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Harishchandra S. Ramane, Kartik S. Pise, and Daneshwari Patil

Department of Mathematics, Karnatak University, Dharwad, India

ABSTRACT

The inverse sum indeg index of a connected graph G is defined as the sum of the ratio $\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$ for all edges uv in the edge set of G, where $d_G(u)$ denotes the degree of a point u. Recently, Pattabiraman [7] has obtained bounds for the inverse sum indeg index of graphs. The upper and lower bounds obtained in terms of harmonic index and second Zagreb index are prone to some error. In this note we correct these errors by establishing the proper bounds in terms of harmonic index and second Zagreb index.

KEYWORDS

Degree of a point; inverse sum indeg index; harmonic index; Zagreb index

1. Introduction

All graphs considered here are simple and connected. A graph G consists of a finite non-empty set $V(G)$ of points together with a prescribed set $E(G)$ of unordered pairs of distinct points of $V(G)$. Each pair $e = (u, v)$ of points of $V(G)$ is an edge of G. The degree of a point u in a graph G, denoted by $d_G(u)$, is the number of edges incident with u. The minimum degree among the points of G is denoted by $\delta(G)$ while $\Delta(G)$ is the largest such number. If $\delta(G) = \Delta(G) = r$, then all points have the same degree and G is called regular graph of degree r. For other graph theoretic terminology, one can refer [1, 5]. The first Zagreb index $M_1(G)$, second Zagreb index $M_2(G)$ [4], harmonic index $H(G)$ [3] and inverse sum indeg index $ISI(G)$ [10] of graphs are defined as follows:

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)],$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v),$$

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u) + d_G(v)}$$

and

$$ISI(G) = \sum_{uv\in E(G)} \frac{d_G(u)d_G(v)}{d_G(u) + d_G(v)}.$$

The work related to inverse sum indeg index is reported in [2, 9]. Recently, Pattabiraman [7] has obtained several bounds for the inverse sum indeg index of graphs. Among these, the bounds obtained in terms of harmonic index and second Zagreb index are prone to some error. In this note we correct these errors by establishing the proper bounds for inverse sum indeg index in terms of harmonic index and second Zagreb index.

2. Bounds on inverse sum indeg index

Lemma 2.1. [6] *Suppose a_i and b_i, $1 \le i \le n$, are positive real numbers, then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le \alpha(n)(A-a)(B-b),$$

where a, b, A and B are constants such that for each i, $1 \le i \le n, a \le a_i \le A$ and $b \le b_i \le B$. Further

$$\alpha(n) = n\left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil\right).$$

In [7], the upper bound for $ISI(G)$ has been given in terms of harmonic index and second Zagreb index as follows.

Theorem 2.2. [7] *Let G be a connected graph with n points and m edges. Then*

$$ISI(G) \le \frac{\alpha(m)(\delta-\Delta)(\Delta^2-\delta^2)}{2m\delta\Delta} + \frac{H(G)M_2(G)}{2m},$$

where $\alpha(m) = m\lceil \frac{m}{2} \rceil \left(1 - \frac{1}{m}\lceil \frac{m}{2} \rceil\right)$ with equality if and only if G is regular.

The inequality in Theorem 2.2 fails for the graph G of Figure 1, because $ISI(G) = 4.15$ and

$$\frac{\alpha(m)(\delta-\Delta)(\Delta^2-\delta^2)}{2m\delta\Delta} + \frac{H(G)M_2(G)}{2m} = 1.608.$$

The correct version of Theorem 2.2 is as follows.

CONTACT Harishchandra S. Ramane hsramane@yahoo.com Department of Mathematics, Karnatak University, Dharwad - 580003, India.

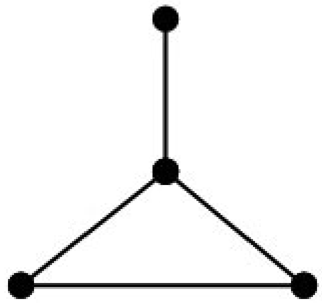

Figure 1. A counter example for the Theorem 2.2.

Theorem 2.3. *Let G be a connected graph with n points and m edges. Then*

$$ISI(G) \leq \frac{\alpha(m)(\Delta-\delta)(\Delta^2-\delta^2)}{2m\delta\Delta} + \frac{H(G)M_2(G)}{2m},$$

where $\alpha(m) = m\lceil\frac{m}{2}\rceil\left(1-\frac{1}{m}\lceil\frac{m}{2}\rceil\right)$ *with equality if and only if G is regular.*

Proof. Choosing $a_i = \frac{1}{d_G(u)+d_G(v)}, b_i = d_G(u)d_G(v), a = \frac{1}{2\Delta}, A = \frac{1}{2\delta}, b = \delta^2, B = \Delta^2$ in Lemma 2.1, we get

$$\left| m\sum_{uv\in E(G)}\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)} - \sum_{uv\in E(G)}\frac{1}{d_G(u)+d_G(v)}\sum_{uv\in E(G)}d_G(u)d_G(v)\right| \leq \alpha(m)\left(\frac{1}{2\delta}-\frac{1}{2\Delta}\right)(\Delta^2-\delta^2). \quad (1)$$

From the definitions of ISI index, harmonic index and second Zagreb index, Eq. (1) reduces to

$$mISI(G) - \frac{H(G)M_2(G)}{2} \leq \alpha(m)\left(\frac{\Delta-\delta}{2\delta\Delta}\right)(\Delta^2-\delta^2).$$

Therefore,

$$ISI(G) \leq \frac{\alpha(m)(\Delta-\delta)(\Delta^2-\delta^2)}{2m\delta\Delta} + \frac{H(G)M_2(G)}{2m}.$$

The equality holds if and only if $\delta = \Delta$, that is, if G is regular. □

Lemma 2.4. (Pólya-Szego Inequality [8]) *Let* $0 < m_1 \leq x_i \leq M_1$ *and* $0 < m_2 \leq y_i \leq M_2$ *for all* $1 \leq i \leq n$. *Then*

$$\sum_{i=1}^n x_i^2\sum_{i=1}^n y_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\sum_{i=1}^n x_iy_i\right)^2.$$

Lemma 2.5. (Cauchy-Schwarz Inequality) *Let* $X = (x_1, x_2, ..., x_n)$ *and* $Y = (y_1, y_2, ..., y_n)$ *be sequences of real numbers. Then*

$$\left(\sum_{i=1}^n x_iy_i\right)^2 \leq \left(\sum_{i=1}^n x_i\right)^2\left(\sum_{i=1}^n y_i\right)^2,$$

with equality if and only if the sequence X and Y are proportional. That is, there exists a constant c such that $x_i = cy_i$ *for all* $1 \leq i \leq n$.

As a special case of Cauchy-Schwarz inequality, when $y_1 = y_2 = \cdots = y_n$ we get the following result.

Corollary 2.6. *Let* $x_1, x_2, ..., x_n$ *be real numbers. Then*

$$\left(\sum_{i=1}^n x_i\right)^2 \leq n\sum_{i=1}^n x_i^2,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

In [7], the lower bound for *ISI*(*G*) has been given in terms of harmonic index and second Zagreb index and is as follows.

Theorem 2.7. [7] *Let G be a connected graph on n points and m edges. Then*

$$\frac{\sqrt{\delta\Delta}}{m(\delta+\Delta)}H(G)M_2(G) \leq ISI(G),$$

with equality if and only if G is regular.

The inequality in Theorem 2.7 fails to some class of graphs, such as wheel graph W_{n+1} on $n+1$ points for all $n \geq 4$, Dutch windmill graph $D_p^{(q)}$ for all $q \geq 3$ etc. The graph $D_p^{(q)}$ is a graph that can be constructed by coalescence q copies of the cycle C_p of length p with a common vertex.

The following theorem is the correct version of Theorem 2.7.

Theorem 2.8. *Let G be a connected graph on n points and m edges. Then*

$$\frac{\sqrt{\delta^3\Delta^3}}{m(\delta^3+\Delta^3)}H(G)M_2(G) \leq ISI(G),$$

with equality if and only if G is regular.

Proof. Note that, $2\delta \leq d_G(u)+d_G(v) \leq 2\Delta$ for any edge $uv \in E(G)$. Taking, $m_1 = \frac{1}{2\Delta}, m_2 = \delta^2, M_1 = \frac{1}{2\delta}, M_2 = \Delta^2$, for all $1 \leq i \leq m, x_i = \frac{1}{d_G(u)+d_G(v)}$ and $y_i = d_G(u)d_G(v)$ for all $1 \leq i \leq m$, in Lemma 2.4, we get

$$\sum_{uv\in E(G)}\left(\frac{1}{d_G(u)+d_G(v)}\right)^2\sum_{uv\in E(G)}(d_G(u)d_G(v))^2 \leq \frac{1}{4}\left(\sqrt{\frac{\frac{\Delta^2}{2\delta}}{\frac{\delta^2}{2\Delta}}}+\sqrt{\frac{\frac{\delta^2}{2\Delta}}{\frac{\Delta^2}{2\delta}}}\right)^2\sum_{uv\in E(G)}\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}\right)^2. \quad (2)$$

And by Corollary 2.6, we have

$$\sum_{uv\in E(G)}\left(\frac{1}{d_G(u)+d_G(v)}\right)^2\sum_{uv\in E(G)}(d_G(u)d_G(v))^2 \geq \frac{1}{m^2}\left(\sum_{uv\in E(G)}\frac{1}{d_G(u)+d_G(v)}\right)^2\left(\sum_{uv\in E(G)}d_G(u)d_G(v)\right)^2. \quad (3)$$

Combining inequalities (2) and (3), we have

$$\frac{1}{4m^2}(H(G))^2(M_2(G))^2 \leq \frac{1}{4}\left(\sqrt{\frac{\frac{\Delta^2}{2\delta}}{\frac{\delta^2}{2\Delta}}}+\sqrt{\frac{\frac{\delta^2}{2\Delta}}{\frac{\Delta^2}{2\delta}}}\right)^2(ISI(G))^2,$$

which implies,

$$\frac{\sqrt{\delta^3\Delta^3}}{m(\delta^3+\Delta^3)}H(G)M_2(G) \leq ISI(G).$$

The equality holds if and only if G is regular. □

Disclosure statement

No conflicts of interest have been reported by the authors.

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References

[1] Buckley, F., Harary, F. (1990). *Distance in Graphs*. Redwood, CA: Addison-Wesley.

[2] Došlić, T., Azari, M., Falahati-Nezhad, F. (2017). Sharp bounds on the inverse sum indeg index. *Discrete Appl. Math.* 217: 185–195.

[3] Fajtlowicz, S. (1987). On conjectures on Graffiti-II. *Congr. Numer.* 60:187–197.

[4] Gutman, I., Trinajstić, N. (1972). Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons. *Chem. Phys. Lett.* 17(4):535–538.

[5] Harary, F. (1999). *Graph Theory*. New Delhi, India: Narosa Publishing House.

[6] Milovanovic, I. Z., Milovanovic, E. I., Zakić, A. (2014). A short note on graph energy. *MATCH Commun. Math. Comput. Chem.* 72:179–182.

[7] Pattabiraman, K. (2018). Inverse sum indeg index of graphs. *AKCE Int. J. Graphs Comb.* 15(2):155–167.

[8] Pólya, G., Szego, G. (1972). *Problems and Theorems in Analysis, Series, Integral Calculus, Theory of Functions*. Berlin, Germany: Springer-Verlag.

[9] Sedlar, J., Stevanović, D., Vasilyev, A. (2015). On the inverse sum indeg index. *Discrete Appl. Math.* 184:202–212.

[10] Vukičević, D., Gašperov, M. (2011). Bond additive mdelling 1. Adriatic indices. *Croat. Chem. Acta* 83:87–260.