, let z be any vertex in G such that $z \notin V(P)$. Then, we have that $I[z, u] \subseteq V(P)$. Moreover, by Theorem 2.3, d(z, u) = d(z, x). If d(z, x) = 2, then $I[z, x] \subseteq V(P)$, we have that z is adjacent to y. But in this case d(z, u) cannot be equal to 2. Similarly, if d(z, x) = 3 then z is adjacent with v but not y. Then it is clear that $d(z, u) \neq 3$. Hence it follows that d(z, u) = d(z, x) = 1. Again by Theorem 2.3, $V(G) \setminus \{y, v\}$ induces a clique. Hence the graph reduces to the class \mathcal{F}_3 .

Subcase 1.3: $u, y \in S$. Then $x, v \notin S$ and $S = V(G) \setminus \{x, v\}$. Now, for any $z \notin V(P)$, we have that $I[z, y] \subseteq V(P)$ and $I[z, u] \subseteq V(P)$. Thus $d(z, y) \leq 3$ for all $z \notin V(P)$. If d(z, y) = 3, then z must be adjacent to u and so by Theorem 2.3, d(u, y) = 3, a contradiction. Thus $d(z, y) \in \{1, 2\}$. If d(z, y) = 1, then again by Theorem 2.3, we have that d(u, z) = 2 and so z must be adjacent to x. Moreover, $\{z \notin V(P) : d(z, y) = 1\}$ induces a clique. Now, if d(z, y) = 2, then by using the same argument, we have that z is either adjacent to x or z is adjacent to both x and v. Hence the graph reduces to class \mathcal{F}_4 .

Subcase 1.4: $x, y \in S$. Then $u, v \notin S$ and $S = V(G) \setminus \{u, v\}$. Now, for any $z \notin V(P)$, as in the previous case we have that $I[z, x] \subseteq V(P)$ and $I[z, y] \subseteq V(P)$. Moreover, by Theorem 2.3, d(z, x) = d(z, y). Now, if $d(z, x) \neq 1$, then $d(z, y) \neq 1$. This shows that z must be adjacent to both u and v, which is impossible. Hence d(z, x) = d(z, y) = 1. Hence it follows from Theorem 2.3, $V(G) \setminus \{u, v\}$ induces a clique. Moreover, since both x and y belong to S, it is clear that d(u, z) = d(v, z) = 2 for all $z \notin V(P)$. Hence in this case the graph reduces to the family \mathcal{F}_2 .

Case 2: diam(G) = 2. Then by Theorem 2.4, we have $gp(G) = max\{\omega(G), \eta(G)\} = n - 2$. We consider the following two subcases.

Subcase 2.1: $\omega(G) \ge \eta(G)$. Then $\operatorname{gp}(G) = \omega(G) = n - 2$. Let K be a clique of order n - 2 and let $u, v \in V(G)$ be such that $u, v \notin V(K)$. Then it is clear that $1 \le \operatorname{deg}(u) \le n - 3$ and $1 \le \operatorname{deg}(v) \le n - 3$. Now, if u and v are adjacent in G, then G belongs to the family \mathcal{F}_6 . Otherwise, G belongs to the family \mathcal{F}_5 .

Subcase 2.2: $\eta(G) > \omega(G)$. Then $gp(G) = \eta(G) = n - 2$. This shows that the complement of *G* has complete mulipartite subgraph *H* of order n - 2. Thus the components of the induced subgraphs of *H* in *G* are cliques, say $K_{n_1}, K_{n_2}, ..., K_{n_r}$. Moreover d(u, v) = 2 for all $u \in V(K_{n_i})$ and $v \in V(K_{n_j})$. Now, let *x* and *y* be the vertices in *G* such that $x, y \notin V(H)$. Then it is clear that the graph reduces to the family \mathcal{F}_8 , when *x* and *y* are adjacent in *G*. Otherwise it belongs to the family \mathcal{F}_7 .

On the other hand, if G belongs to the family $\bigcup_{i=1}^{8} \mathcal{F}_i$, by Theorems 2.1 and 2.3, one can easily verify that gp(G) = n - 2. This completes the proof.

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