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Some results on one type of graph family with some special number sequences

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ABSTRACT

In this study, we introduce a new graph family. Then, we calculate eigenvalues of the adjacency and the Laplacian matrix of this graph family. Moreover, we show that the perfect matching number of this graph family equals to special second order recurrence by hafnian method. For some special kinds of this family, we obtain that the perfect matching number of corresponding graphs equals to some famous number sequences such as Fibonacci, Pell and Jacobsthal numbers. Also, we find energies and obtain upper bounds for Laplacian energies of these graphs.

KEYWORDS

Hafnian; eigenvalue; energy; Fibonacci number; Pell number

1. Introduction

In here, all graphs are considered as undirected, connected and without loops.

The adjacency matrix [12] of the graph $G=(V,E)$ is an $n\times n$ matrix $A=(a_{ij})$, where n is the number of vertices in G and $a_{ij}=$ number of edges between v_i and v_j. Let the eigenvalues of A be $\lambda_1,\lambda_2,\ldots,\lambda_n$. In [6], the *energy* of G is defined as

$$E(G)=\sum_{i=1}^{n}|\lambda_i|.$$

The degree of v_i, $\deg(v_i)$, is the number of edges incident to v_i. Then, the degree matrix D is defined as $D=diag(\deg(v_1),\ \deg(v_2),\ldots,\deg(v_n))$ in [5].

In [11], the Laplacian matrix of G is $L=D-A$ and the Laplacian energy of G is defined as

$$LE(G)=\sum_{i=1}^{n}\left|\mu_i-\frac{2m}{n}\right|,$$

where μ_i's are the eigenvalues of L and n and m are the number of vertices and edges, respectively [6].

Recently, many scientists study on relations of graph theory, chemistry and number theory. For example, Zhao and Li investigated the orderings of two classes of trees by their Fibonacci numbers. Using these orderings, they determined the unique tree with the second, and respectively the third smallest Fibonacci number among all trees with n vertices, where Fibonacci number is defined as $F_n=F_{n-1}+F_{n-2}$ with initial conditions $F_0=0$ and $F_1=1$ [14]. In [3], the authors proved that the number of 1-factorizations of a generalized Petersen graph of the type $GP(3k,k)$ is equal to the $k-$ th Jacobsthal number J_k when k is odd, and equal to $4J_k$ when k is even, where Jacobsthal number is defined as $J_n=J_{n-1}+2J_{n-2}$ with initial conditions $J_0=0$ and $J_1=1$. In [8], the authors showed that the perfect matchin number of one type of graph family equals to the $(k+1)-$th Pell number P_{k+1}, where Pell number is defined as $P_n=2P_{n-1}+P_{n-2}$ with initial conditions $P_0=0$ and $P_1=1$. In [1], the authors considered different types of fullerene graphs. Then, they gave an overview of some graph invariants that can possibly correlate with the fullerene molecule stability, such as: the bipartite edge frustration, the independence number, the saturation number, the number of perfect matching, etc.

In this study, we define one type of graph family. Then, we explore some special properties of this graphs such as energy, Laplacian energy and perfect matching. Moreover, we show that the perfect matching numbers for some special cases of the family are equal to some special numbers.

Now, we introduce the following graph family:where a, b, c are the number of edges between v_i and v_j $(1\le i,j\le 2k)$ (Figure 1).

So, we can easily write that the adjacency matrix of this graph family has the following form:

$$A=\begin{pmatrix} E_k & B_k\\ B_k & E_k\end{pmatrix},$$

where

$$E_k=\begin{pmatrix} 0 & b & \\ b & 0 & \ddots\\ & \ddots & \ddots\end{pmatrix}$$

and

$$B_k=\begin{pmatrix} a & c & \\ c & a & \ddots\\ & \ddots & \ddots\end{pmatrix}. \tag{1}$$

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Let us consider zero diagonal and symmetric matrix $T = [t_{ij}]$ with the order $2k$ $(k \geq 1)$. The *hafnian* of T is

$$\mathbf{haf}(T) = \sum_{\left\{\begin{smallmatrix} 1 & 2 & \cdots & 2k\bar{1} & 2k \\ i_1 & j_1 & \cdots & i_k & j_k \end{smallmatrix}\right\}}^{2k} t_{i_1 j_1} t_{i_2 j_2} \ldots t_{i_k j_k} \ ,$$

where $1 \leq i_s \leq j_s \leq k$ and $1 \leq s \leq k$ [2]. Further, from Laplace expansion along with last column for hafnian, we can write

$$\mathbf{haf}(T) = \sum_{r=1}^{2k\bar{1}} t_{r,2k} haf(T_r) \ ,$$

where T_r matrix is obtained by deleting $r - th$ and $2k - th$ rows and columns of T [2].

Theorem 1. [10] *Let X and Y are $k \times k$ real matrices and S is block matrix as below:*

$$S = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}.$$

Then, the eigenvalues of $X + Y$ and $X - Y$ matrices are the eigenvalues of S.

Lemma 2. *[8] Let us consider Q_n block matrix as below:*

$$Q_n = \begin{pmatrix} I_k & I_k \\ I_k & -I_k \end{pmatrix},$$

where I_k is identity matrix. So,

$$Q_n^{-1} = \frac{1}{2}\begin{pmatrix} I_k & I_k \\ I_k & -I_k \end{pmatrix}.$$

Lemma 3. *Let H be block matrix as in the following:*

$$H = \left(\begin{array}{c|c} F_k & 0 \\ 0 & S_k \end{array}\right),$$

where

$$F_k = \begin{pmatrix} a & b+c & & \\ b+c & a & \ddots & \\ & \ddots & \ddots & \\ & & & \end{pmatrix}, \quad S_k = \begin{pmatrix} -a & b-c & & \\ b-c & -a & \ddots & \\ & \ddots & \ddots & \\ & & & \end{pmatrix}.$$

Then, H and A matrices are similar matrices.

Proof. From matrix multiplication, we have $H = Q_n A Q_n^{-1}$, where Q_n and Q_n^{-1} are in Lemma 2. So which is desired. □

Therefore, we have following result.

Corollary 4. *Let A be the adjacency matrix of the graph G as in Figure 1 of order $2k$. For $k \geq 1$, i) $\det A = x_k y_k$ where $n = 2k$ and $x_k = a x_{k-1} - (b+c)^2 x_{k-2}$ and $y_k = -a y_{k-1} - (b-c)^2 y_{k-2}$ with initial conditions $x_0 = 1, x_1 = a, y_0 = 1$ and $y_1 = -a$.ii) The eigenvalues of A are (for $t = 1, 2, \ldots, k$)*

$$\eta_t = a - 2(b+c)\cos\left(\frac{t\pi}{k+1}\right)$$

and

$$\gamma_t = -a - 2(b-c)\cos\left(\frac{t\pi}{k+1}\right).$$

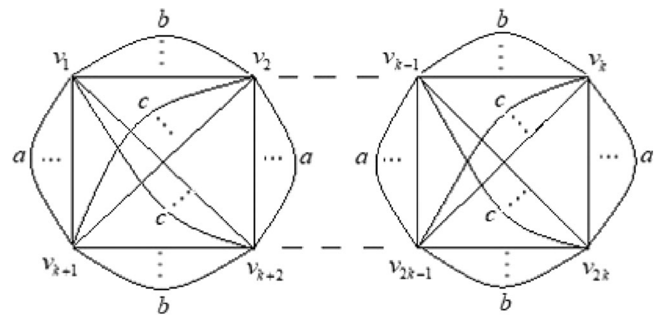

Figure 1. One type of graph family.

Proof. i) From [7], we have $\det F_k = x_k$ and $\det S_k = y_k$, where $x_k = a x_{k-1} - (b+c)^2 x_{k-2}$ and $v_k = -a v_{k-1} - (b-c)^2 v_{k-2}$ with initial conditions $x_0 = 1, x_1 = a$, $y_0 = 1$ and $y_1 = -a$. So, $\det H = \det A = x_k y_k$.*ii)* From Theorem 1, the set of eigenvalue of A are union of the set of eigenvalues of $F_k = E_k + B_k$ and $S_k = E_k - B_k$ matrices. From [9], the eigenvalues of A are (for $t = 1, 2, \ldots, k$)

$$\eta_t = a - 2(b+c)\cos\left(\frac{t\pi}{k+1}\right)$$

and

$$\gamma_t = -a - 2(b-c)\cos\left(\frac{t\pi}{k+1}\right). \qquad \square$$

2. Main results

In this section, we calculate eigenvalues of the Laplacian matrix of of the graph G in Figure 1. Also, we show that the perfect matching number of the graph equals to special second order recurrence by hafnian method. Finally, for some special kinds of the graph G in Figure 1, we obtain that the perfect matching number of corresponding graphs equals to some famous number sequences such as Fibonacci, Pell and Jacobsthal numbers. Moreover, we find energies and obtain upper bounds for Laplacian energies of these special kinds graphs.

Lemma 5. *The eigenvalues of Laplacian matrix L of the graph G in Figure 1 are*

$$\mu_t = 2(b+c)\left(1 + \cos\frac{t\pi}{k}\right)$$

and

$$\mu_t^* = 2(a+b+c) + 2(c-b)\cos\theta_t,$$

for $t = 1, 2, \ldots, k$, where θ_t's are the roots of

$$\begin{aligned} &(c-b)^2 \sin(n+1)\theta + 2(c^2 - b^2)\sin n\theta \\ &+ (b+c)^2 \sin(n-1)\theta \\ &= 0, \end{aligned}$$

for $\sin\theta \neq 0$.

Proof. The degree matrix D of the graph family in Figure 1 is

$$D=\begin{pmatrix} D^{*} & 0\\ 0 & D^{*}\end{pmatrix},$$

where

$$D^{*}=\begin{pmatrix} a+b+c &&&&\\ & a+2b+2c &&&\\ &&\ddots&&\\ &&&a+2b+2c&\\ &&&&a+b+c\end{pmatrix}_{k\times k}.$$

Then,

$$L=D-G=\begin{pmatrix} C_k & -B_k\\ -B_k & C_k\end{pmatrix},$$

where

$$C_k=\begin{pmatrix} a+b+c & -b &&&\\ -b & a+2b+2c & -b &&\\ & \ddots & \ddots & \ddots &\\ && -b & a+2b+2c & -b\\ &&& -b & a+b+c\end{pmatrix}$$

and B_k is as in (1). Then, we can write

$$C_k-B_k=\mathcal{U}_1=\begin{pmatrix} b+c & -b-c &&&\\ -b-c & 2b+2c & -b-c &&\\ & \ddots & \ddots & \ddots &\\ && -b-c & 2b+2c & -b-c\\ &&& -b-c & b+c\end{pmatrix}_{k\times k}$$

and

$$C_k+B_k=\mathcal{U}_2=\begin{pmatrix} 2a+b+c & c-b &&&\\ c-b & 2a+2b+2c & c-b &&\\ & \ddots & \ddots & \ddots &\\ && c-b & 2a+2b+2c & c-b\\ &&& c-b & 2a+b+c\end{pmatrix}_{k\times k}.$$

From Theorem 1, it is clear that the eigenvalues of $\mathcal{U}_1$ and $\mathcal{U}_2$ are also the eigenvalues of L. From ([13], pp. 4-8), we can easily see that the eigenvalues of L (that is, Laplacian eigenvales) are

$$\mu_t=2(b+c)\left(1+\cos\frac{t\pi}{k}\right)$$

and

$$\mu_t^{*}=2(a+b+c)+2(c-b)\cos\theta_t,$$

for $t=1,2,\dots,k,$ where θ_t' s are the roots of

$$\begin{aligned}&(c-b)^2\sin(n+1)\theta+2(c^2-b^2)\sin n\theta\\ &\quad+(b+c)^2\sin(n-1)\theta\\ &\quad=0,\end{aligned}$$

for $\sin\theta\neq 0$. □

Theorem 6. *Let A_{2k}be the adjacency matrix of the graph G as in Figure 1 of order 2k. Then,* $\mathbf{haf}(A_{2k})=z_{k+1}$, *where* $z_{k+1}=az_k+(b^2+c^2)z_{k-1}$ *with initial conditions* $z_1=1,\ z_2=a$.

Proof. We use induction method (PMI) for the proof. For $k=1$,

$$A_2=\begin{pmatrix} 0 & a\\ a & 0\end{pmatrix}.$$

In this case,

$$\mathbf{haf}(A_2)=z_2=a.$$

For all $t\leq k$, assume that this recurrences is true. Then, we must show that the equation is true for $k=t+1$. So, we can write that the adjacency matrix of the graph G (for $k=t+1$) as in Figure 1 has the following form:

$$A_{2t+2}=\left(\begin{array}{cccc|cccc} 0 & b & & 0 & a & c & & 0\\ b & 0 & \ddots & & c & a & \ddots & \\ & \ddots & \ddots & b & & \ddots & \ddots & c\\ 0 & & b & 0 & 0 & & c & a\\ \hline a & c & & 0 & 0 & b & & 0\\ c & a & & & b & 0 & \ddots & \\ & & \ddots & c & & \ddots & \ddots & b\\ 0 & & c & a & 0 & & b & 0\end{array}\right). \tag{2}$$

If we use Laplace expansion for hafnian along with $2t+2$ column of (2), we can write that

$$\begin{aligned}\mathbf{haf}(A_{2t+2})&=a\,\mathbf{haf}(A_{2t})+b\,\mathbf{haf}(\mathcal{Y}_{2t})+c\,\mathbf{haf}(\mathcal{Z}_{2t})\\ &=az_{t+1}+b\,\mathbf{haf}(\mathcal{Y}_{2t})+c\,\mathbf{haf}(\mathcal{Z}_{2t}),\end{aligned}\tag{3}$$

where

$$\mathcal{Y}_{2t}=\left(\begin{array}{ccccc|cc|ccccc} &&&&& 0 & 0 &&&&&\\ &&&&& 0 & 0 &&&&&\\ &&E_{t-1}&&& \vdots & \vdots &&&B_{t-1}&&\\ &&&&& 0 & 0 &&&&&\\ &&&&& b & 0 &&&&&\\ \hline 0 & 0 & \dots & 0 & b & 0 & b & 0 & 0 & \dots & 0 & c\\ 0 & 0 & \dots & 0 & 0 & b & 0 & 0 & 0 & \dots & 0 & 0\\ \hline &&&&& 0 & 0 &&&&&\\ &&&&& 0 & 0 &&&&&\\ &&B_{t-1}&&& 0 & 0 &&&E_{t-1}&&\\ &&&&& \vdots & \vdots &&&&&\\ &&&&& c & 0 &&&&&\end{array}\right).$$

and

$$\mathcal{Z}_{2t}=\begin{pmatrix} &&&&& 0 &&&&& 0\\ &&&&& 0 &&&&& 0\\ &&E_{t-1}&&& \vdots &&&B_{t-1}&& \vdots\\ &&&&& 0 &&&&& 0\\ &&&&& 0 &&&&& c\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & c\\ &&&&& 0 &&&&& 0\\ &&&&& 0 &&&&& 0\\ &&B_{t-1}&&& \vdots &&&E_{t-1}&& \vdots\\ &&&&& 0 &&&&& 0\\ &&&&& 0 &&&&& b\\ 0 & 0 & \dots & 0 & c & c & 0 & 0 & \dots & 0 & b & 0\end{pmatrix}$$

We know that simultaneous interchange of two different rows and corresponding columns doesn't change the sign and value of hafnian. By applying this property $(t-1)$ and t times to $\mathcal{Y}_{2t}$ and $\mathcal{Z}_{2t}$, respectively, we obtain the following matrices:

$$\mathcal{R}_{2t}=\left(\begin{array}{ccccccccccccc} & & & & & 0 & & & & & & & 0\\ & & & & & 0 & & & & & & & 0\\ & & E_{t-1} & & & \vdots & & & B_{t-1} & & & & \vdots\\ & & & & & 0 & & & & & & & 0\\ & & & & & b & & & & & & & 0\\ 0 & 0 & \dots & 0 & b & 0 & 0 & 0 & \dots & 0 & c & & b\\ & & & & & 0 & & & & & & & 0\\ & & & & & 0 & & & & & & & 0\\ & & B_{t-1} & & & \vdots & & & E_{t-1} & & & & \vdots\\ & & & & & 0 & & & & & & & 0\\ & & & & & c & & & & & & & 0\\ 0 & 0 & \dots & 0 & c & b & 0 & 0 & \dots & 0 & 0 & & 0 \end{array}\right) \tag{4}$$

and

$$\mathcal{J}_{2t}=\left(\begin{array}{cccccccccccc} & & & & & & & & & & 0 & 0\\ & & & & & & & & & & 0 & 0\\ & & E_{t-1} & & & & & B_{t-1} & & & \vdots & \vdots\\ & & & & & & & & & & 0 & 0\\ & & & & & & & & & & c & 0\\ & & & & & & & & & & 0 & 0\\ & & & & & & & & & & 0 & 0\\ & & B_{t-1} & & & & & E_{t-1} & & & \vdots & \vdots\\ & & & & & & & & & & 0 & 0\\ & & & & & & & & & & b & 0\\ 0 & 0 & \cdots & 0 & c & 0 & \cdots & 0 & 0 & b & 0 & c\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c & 0 \end{array}\right). \tag{5}$$

So, we have $\mathbf{haf}(\mathcal{Y}_{2t})=\mathbf{haf}(\mathcal{R}_{2t})$ and $\mathbf{haf}(\mathcal{Z}_{2t})=\mathbf{haf}(\mathcal{J}_{2t})$. Using expansion method along with last columns of (4) and (5), we obtain

$$\mathbf{haf}(\mathcal{R}_{2t})=b\,\mathbf{haf}(A_{2t-2})=bz_t,$$

and

$$\mathbf{haf}(\mathcal{J}_{2t})=c\,\mathbf{haf}(A_{2t-2})=cz_t.$$

As a consequence, we have

$$\mathbf{haf}(A_{2t+2})=az_{t+1}+(b^2+c^2)z_t=z_{t+2}.$$

Furthermore, the number of perfect matching of this type graph family with $2k$ vertices is equal to z_{k+1}. □

2.1. Special cases

Case 1: For $a=b=1$ and $c=0$, the graph family in Figure 1 turns into ladder graph [4]. That is, this graph becomes Figure 2.

It is clear that $\mathbf{haf}(A_{2k})=F_{k+1}$, that is, the number of perfect matching of ladder graph equals to F_{k+1}, where F_{k+1} is $(k+1)$ th Fibonacci number. For this a, b and c values, the eigenvalues of A_{2k} and L are

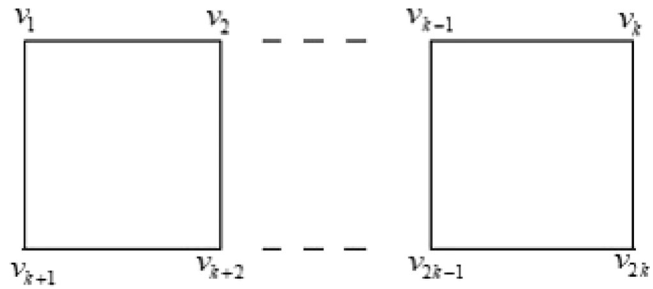

Figure 2. Ladder graph.

$$\eta_t=1-2\cos\left(\frac{t\pi}{k+1}\right)$$
$$\gamma_t=-1-2\cos\left(\frac{t\pi}{k+1}\right)$$

and

$$\mu_t=2+2\cos\left(\frac{t\pi}{k}\right)$$
$$\mu_t^*=4+2\cos\left(\frac{t\pi}{k}\right),$$

for $t=1,2,\dots,k,$ respectively. For this graph,

$$E(G)=\sum_{t=1}^{k}|\eta_t|+\sum_{t=1}^{k}|\gamma_t|$$
$$=\sum_{t=1}^{k}\left|1-2\cos\left(\frac{t\pi}{k+1}\right)\right|+\sum_{t=1}^{k}\left|-1-2\cos\left(\frac{t\pi}{k+1}\right)\right|.$$

For $k\equiv 0,1 \pmod 3$ and $1\le x\le\lfloor\frac{k}{3}\rfloor$, we have $1-2\cos\left(\frac{x\pi}{k+1}\right)\le 0$. So, for $\lfloor\frac{k}{3}\rfloor<t\le k$, we can see that $1-2\cos\left(\frac{t\pi}{k+1}\right)>0$. Also, for $z=1,2,\dots,k-\lfloor\frac{k}{3}\rfloor$, we have $\frac{z\pi}{k+1}\le\frac{2\pi}{3}$ and $-1-2\cos\left(\frac{z\pi}{k+1}\right)\le 0$. Then, for $k-\lfloor\frac{k}{3}\rfloor<t\le k$, we can write that $-1-2\cos\left(\frac{t\pi}{k+1}\right)>0$. Consequently,

$$E(G)=\sum_{t=1}^{\left\lfloor\frac{k}{3}\right\rfloor}\left[-1+2\cos\left(\frac{t\pi}{k+1}\right)\right]+\sum_{t=\left\lfloor\frac{k}{3}\right\rfloor+1}^{k}\left[1-2\cos\left(\frac{t\pi}{k+1}\right)\right]$$
$$+\sum_{t=1}^{k-\left\lfloor\frac{k}{3}\right\rfloor}\left[1+2\cos\left(\frac{t\pi}{k+1}\right)\right]+\sum_{t=k-\left\lfloor\frac{k}{3}\right\rfloor+1}^{k}\left[-1-2\cos\left(\frac{t\pi}{k+1}\right)\right]$$
$$=2k-4-4\left\lfloor\frac{k}{3}\right\rfloor+4\left[\sin\left(\frac{\left\lfloor\frac{k}{3}\right\rfloor\pi}{k+1}\right)\cot\frac{\pi}{2k+2}+\cos\left(\frac{\left\lfloor\frac{k}{3}\right\rfloor\pi}{k+1}\right)\right]$$
$$-4\left[\cos\left(\frac{\left(\left\lfloor\frac{k}{3}\right\rfloor+1\right)\pi}{k+1}\right)+\cos\left(\frac{\left(\left\lfloor\frac{k}{3}\right\rfloor+2\right)\pi}{k+1}\right)+\dots+\cos\left(\frac{\left(k-\left\lfloor\frac{k}{3}\right\rfloor\right)\pi}{k+1}\right)\right]$$
$$=2k-4-4\left\lfloor\frac{k}{3}\right\rfloor+4\left[\sin\left(\frac{\left\lfloor\frac{k}{3}\right\rfloor\pi}{k+1}\right)\cot\frac{\pi}{2k+2}+\cos\left(\frac{\left\lfloor\frac{k}{3}\right\rfloor\pi}{k+1}\right)\right].$$

For $k\equiv 2 \pmod 3$ and $1\le x\le\frac{k+1}{3}$, we have $1-2\cos\left(\frac{x\pi}{k+1}\right)\le 0$. So, for $\frac{k+1}{3}<t\le k$, we can see that $1-2\cos\left(\frac{t\pi}{k+1}\right)>0$. Also, for $z=1,2,\dots,k+1-\frac{k+1}{3}$, we have

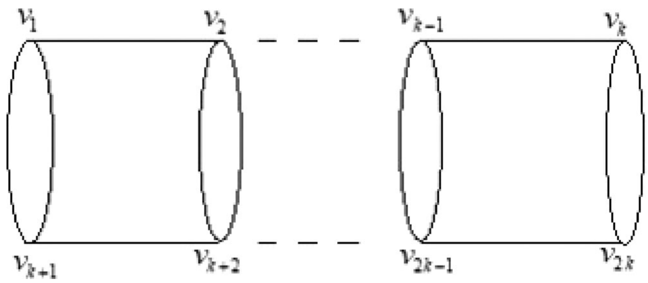

Figure 3. Cylinder type graph.

$\frac{z\pi}{k+1} \leq \frac{2\pi}{3}$ and $-1-2\cos\left(\frac{z\pi}{k+1}\right) \leq 0$. Then, we can write that $-1-2\cos\left(\frac{t\pi}{k+1}\right) > 0$, for $\frac{2k+2}{3} < t \leq k$. Consequently,

$$E(G) = \sum_{t=1}^{\frac{k+1}{3}} \left[-1+2\cos\left(\frac{t\pi}{k+1}\right)\right] + \sum_{t=\frac{k+4}{3}}^{k} \left[1-2\cos\left(\frac{t\pi}{k+1}\right)\right]$$

$$+ \sum_{t=1}^{\frac{2k+2}{3}} \left[1+2\cos\left(\frac{t\pi}{k+1}\right)\right] + \sum_{t=\frac{2k+5}{3}}^{k} \left[-1-2\cos\left(\frac{t\pi}{k+1}\right)\right]$$

$$= \frac{2k-10}{3} + 2\sqrt{3}\cot\frac{\pi}{2k+2}.$$

It is clear that the number of edges of this graph equals to $3k-2$ for $n=2k$. So, we have

$$LE(G) = \sum_{i=1}^{k}\left|\mu_i - \frac{3k-2}{k}\right| + \sum_{i=1}^{k}\left|\mu_i^* - \frac{3k-2}{k}\right|.$$

It is easy to see $LE(G) = 2$, for $k=1$. If $k \geq 2$, we obtain

$$\begin{aligned}
LE(G) &= \sum_{i=1}^{k}\left|\mu_i - \frac{3k-2}{k}\right| + \sum_{i=1}^{k}\left|\mu_i^* - \frac{3k-2}{k}\right| \\
&= \sum_{i=1}^{k}\left|2+2\cos\left(\frac{i\pi}{k}\right) - \frac{3k-2}{k}\right| + \sum_{i=1}^{k}\left|4+2\cos\left(\frac{t\pi}{k}\right) - \frac{3k-2}{k}\right| \\
&= \sum_{i=1}^{k}\left|-1+\frac{2}{k}+2\cos\left(\frac{i\pi}{k}\right)\right| + \sum_{i=1}^{k}\left|1+\frac{2}{k}+2\cos\left(\frac{t\pi}{k}\right)\right| \\
&\leq \sum_{i=1}^{k}\left|-1+\frac{2}{k}\right| + 2\sum_{i=1}^{k}\left|\cos\left(\frac{i\pi}{k}\right)\right| \\
&\quad + \sum_{i=1}^{k}\left(1+\frac{2}{k}\right) + 2\sum_{i=1}^{k}\left|\cos\left(\frac{i\pi}{k}\right)\right|.
\end{aligned}$$

Consequently, we get following inequality:

$$\begin{aligned}
LE(G) &\leq 2k + 4\sum_{i=1}^{k}\left|\cos\left(\frac{i\pi}{k}\right)\right| \\
&= 2k+4+8\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\cos\left(\frac{i\pi}{k}\right) \\
&= 2k+4\left(\sin\frac{\left\lfloor\frac{k}{2}\right\rfloor\pi}{k}\cot\frac{\pi}{2k} + \cos\frac{\left\lfloor\frac{k}{2}\right\rfloor\pi}{k}\right).
\end{aligned}$$

Case 2: For $a=2$, $b=1$ and $c=0$, the graph family in Figure 1 turns into cylinder type graph [8]. That is, this graph is Figure 3.

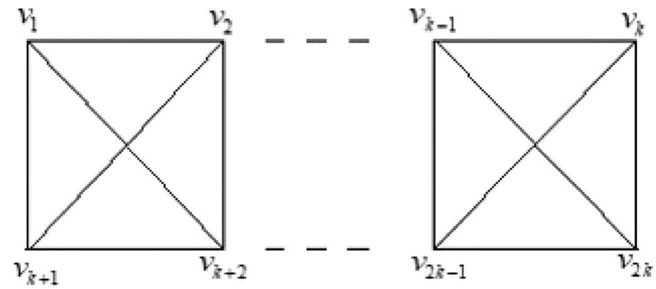

Figure 4. Special subgraph of graph family in Figure 1.

It is clear that $\mathbf{haf}(A_{2k}) = P_{k+1}$ and the number of perfect matching of the cylinder type graph equals to P_{k+1}, where P_{k+1} is $(k+1)$ th Pell number [8]. For this a, b and c values, the authors obtained eigenvalues of A_{2k} and L and energy and upper bound for Laplacian energy of this graph [8].

Case 3: For $a = b = c = 1$, the graph family in Figure 1 turns into as below (Figure 4):

It is clear that $\mathbf{haf}(A_{2k}) = J_{k+1}$ and the number of perfect matching of this graph equals to J_{k+1}, where J_{k+1} is $(k+1)$ th Jacobsthal number. For this a, b and c values, the eigenvalues of A_{2k} and L are

$$\eta_t = 1-4\cos\left(\frac{t\pi}{k+1}\right),$$
$$\gamma_t = -1-4\cos\left(\frac{t\pi}{k+1}\right)$$

and

$$\mu_t = 4+4\cos\left(\frac{t\pi}{k}\right)$$
$$\mu_t^* = 4 \text{ and } \mu_{k-1}^* = \mu_k^* = 6 \ (1 \leq t \leq k-2),$$

for $t = 1, 2, \ldots, k$, respectively. For this graph,

$$\begin{aligned}
E(G) &= \sum_{t=1}^{k}|\eta_t| + \sum_{t=1}^{k}|\gamma_t| \\
&= \sum_{t=1}^{k}\left|1-4\cos\left(\frac{t\pi}{k+1}\right)\right| + \sum_{t=1}^{k}\left|-1-4\cos\left(\frac{t\pi}{k+1}\right)\right|.
\end{aligned}$$

For $x = 1, 2, \ldots, t < k$ and $\cos\left(\frac{x\pi}{k+1}\right) \geq \frac{1}{4}$, we can write $1-4\cos\left(\frac{x\pi}{k+1}\right) \leq 0$ and $1-4\cos\left(\frac{z\pi}{k+1}\right) \geq 0$, where $t < z \leq k$. Also, for $y = 1, 2, \ldots, q < k$ and $\cos\left(\frac{y\pi}{k+1}\right) \geq -\frac{1}{4}$, we have $-1-4\cos\left(\frac{y\pi}{k+1}\right) \leq 0$ and $-1-4\cos\left(\frac{p\pi}{k+1}\right) \geq 0$, where $q < p \leq k$. Then, we obtain

$$\begin{aligned}
E(G) &= \sum_{i=1}^{t}\left[-1+4\cos\left(\frac{i\pi}{k+1}\right)\right] + \sum_{i=t+1}^{k}\left[1-4\cos\left(\frac{i\pi}{k+1}\right)\right] \\
&\quad + \sum_{i=1}^{q}\left[1+4\cos\left(\frac{i\pi}{k+1}\right)\right] + \sum_{i=q+1}^{k}\left[-1-4\cos\left(\frac{i\pi}{k+1}\right)\right] \\
&= 2\left\{q-t-4+2\left(\cot\left(\frac{\pi}{2k+2}\right)\left[\sin\left(\frac{x\pi}{k+1}\right) + \sin\left(\frac{(k-y)\pi}{k+1}\right)\right]\right.\right. \\
&\quad \left.\left. + \cos\left(\frac{x\pi}{k+1}\right) + \cos\left(\frac{(k-y)\pi}{k+1}\right)\right)\right\}.
\end{aligned}$$

For example, we obtain $q=3$ and $t=2$, for $k=5$. Therefore,

$$\begin{aligned} E(G) &= 2\left[3-2-4+4\left(\cot\frac{\pi}{12}\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)\right] \\ &= 10+4\sqrt{3}. \end{aligned}$$

It is clear that the number of edges of this graph equals to $5k-4$ for $n=2k$. So,

$$\begin{aligned} LE(G) &= \sum_{i=1}^{k}\left|\mu_i-\frac{5k-4}{k}\right|+\sum_{i=1}^{k}\left|\mu_i^*-\frac{5k-4}{k}\right| \\ &= \sum_{i=1}^{k}\left|4+4\cos\left(\frac{i\pi}{k}\right)-\frac{5k-4}{k}\right|+\sum_{i=1}^{k}\left|\mu_i^*-\frac{5k-4}{k}\right| \\ &= 2+\frac{8}{k}+\sum_{i=1}^{k}\left|-1+\frac{4}{k}+4\cos\left(\frac{i\pi}{k}\right)\right|+\sum_{i=1}^{k\bar{2}}\left|-1+\frac{4}{k}\right| \\ &\leq 2+\frac{8}{k}+\sum_{i=1}^{k}\left|-1+\frac{4}{k}\right|+\sum_{i=1}^{k\bar{2}}\left|-1+\frac{4}{k}\right|+4\sum_{i=1}^{k}\left|\cos\left(\frac{i\pi}{k}\right)\right| \end{aligned}$$

For $k\leq 3,$ we can easily calculate $LE(G)$. If $k\geq 4,$ we have

$$\begin{aligned} LE(G) &\leq 2+\frac{8}{k}+\sum_{i=1}^{k}\left|-1+\frac{4}{k}\right|+\sum_{i=1}^{k\bar{2}}\left|-1+\frac{4}{k}\right|+4\sum_{i=1}^{k}\left|\cos\left(\frac{i\pi}{k}\right)\right| \\ &= 8+2\sum_{i=1}^{k\bar{2}}\left(1-\frac{4}{k}\right)+8\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\cos\left(\frac{i\pi}{k}\right) \\ &= 2k-8+\frac{16}{k}+4\left(\sin\frac{\left\lfloor\frac{k}{2}\right\rfloor\pi}{k}\cot\frac{\pi}{2k}+\cos\frac{\left\lfloor\frac{k}{2}\right\rfloor\pi}{k}\right). \end{aligned}$$

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