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Note on diagonal construction of Z_{2nm}-supermagic labeling of $C_n \square C_m$

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ABSTRACT

A Γ-supermagic labeling of a graph $G(V, E)$ with $|E| = k$ is a bijection from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. An existence of a Z_{2nm}-supermagic labeling of Cartesian product of two cycles, $C_n \square C_m$ for n odd was proved recently. This along with an earlier result by Ivančo proved the existence of a Z_{2nm}-supermagic labeling of $C_n \square C_m$ for every $n, m \geq 3$. In this paper we present a simple unified labeling method for all $n, m \geq 3$.

KEYWORDS

Magic-type labeling; supermagic labeling; vertex-magic edge labeling; group supermagic labeling; Cartesian product of cycles

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1. Motivation

The similarity between the structure of Abelian groups and Cartesian product of cycles $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$, which can be viewed as the Cayley graph of group $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s}$ generated by group elements $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ leads to the following question.

Let $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$ be a Cartesian product of cycles with vertex set V and edge set E and $\Gamma = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_t}$ a finite Abelian group. We can ask when we can label the vertices, edges, or vertices and edges of G bijectively with elements of Γ, where $|\Gamma| = |V|$, or $|E|$, or $|V \cup E|$, respectively, so that the sum of the labels of the elements incident or adjacent to every edge or vertex is the same group element μ? We provide exact definitions of the above notions in Section 2.

The question has been so far studied for products of two cycles for both vertex and edge labelings. For edge labelings of $C_n \square C_m$, three different constructions depending on parities of n and m were used in [3] (the case of n, m both even is based on a related result by Ivančo [4]).

We present a construction that covers all cases in a uniform way. The construction is a modification of a similar construction used for a vertex labeling in [2].

2. Definitions and known results

Although the Cartesian product of graphs is a well known notion, we define it here for completeness.

Definition 1. The *Cartesian product* $G = G_1 \square G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and u_2 is adjacent with v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent with v_1 in G_1.

The notion of a Γ-*supermagic labeling* was introduced in [3].

Definition 2. A Γ-*supermagic labeling* of a graph $G(V, E)$ with $|E| = k$ is a bijection f from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant*. That is,

$$w(x) = \sum_{xy \in E} f(xy) = \mu$$

for every vertex $x \in V$.

The definition is a generalization of previously studied notion of a *vertex-magic edge labeling*, where the labels are consecutive positive integers $1, 2, \ldots, |E|$ and the addition is performed in $\mathbb{Z}$. This type of labeling is also often called a *supermagic labeling*.

The vertex version of the labeling, called Γ-*distance magic labeling* was introduced in [2].

Definition 3. A Γ-*distance magic labeling* of a graph $G(V, E)$ with $|V| = p$ is a bijection g from V to an Abelian group Γ of order p such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant*. That is,

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$$w(x) = \sum_{y:xy \in E} g(y) = \mu$$

for every vertex $x \in V$.

Again, results on a labeling with positive integers $1, 2, \ldots, |V|$ and addition in $\mathbb{Z}$ preceded those on a labeling with group elements. Such labeling is called simply a *distance magic labeling*.

Rao et al. proved the following [5].

Theorem 4. *The graph $C_n \square C_m$ is distance magic if and only if $n = m \geq 6$ and $n, m \equiv 2 \pmod 4$.*

Based on this notion of distance magic graphs, Froncek in [2] introduced the concept of group distance magic labeling and proved a complete result on Γ-distance magic labeling of Cartesian product of two cycles with cyclic groups.

Theorem 5. *The Cartesian product $C_n \square C_m$ has a Z_{nm}-distance magic labeling if and only if $n, m \geq 3$ and nm is even.*

Cichacz [1] then extended the result to a wider class of Abelian groups.

Theorem 6. *Let n, m, t, s be positive integers, $n, m \geq 3$ and $l = \operatorname{lcm}(n, m)$. Let $\Gamma = Z_{lt} \times A$, where A is an Abelian group of order s and $nm = lts$. Then the Cartesian product $C_n \square C_m$ has a Γ-distance magic labeling.*

Results analogical to Theorem 4 for supermagic labeling were proved by Ivančo [4].

Theorem 7. *Let n, m be positive integers. Then $C_{2n} \square C_{2m}$ has a supermagic labeling for any $n, m \geq 2$ and $C_n \square C_n$ for any $n \geq 3$.*

Ivančo also conjectured that the Cartesian product $C_n \square C_m$ allows a vertex-magic edge labeling for any $n, m \geq 3$.

Froncek et al. [3] proved that a group supermagic labeling equivalent of the conjecture is true for the cyclic group Z_{2nm}.

Theorem 8. *The Cartesian product $C_n \square C_m$ admits a Z_{2nm}-supermagic labeling for all odd $n \geq 3$ and any $m \geq 3$.*

This along with Ivančo's theorem gives a complete result.

Theorem 9. *The Cartesian product $C_n \square C_m$ admits a Z_{2nm}-supermagic labeling for all $n, m \geq 3$.*

The results obtained in Theorem 8 were based on two different (yet similar) constructions for n, m both odd, and n odd and m even. The last case of n, m both even relied on Ivančo's purely existential result in Theorem 7. Since Ivančo did not provide a construction of a labeling whose existence he proved, the construction in the next section is filling the gap for this case. Also, this method provides a unified approach for all three cases for different parities of n and m.

3. Diagonal construction

We use the following notation. Vertices of $C_n \square C_m$ will be denoted x_{st} for $s = 0, 1, \ldots, n-1$ and $t = 0, 1, \ldots, m-1$. Every vertex x_{st} is then incident with two *vertical edges* $x_{(s-1)t}x_{st}$ and $x_{st}x_{(s+1)t}$ and two *horizontal edges* $x_{s(t-1)}x_{st}$ and $x_{st}x_{s(t+1)}$.

By a *diagonal* D^j we mean a cycle consisting alternately of horizontal and vertical edges starting with $x_{0j}x_{0(j+1)}$. More precisely,

$$D^j = (x_{0j}x_{0(j+1)}, x_{0(j+1)}x_{1(j+1)}, x_{1(j+1)}x_{1(j+2)}, x_{1(j+2)}x_{2(j+2)}, \ldots, x_{(n-1)j}x_{0j}).$$

Construction 10. First we observe that each diagonal D^j is a cycle of length $2l$, where $l = \operatorname{lcm}(n, m)$. To see that, we notice that getting back to x_{0j} we need to pass through *am* horizontal edges and *bn* vertical edges, where a, b are positive integers. Because the number of horizontal and vertical edges in the diagonal is the same, we must have $am = bn = \operatorname{lcm}(n, m)$ and the conclusion follows. Since each diagonal has $2l$ edges, the number of diagonals, call it d, is $d = 2nm/2l = nm/l = \gcd(n, m)$.

Now we label each diagonal with two cosets of order l induced by the subgroup $\langle 2d \rangle$. In particular, we label D^j with $\langle 2d \rangle + j$ and $\langle 2d \rangle - j - 1$ for $0 \leq j \leq 2d - 1$. We use $\langle 2d \rangle + j$ for the horizontal edges in increasing order, and $\langle 2d \rangle - j - 1$ for the vertical edges in decreasing order, starting with edges $x_{0j}x_{0(j+1)}$ and $x_{0(j+1)}x_{1(j+1)}$, respectively. To avoid complicated notation in vertex subscripts, we just denote the horizontal edges in D^j consecutively as $h_0^j, h_1^j, \ldots, h_{l-1}^j$ and the vertical ones by $v_0^j, v_1^j, \ldots, v_{l-1}^j$. That is, $h_0^j = x_{0j}x_{0(j+1)}$, $v_0^j = x_{0(j+1)}x_{1(j+1)}$ and so on.

Thus we have $f(h_k^j) = 2dk + j$ for the horizontal edges and $f(v_k^j) = -2dk - j - 1$ for the vertical edges. Then we look at the partial weights created by the labels in each D^j. We will call the pair of consecutive edges (h_k^j, v_k^j) the *HV-corner* and the pair (v_k^j, h_{k+1}^j) the *VH-corner*. Now at the k-th *HV*-corner, we have the partial weight w_{HV} of the appropriate vertex of D^j defined as

$$w_{HV}(x_{st}) = f(h_k^j) + f(v_k^j) = (2dk + j) + (-2dk - j - 1) = -1 \tag{1}$$

and at the k-th *VH*-corner the partial weight w_{VH} is

$$w_{VH}(x_{st}) = f(v_k^j) + f(h_{k+1}^j) = (-2dk - j - 1) + (2d(k+1) + j) = 2d - 1, \tag{2}$$

regardless of the value of j. At each vertex x_{st} the edges $x_{s(t-1)}x_{st}$ and $x_{st}x_{(s+1)t}$ form an *HV*-corner inducing the partial weight $w_{HV}(x_{st}) = -1$ and the edges $x_{(s-1)t}x_{st}$ and $x_{st}x_{s(t+1)}$ form a *VH*-corner inducing the partial weight $w_{VH}(x_{st}) = 2d - 1$. Notice that when $d > 1$, then the two corners come from two consecutive diagonals D^j and D^{j+1}, while when $\gcd(n, m) = 1$ they both come from the same diagonal, as there is only one, D^0.

Adding (1) and (2), we get for every vertex x_{st}

$$w(x_{st}) = w_{HV}(x_{st}) + w_{VH}(x_{st}) = -1 + (2d - 1) = 2d - 2.$$

This means that the labeling is really a vertex-magic edge Z_{2nm}-labeling with magic constant $\mu = 2d - 2$.

To make sure that we were not just re-inventing the wheel by finding the same labeling using different approach, we look for instance at $C_3 \square C_3$. The labeling used in the proof of Theorem 8 assigns the subgroup $\langle 6 \rangle$ to the edges of the same horizontal cycle, similarly as the cosets $\langle 6 \rangle + 2$ and $\langle 6 \rangle + 4$. Each of the odd cosets is then assigned to one of the vertical cycles.

In the diagonal construction, the subgroup $\langle 6 \rangle$ is assigned to the horizontal edges of the diagonal D^0. No two of these edges belong to the same horizontal cycle. Similarly, the elements of the remaining two lower cosets $\langle 6 \rangle + 1$ and $\langle 6 \rangle + 2$ are used for horizontal edges in diagonals D^1 and D^2, while the upper cosets $\langle 6 \rangle + 3, \langle 6 \rangle + 4$ and $\langle 6 \rangle + 5$ label the vertical edges in the three diagonals. Therefore, we can formalize our observation.

Observation 11. *The labelings provided in Theorem 8 and Construction 10 are different.*

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