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ABSTRACT

Let $D = (V(D), A(D))$ be a digraph and S a subset of vertices of D, S is an absorbent set if for every v in $V(D) \setminus S$ there exists a vertex u in S such that $(v, u) \in A(D)$. A subset S of $V(D)$ is a semicomplete absorbent set if S is absorbent and the induced subdigraph $D\langle S\rangle$ is semicomplete. The minimum (respectively maximum) of the cardinalities of the semicomplete absorbent sets is the lower (respectively upper) semicomplete absorbent number, denoted by $\gamma_{sas}(D)$ (respectively $\Gamma_{sas}(D)$). In this paper we introduce the concept of semicomplete absorbent set; we will show some structural properties on the digraphs which have a semicomplete absorbent set and we will present some bounds for $\gamma_{sas}(D)$ and $\Gamma_{sas}(D)$. Then we will study the Cartesian product, the composition of digraphs and the line digraph in relation with those numbers.

KEYWORDS
Semicomplete; absorbent; domination; Cartesian; composition

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05C20; 05C69; 05C76

1. Introduction

For general concepts we refer the rader to [3] and [6]. Let $G = (V(G), E(G))$ be a simple undirected graph. An isolated vertex of G is a vertex whose degree is zero. We denote the path and cycle with n vertices by P_n and C_n, respectively. For a nonempty subset of $V(G)$, say S, the subgraph induced by S is denoted by $\langle S\rangle$. If S is such that $\langle S\rangle$ is complete, then we say that S is a clique of G. We say that S is a dominating set if for every x in $V(G) \setminus S$ there exists z in S such that $xz \in E(G)$.

Throughout the paper, $D = (V(D), A(D))$ denotes a loopless digraph with vertex set $V(D)$ and arc set $A(D)$. For an arc (u,v), u and v are its end-vertices; we say that the end-vertices are adjacent, we also say that u dominates v and v absorbs u. We will say that the arc (u,v) is symmetric if $(v,u) \in A(D)$. Let S be a subset of $V(D)$ and x in $V(D)$, we say that x is absorbed by S (x is dominated by S) if z absorbs x for some z in S (z dominates x for some z in S). We will say that S is an absorbent set (dominating set) if every vertex in $V(D) \setminus S$ is absorbed by S (dominated by S). If x is a vertex of D, then the ex-neighborhood of x is the set $\{z \in V(D) : (x,z) \in A(D)\}$, denoted by $N^+(x)$, while the in-neighborhood of x is the set $\{z \in V(D) : (z,x) \in A(D)\}$, denoted by $N^-(x)$. The neighborhood of x is the set $N^+(x) \cup N^-(x)$ and it is denoted by $N(x)$. The out-degree $d^+_D(x)$ of a vertex x is the number of vertices in $N^+(x)$, while the in-degree $d^-_D(x)$ of a vertex x is the number of vertices in $N^-(x)$. Let S be a subset of $V(D)$, the ex-neighborhood of S is the set $\cup_{x\in S} N^+(x)$, denoted by $N^+(S)$, while the in-neighborhood of S is the set $\cup_{x\in S} N^-(x)$, denoted by $N^-(S)$. A vertex v is called a sink if it has out-degree zero. A vertex v is called isolated if $d^+_D(v) = 0 = d^-_D(v)$. For a subset S of $V(D)$ the subdigraph of D induced by S, denoted by $D\langle S\rangle$, has $V(D\langle S\rangle) = S$ and $A(D\langle S\rangle) = \{(u,v) \in A(D)$: $\{u, v\} \subseteq S\}$. A pair of digraphs D and H are isomorphic, denoted by $D \cong H$, if there exists a bijection $f : V(D) \to V(H)$ such that $(u,v) \in A(D)$ if an only if $(f(u), f(v)) \in A(H)$. Let S_1 and S_2 be subsets of $V(D)$, an arc (u,v) of D will be called an $\boldsymbol{S_1S_2}$−arc whenever $u \in S_1$ and $v \in S_2$. If $S_1 = \{x\}$ or $S_2 = \{x\}$, then we will write $\boldsymbol{xS_2}$−arc or $\boldsymbol{S_1x}$−arc, respectively.

A directed walk W in D is a sequence of vertices $(x_0, x_1, \ldots, x_n)$ such that $(x_i, x_{i+1}) \in A(D)$ for every i in $\{0, 1\ldots, n-1\}$. We will say that W is a $\boldsymbol{x_0x_n}$−walk. The length of W is the number n. If $x_i \neq x_j$ for all i and j such that $\{i, j\} \subseteq \{0, \ldots, n\}$ and $i \neq j$, then W is called a directed path ($\boldsymbol{x_0x_n}$−path). W is a Hamiltonian directed path if W is a directed path and $V(W) = V(D)$. Let $\{x_i, x_j\}$ be a subset of $V(W)$, with $i < j$, the x_ix_j−walk $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$ contained in W will be denoted by $(\boldsymbol{x_i}, \boldsymbol{W}, \boldsymbol{x_j})$. A directed cycle is a directed walk $(v_1, v_2, \ldots, v_n, v_1)$ such that $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$. We will say that D is weak connected if for every subset $\{x, z\}$ of $V(D)$ there exists a walk between x and z which is not necessarily a directed walk. D is strong if, for every pair of vertices u and v in D, there exists a uv-walk and there exists a vu-walk in D. A strong component of D is a maximal induced subdigraph of D which is strong. We will say that H is a terminal strong component (initial strong component) of D if $N^+(x) \subseteq V(H)$ ($N^-(x) \subseteq V(H)$) for every x in V(H). D is unilaterally connected if, for every pair of vertices x and z of D, either there exists a xz−walk or there exists a zx−walk (or both). Let S_1 and S_2 be two subsets of $V(D)$, a uv-walk

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in D will be called an $\boldsymbol{S_1S_2}$−path whenever $u \in S_1$ and $v \in S_2$. If $S_1 = \{x\}$ or $S_2 = \{x\}$, then we will write $\boldsymbol{xS_2}$−path or $\boldsymbol{S_1x}$−path, respectively. A vertex v in D is a $\boldsymbol{k}$−king if for every u in $V(D) \setminus \{v\}$ there exists a uv−path of length at most k. Landau proved in [9] the following result.

Theorem 1.1. ([9]) *Every tournament has a 2-king.*

For a graph G, a digraph D is called an orientation of G if D is obtained from G by replacing each edge xy of G by the arc either (x,y) or (y,x).

For three different vertices u, v and w, will say that D is a transitive digraph whenever $\{(u,v), (v,w)\} \subseteq A(D)$ implies $(u,w) \in A(D)$. A digraph is semicomplete if for every u and v in $V(D)$ we have that $\{(u,v),(v,u)\} \cap A(D) \neq \emptyset$. Let S be a subset of vertices of D, S is said to be a semicomplete set if $D\langle S\rangle$ is a semicomplete digraph. The semicomplete number of D, denoted by $\omega^*(D)$, is defined as $max\{|S| : S \text{ is a semicomplete set of } D\}$. A tournament is a semicomplete digraph without symmetric arcs. We will say that D is a transitive tournament if D is a tournament and D is a transitive digraph. The transitive tournament of order three is denoted by T_3. A digraph D is an oriented tree if D is an orientation of a tree without symmetric arcs. A digraph T is an out-tree (an in-tree) if T is an oriented tree with just one vertex s of in-degree zero (out-degree zero). If an out-tree (in-tree) T is a spanning subdigraph of a digraph D, T is called an out-branching (in-branching). The line digraph of D, denoted by $L(D)$, is the digraph such that $V(L(D)) = A(D)$, and $((u,v),(w,z)) \in A(L(D))$ if and only if $v = w$.

The Cartesian product of two digraphs D and H, denoted by $D\Box H$, is the digraph whose vertex set is $V(D) \times V(H)$ and $((x,z),(u,v))$ is an arc of $D\Box H$ if and only if either $x = u$ and $(z,v) \in A(H)$ or $z = v$ and $(x,u) \in A(D)$. The horizontal level of the vertex x_0 in $D\Box H$ is $H_{x_0} = \{(x_0,y) \in V(D\Box H) : y \in V(H)\}$ and the vertical level of the vertex y_0 is $D_{y_0} = \{(x,y_0) \in V(D\Box H) : x \in V(D)\}$. Notice that $(D\Box H)\langle H_{x_0}\rangle \cong H$ and $(D\Box H)\langle D_{z_0}\rangle \cong D$ for every x_0 in $V(D)$ and every z_0 in $V(H)$, respectively. Let D be a digraph and $\alpha = (D_v)_{v\in V(D)}$ be a sequence of digraphs which are pairwise vertex disjoint. The composition of D respect to α, denoted by $D[\alpha]$, is the digraph obtained from D replacing every vertex v of D by the digraph D_v and joining every vertex from $V(D_v)$ to every vertex in $V(D_u)$ if and only if $(v,u) \in A(D)$. Let D be a digraph of order p, with $V(D) = \{u_1, \ldots, u_p\}$, and let $\psi = (D_1, \ldots, D_p)$ be a sequence of vertex disjoint digraphs such that $V(D_i) \cap V(D) = \emptyset$ for every i in $\{1, \ldots, p\}$. The corona of D with the sequence ψ, denoted by $\boldsymbol{D \circ \psi}$, is the digraph such that $V(D \circ \psi) = V(D) \cup (\cup_{i=1}^{p} V(D_i))$ and $A(D \circ \psi) = A(D) \cup (\cup_{i=1}^{p} A(D_i)) \cup (\cup_{i=1}^{p} \{(x, u_i) : x \in V(D_i)\})$.

Several kinds of dominating sets in graphs have been studied for some researches by adding conditions on the induced subgraph $\langle S\rangle$, where S is a dominating set. For example, a subset S of vertices of G is a total dominating set if S is a dominating set and $\langle S\rangle$ has no isolated vertices. This concept was introduced by Cockayne, Dawes and Hedetniemi in [5] and similar concepts were given in digraphs. A subset S of vertices of a digraph D is a total dominating set if S is an absorbent set and $\langle S\rangle$ has no isolated vertices. S is an open dominating set if every vertex in $V(D)$ is dominated by S. Both concepts were given by Arumugam, Jacob and Volkmann in [1]. A subset S of vertices of a graph is a connected dominating set if S is a dominating set and $\langle S\rangle$ is connected, this concept was introduced by Sampathkumar and Walinkar in [13] and it is a particular case of total dominating sets. In [1] Arumugam, Jacob and Volkmann established the concept of connected dominating set in digraphs. A dominating set S in a digraph is a weak connected set if $D\langle S\rangle$ is weak connected; in the same way, a unilaterally connected set and a strong connected set are defined. For a comprehensive survey see [6].

Another kind of dominating set is the dominating clique set which is a subset of vertices S of a graph G such that S is a dominating set and $\langle S\rangle$ is a clique. The minimum of the cardinalities of the dominating clique sets is the clique dominating number, denoted by $\gamma_{cl}(G)$. Notice that every dominating clique is a connected dominating set. In [10] Cozzens and Kelleher introduced these concept and proved the following theorem:

Theorem 1.2. *If G is a connected graph without neither induced P_5 nor induced C_5, then G has a dominating clique.*

Also, in [10], Cozzens and Kelleher gave a polynomial time algorithm in order to find a dominating clique set in graphs without neither induced P_5 nor C_5 and, in [7], they gave some applications of dominating cliques in social networks. Moreover, in [2], Bascó and Tuza proved the following:

Theorem 1.3. *In a connected graph G, every connected subgraph contains a dominating clique if and only if G has no neither induced P_5 nor C_5.*

On the other hand, in [8], Kratsch proved that find a dominating clique set is an NP-complete problem.

In digraph theory also there exist similar concepts of domination, for example, the absorbent number of a digraph D, denoted by $\boldsymbol{\gamma_a(D)}$, is the the minimum cardinality of an absorbent set of D. An absorbent set that is an independent set is called a kernel. This concept was introduced by von Neumann and Morgernstern in [11] and Chvátal proved in [4] that find a kernel is an NP-complete problem.

In this paper we introduce the concept of clique dominating number for a digraph. If D is a digraph and S is a subset of $V(D)$, S is a semicomplete absorbent set if S is an absorbent set and $D\langle S\rangle$ is a semicomplete digraph. The minimum (maximum) of the cardinalities of the semicomplete absorbent sets in a digraph D is the lower (upper) semicomplete absorbent number, denoted by $\boldsymbol{\gamma_{sas}(D)}$ ($\boldsymbol{\Gamma_{sas}(D)}$). A semicomplete absorbent set of D with $\gamma_{sas}(D)$ vertices is called a $\boldsymbol{\gamma_{sas}}$−***set*** and in the same way a semicomplete absorbent set with $\Gamma_{sas}(D)$ vertices is called a $\boldsymbol{\Gamma_{sas}}$−***set***. In this paper we will present some basic results for semicomplete absorbent sets in digraphs. We will give some bounds for the lower and upper semicomplete absorbent number; in addition we will establish a relation between $\gamma_a(\bar{D})$ and $\boldsymbol{\Gamma_{sas}}(D)$. We are going to present a characterization of tournaments with relation to $\Gamma_{sas}(D)$. We will study structural properties of semicomplete absorbent sets in the composition, the Cartesian product and the line digraph and then we will show some bounds for the upper semicomplete absorbent set in those digraphs.

We will need the following results.

Theorem 1.4 ([3]). *Every transitive tournament has a sink.*

Theorem 1.5 ([3]). *Every semicomplete digraph has a Hamiltonian directed path.*

The following result characterize the line digraphs.

Corollary 1.1 ([3]). *A digraph D is a line digraph if and only if D does not contain, as an induced subdigraph, any directed pseudograph*[1] *that can be obtained from one of the pseudographs in Figure 1 (dotted arcs are missing) by adding zero or more arcs (other than the dotted ones).*

Theorem 1.6 ([12]). *Let D be a digraph. If D is a transitive digraph, then D has a kernel. Moreover, every kernel is obtained by choosing one vertex in each terminal strong component of D. So, all the kernels of D have the same cardinality.*

2. Existence

Proposition 2.1. *Let* $\{p, s, t\}$ *be a subset of* $\mathbb{N}$ *such that* $1 \le s \le t \le p$ *and* $p = s + t$. *There exists a digraph D of order p such that* $\Gamma_{sas}(D) = t$ *and* $\gamma_{sas}(D) = s$.

Proof. Let $W = \{w_1, w_2, ..., w_s\}$ and $X = \{x_1, x_2, ..., x_s\}$ be two sets, and $U = \{u_1, ..., u_k\}$ a possibly empty set, where $k = t - s$. Consider the following sets $A_1 = \{(w_i, w_j) \ : \ i \neq j\}$, $A_2 = \{(u_i, u_j) : i \neq j\}$, $A_3 = \{(u_i, w_j) \ : \ i \in \{1, 2, ..., s\}, j \in \{1, 2, ..., k\}\}$ and $A_4 = \{(x_i, w_i) \ : \ i \in \{1, 2, ..., s\}\}$. Let D the digraph with set of vertices $W \cup U \cup X$ and set of arcs $A_1 \cup A_2 \cup A_3 \cup A_4$. By construction, D has order p and $W \cup U$ is a $\Gamma_{sas} - set$ of D, which implies that $\Gamma_{sas}(D) = t$. On the other hand, if S is an absorbent set of D, by construction of D it holds that either $x_i \in S$ or $w_i \in S$ for every i in $\{1, ..., s\}$. Therefore $s \le |S|$, in particular, $s \le \gamma_{sas}(D)$. Since W is a semicomplete absorbent set, by construction of D, of order s, we conclude that $\gamma_{sas}(D) = s$. □

The relation $\gamma_{sas}(D) \le \Gamma_{sas}(D)$ can be strict as shows the following result.

Corollary 2.1. *If* $1 < p$, *then there exists a digraph D of order p such that*

$$\gamma_{sas}(D) < \Gamma_{sas}(D).$$

Proof. We will consider two possible cases on p. If $p = 2$, then K_2 is the desire digraph. If $p > 2$, there exist s and t positive integers such that $s < t$ and $p - s = t$. It follows from Proposition 2.1 that there exists a digraph D of order p such that $\Gamma_{sas}(D) = t$ and $\gamma_{sas}(D) = s$. □

In the following proposition we use Theorem 1.2 in order to show a family of digraphs with a semicomplete absorbent set.

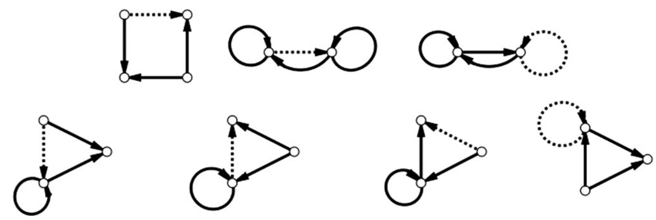

Figure 1. Forbbiden directed pseudographs.

Proposition 2.2. *If G is a graph without neither induced* P_5 *nor induced* C_5, *then there exists an orientation D of G such that D has a semicomplete absorbent set.*

Proof. If G is a graph without neither P_5 nor C_5 as induced subgraphs, then G has a dominating clique S, which implies that for every x in $V(G) \setminus S$ there exists a vertex $s(x)$ in S such that $xs(x) \in E(G)$. If D is an orientation of G such that the edge $xs(x)$ is oriented from x to $s(x)$ in D and the remaining edges have an arbitrary orientation, then S is a semicomplete absorbent set in D. □

Proposition 2.3. *Let* $p \ge 4$ *be an even natural number. There exists a digraph D of order p such that* $\Gamma_{sas}(D) = \gamma_{sas}(D)$.

Proof. Let D be a tournament of order $\frac{p}{2}$, with $V(D) = \{w_1, w_2, ..., w_{\frac{p}{2}}\}, \psi = \left(D_1, ..., D_{\frac{p}{2}}\right)$ a sequence of vertex disjoint digraphs such that $V(D_i) = \{u_i\}$. We claim that $\gamma_{sas}(D \circ \psi) = \Gamma_{sas}(D \circ \psi)$.

Notice that $V(D)$ is a Γ_{sas}-set in $D \circ \psi$ (by definition of corona and because D is a tournament). Since $\gamma_{sas}(D \circ \psi) \le |V(D)|$, then in order to prove that $\gamma_{sas}(D \circ \psi) = |V(D)|$, we can proceed by contradiction and suppose that $\gamma_{sas}(D \circ \psi) < |V(D)|$.

Let N be a γ_{sas}-set in $D \circ \psi$, by our supposition we have that $|N| < |V(D)|$, which implies that there exists w_i in $V(D) \setminus N$ such that $u_i \notin N$. On the other hand, by definition of $D \circ \psi$, since $N^+_{D \circ \psi}(u_i) = \{w_i\}$, we have that u_i is not absorbed by N, a contradiction. Therefore, $\gamma_{sas}(D \circ \psi) = |V(D)|$; that is, $\gamma_{sas}(D \circ \psi) = \Gamma_{sas}(D \circ \psi)$. □

3. Some properties of the digraphs which have semicomplete absorbent sets

Proposition 3.1. *If D is a digraph with at least one semicomplete absorbent set, then D has a unique terminal strong component.*

Proof. Let S be a semicomplete absorbent set in D and H a terminal strong component of D. By contradiction, suppose that there exists a terminal strong component H' of D such that $H' \neq H$. Notice that $V(H) \cap S \neq \emptyset$, because $N^+(V(H)) \subseteq V(H)$ and analogously, $V(H') \cap S \neq \emptyset$. Let $w \in V(H) \cap S$ and $w' \in V(H') \cap S$, since $D\langle S \rangle$ is semicomplete, either $(w, w') \in A(D)$ or $(w', w) \in A(D)$, which contradicts that either H is a terminal strong component or H' is a terminal strong component. □

If a digraph has only one terminal strong component, then it does not necessarily have a semicomplete absorbent set. $\vec{P}_4$ has only one terminal strong component but it has not a semicomplete absorbent set. However the converse of Proposition 3.1 is true in transitive digraphs.

Proposition 3.2. *Let D be a transitive digraph. D has a unique terminal strong component if and only if D has a semicomplete absorbent set.*

Proof. Suppose that D has a semicomplete absorbent set. It follows from Proposition 3.1 that D has a unique terminal strong component.

On the other hand, suppose that D has a unique terminal strong component.

It follows from Theorem 1.6 that D has a kernel K and since D has a unique terminal strong component then we get that $|K| = 1$. Therefore, K is a semicomplete absorbent set in D. □

Proposition 3.3. *Let D be a digraph of order $p > 1$. Suppose that D has at least a semicomplete absorbent set. Then*

1. $\Gamma_{sas}(D) > 1$,
2. if M is a $\Gamma_{sas} - set$, then for every u in $V(D) \setminus M$ there exists v in M such that u and v are not adjacent,
3. if M is a $\gamma_{sas} - set$ and $|M| > 1$, then there exist u in M and w in $N(u) \cap (V(D) \setminus M)$ such that for every v in $M \setminus \{u\}, (w, v) \notin A(D)$.

Proof.

1. Let S be a $\gamma_{sas}-$set in D. Consider two cases on S.
   Case 1. $|S| > 1$.
   In this case we have that $1 < |S| = \gamma_{sas}(D) \leq \Gamma_{sas}(D)$, which implies that $1 < \Gamma_{sas}(D)$.
   Case 2. $|S| = 1$.
   Suppose that $S = \{x\}$. Since $p \geq 2$, then there exists z in $V(D) \setminus S$, which implies that $(z, x) \in A(D)$ (because S is an absorbent set). Therefore, $\{x, z\}$ is a semicomplete absorbent set in D and so $\Gamma_{sas}(D) \geq |\{x, z\}|$, that is, $\Gamma_{sas}(D) > 1$.
2. Proceeding by contradiction, suppose that M is a $\Gamma_{sas}-$set in D such that for every v in $V(D) \setminus M$ and for every x in M, v and x are adjacent. Therefore, for every v in $V(D) \setminus M$ we have that $M' = M \cup \{v\}$ is a semicomplete absorbent set such that $|M'| > |M| = \Gamma_{sas}(D)$, which is not possible. Thus, for every v in $V(D) \setminus M$ there exists x in M such that v and x are not adjacent.
3. Proceeding by contradiction, suppose that M is a γ_{sas}-set, with $|M| > 1$, such that for every x in M and for every w in $N(x) \cap (V(D) \setminus M)$ there exists y in $M \setminus \{x\}$ such that $(w, y) \in A(D)$. Since $|M| > 1$ and M is a semicomplete set, we have that there exists x in M such that $d^+_{\langle M \rangle}(x) > 1$.
   We claim that $M' = M \setminus \{x\}$ is a semicomplete absorbent set. Clearly M' is a semicomplete set because M is semicomplete; on the other hand, by our supposition M' absorbs $N(x)$, which implies that M' is an absorbent set (because the fact M absorbent implies that M' also absorbs $V(D) \setminus (M \cup N(x))$.
   Therefore, since $|M'| < |M| = \gamma_{sas}(D)$, we obtain a contradiction. Thus, there exists w in $N(x) \cap (V(D) \setminus M)$ such that for every y in $M \setminus \{x\}$ we have that $(w, y) \notin A(D)$. □

Corollary 3.1. *If D is a digraph with at least one semicomplete absorbent set, then every $\Gamma_{sas}-$set of D is an absorbent set in $\bar{D}$. In particular, $\gamma_a(\bar{D}) \leq \Gamma_{sas}(D)$.*

Proof. Let D be a digraph with at least one semicomplete absorbent set and S a $\Gamma_{sas}-$set of D. By Proposition 3.3 (2) we have that for every vertex u in $V(D) \setminus S$ there exists v in S such that u and v are not adjacent, so (u, v) is an arc of $\bar{D}$, that is, S is an absorbent set in $\bar{D}$. Therefore, $\gamma_a(\bar{D}) \leq \Gamma_{sas}(D)$. □

Corollary 3.2. *If D is an asymmetric digraph with at least one semicomplete absorbent set, then $\gamma_{sas}(\bar{D}) \leq \Gamma_{sas}(D)$.*

Proof. Let D be an asymmetric digraph and S a semicomplete absorbent set of D. Since D is an asymmetric digraph, we have that $D\langle S \rangle$ is a tournament, so $\bar{D}\langle S \rangle$ is a tournament. On the other hand, by Corollary 3.1 we have that S is an absorbent set in $\bar{D}$, which implies that S is a semicomplete absorbent set in D, hence $\gamma_{sas}(\bar{D}) \leq \Gamma_{sas}(D)$. □

Corollary 3.3. *Let D be a digraph and S a $\Gamma_{sas}-$set of D. If S is a complete set, then S is a kernel of $\bar{D}$.*

Proof. Since S is a complete set in D, then S is an independent set in $\bar{D}$, which implies that S is a kernel in $\bar{D}$, by Corollary 3.1.

Proposition 3.4. *Let D be a digraph and M a semicomplete absorbent set in D. For every x in $V(D) \setminus M$ we have that $M \cup \{x\}$ is not a semicomplete absorbent set in D if and only if for every w in $V(D) \setminus M$ there exists z in M such that w and z are not adjacent in D.*

Proof. Suppose that for every w in $V(D) \setminus M$ there exists z_w in M such that w and z_w are not adjacent in D. Then, it follows that for every w in $V(D) \setminus M, M \cup \{w\}$ is not a semicomplete set in D (because w and z_w are not adjacent in D). Therefore, for every x in $V(D) \setminus M$ we have that $M \cup \{x\}$ is not a semicomplete absorbent set in D.

Suppose that for every x in $V(D) \setminus M$ we have that $M \cup \{x\}$ is not a semicomplete absorbent set in D. Since M is an absorbent set, we get that $M \cup \{x\}$ is an absorbent set for every x in $V(D) \setminus M$. Therefore, $M \cup \{x\}$ is not a semicomplete set for every x in $V(D) \setminus M$; that is, for every x in $V(D) \setminus M$ there exists u_x in M such that x and u_x are not adjacent in D (because M is semicomplete). □

Proposition 3.5. *Let D be a digraph. If D has a semicomplete absorbent set S, then D has a 3−king. Moreover, if S is a semicomplete absorbent set such that $|S| = t$, for some $t \in \mathbb{N}$, then D has a t-king.*

Proof. Let S be a semicomplete absorbent set of D. By Theorem 1.1 we have that $D\langle S\rangle$ has a $2-$king, say x. It is straightforward to see that x is a $3-$king of D.

On the other hand, suppose that $|S| = T$ and consider $P = (x_0, \ldots, x_{t-1})$ be a Hamiltonian path in $D\langle S\rangle$ (there exists P by Theorem 1.5). We claim that x_{t-1} is a t-king. Since for every u in $V(D) \setminus S$ there exists w_u in S such that $(u, w_u) \in A(D)$ then we get that $(u, w_u) \cup (w_u, P, x_{t-1})$ is a ux_{t-1}-path of length at most t. Therefore, x_{t-1} is a t-king. □

In [3] we can found the following proposition.

Proposition 3.6. *Let D be a weak connected digraph. D contains an out-branching (in-branching) if and only if D has only one initial (terminal) strong component.*

It follows from Proposition 3.1 and Proposition 3.6 that every digraph D with at least one semicomplete absorbent set has an in-branching. Moreover, Proposition 3.5 shows that the in-branching T is the digraph such that $V(T) = V(D)$ and $A(T) = A(P) \cup \{(u, w_u) \in A(D) : u \in V(D) \setminus S \text{ and } w_u \text{ in } V(P)\}$. On the other hand, we get from Proposition 3.2 and Proposition 3.6 that for a transitive digraph D it holds that D has at least one semicomplete absorbent set if and only if D has an in-branching.

Corollary 3.4. *Let D be a digraph.*

1. *If D is transitive then D has at least one semicomplete absorbent set if and only if D has an in-branching.*
2. *Every digraph D with at least one semicomplete absorbent set has an in-branching.*

4. Some bounds for $\Gamma_{sas}(D)$ y $\gamma_{sas}(D)$

Proposition 4.1. *Let D be a digraph of size q. If D has a semicomplete absorbent set, then* $\Gamma_{sas}(D)^2 - \Gamma_{sas}(D) \leq 2q$.

Proof. Let S be a $\Gamma_{sas}-$set of D and p the order of D. Since $D\langle S\rangle$ is a semicomplete digraph, then $|A(D\langle S\rangle)| \geq \frac{|S|(|S|-1)}{2}$. On the other hand, for every x in $V(D) \setminus S$ there exists y in S such that $(x, y) \in A(D)$, because S is an absorbent set, then

$$q \geq |V(D) \setminus S| + |A(D\langle S\rangle)| \geq (p - |S|) + \frac{|S|(|S| - 1)}{2}$$
$$= \frac{|S|^2 - 3|S| + 2p}{2},$$

which implies that $2q \geq \Gamma_{sas}(D)^2 - 3\Gamma_{sas}(D) + 2p \geq \Gamma_{sas}(D)^2 - \Gamma_{sas}(D)$. □

Proposition 4.2. *Let D be a digraph with at least one semicomplete absorbent set. Then D is a tournament if and only if* $2q = \Gamma_{sas}(D)^2 - \Gamma_{sas}(D)$.

Proof. If D is a tournament of order p, then $p = \Gamma_{sas}(D)$ and so,

$$2q = p(p-1) = \Gamma_{sas}(D)^2 - \Gamma_{sas}(D).$$

On the other hand, let D be a digraph such that $2q = \Gamma_{sas}(D)^2 - \Gamma_{sas}(D)$ and by contradiction, suppose that D is not a tournament. Notice that D is not a semicomplete digraph, otherwise since D is not a tournament it has at least one symmetric arc, then $p = \Gamma_{sas}(D)$ and $2q > p(p-1)$, that is $2q > \Gamma_{sas}(D)^2 - \Gamma_{sas}(D)$, which contradicts the hypothesis. Therefore D is not a semicomplete digraph. Let S be a $\Gamma_{sas}-$set of D, then $2q_S \geq \Gamma_{sas}(D)^2 - \Gamma_{sas}(D)$, where q_S is the size of $D\langle S\rangle$, since D is not a semicomplete set, then $S \subset V(D)$, which implies that there exists u in $V(D) \setminus S$ and v in S such that $(u, v) \in A(D)$. Thus $2q > 2q_S \geq \Gamma_{sas}(D)^2 - \Gamma_{sas}(D)$, a contradiction. Therefore, D is a tournament. □

Proposition 4.3. *If D is a digraph of order p, such that D has at least one semicomplete absorbent set, then* $\gamma_{sas}(D) \leq \frac{p+1}{2}$. *Moreover, if there exists a* $\gamma_{sas}-$*set S of D such that* $D\langle S\rangle$ *has no sinks, then* $\gamma_{sas}(D) \leq \frac{p}{2}$.

Proof. Let S be a $\gamma_{sas}-$set of D and $S' = \{x \in S \ : \ d^+_{\langle S\rangle}(x) \neq 0\}$. Notice that for every x in S' there exists y in $V(D) \setminus S$ that satisfies $N^+(y) \cap S = \{x\}$, otherwise, if there exists x in S' such that for every y in $V(D) \setminus S, N^+(y) \cap S \neq \{x\}$, then by choice of x, every vertex in $V(D) \setminus S$ is absorbed by a vertex in $S \setminus \{x\}$ and by definition of S' we have that x is absorbed by $S \setminus \{x\}$. Moreover, since $S \setminus \{x\} \subset S$, then $S \setminus \{x\}$ is a semicomplete set and therefore $S \setminus \{x\}$ is a semicomplete absorbent set, a contradiction. Therefore for every x in S' there exists y in $V(D) \setminus S$ that satisfies $N^+(y) \cap S = \{x\}$ and thus $|S'| \leq |V(D) \setminus S|$; that is, $|S'| + |S| \leq |V(D)|$ and by choice of S we have that $\gamma_{sas}(D) + |S'| \leq |V(D)|$.

On the other hand, since $D\langle S\rangle$ is a semicomplete digraph, it has at most one sink, then $|S'| \geq |S| - 1$, concluding that $\gamma_{sas}(D) \leq \frac{p+1}{2}$. In particular, if $D\langle S\rangle$ has no sinks, then $|S| = |S'|$ and therefore $\gamma_{sas}(D) \leq \frac{p}{2}$. □

Let p an odd natural number, say $p = 2n + 1$ for some natural number n, D a transitive tournament of order $n + 1$, with $V(D) = \{w_1, w_2, \ldots, w_{n+1}\}$, and $\psi = (D_1, \ldots, D_{n+1})$ a sequence of vertex disjoint digraphs such that $V(D_i) = \{u_i\}$. Since D has a sink (by Theorem 1.4), suppose without loss of generality that $d^+(w_{n+1}) = 0$. Consider the digraph $H = D \circ \psi\langle V(D) \cup \{u_1, \ldots, u_n\}\rangle$. It is straightforward to see that $V(D)$ is the only one semicomplete absorbent set in H, which implies that $\gamma_{sas}(H) = \frac{p+1}{2}$, that is, H achieve the first bound in Proposition 4.3.

Let p be an even natural number, D a complete digraph of order $\frac{p}{2}$, with $V(D) = \{w_1, w_2, \ldots, w_{\frac{p}{2}}\}$, and $\psi = \left(D_1, \ldots, D_{\frac{p}{2}}\right)$ a sequence of vertex disjoint digraphs such that $V(D_i) = \{u_i\}$. Notice that $V(D)$ is the only one semicomplete absorbent set in $D \circ \psi$, which implies that $\gamma_{sas}(D \circ \psi) = \frac{p}{2}$, that is, $D \circ \psi$ achieve the second bound in Proposition 4.3.

In Proposition 2.3, we proved that given an even natural number $p \geq 4$ there exists a digraph D of order p such that $\Gamma_{sas}(D) = \gamma_{sas}(D)$. In the following corollary we show an uper bound for $\Gamma_{sas}(D)$ for this kind of digraphs.

Corollary 4.1. *If D is a digraph such that* $\Gamma_{sas}(D) = \gamma_{sas}(D)$, *then* $\Gamma_{sas}(D) \leq \frac{p+1}{2}$.

Proof. If follows from Proposition 4.3 that $\frac{p+1}{2} \geq \gamma_{sas}(D)$, which implies that $\frac{p+1}{2} \geq \Gamma_{sas}(D)$ (by hypothesis). □

5. Semicomplete absorbent numbers in the Cartesian product, the composition of digraphs and the line digraph

We give some bounds for the lower and upper semicomplete absorbent number in these operations.

Lemma 5.1. *Let D and H be two digraphs, $D\square H$ the Cartesian product and $U \subseteq V(D\square H)$. If U is a semicomplete set, then U is contained either in H_{x_0} for some $x_0 \in V(D)$ or U is contained in D_{z_0} for some $z_0 \in V(H)$.*

Proof. Notice that if $|U| = 1$, the result holds. Otherwise, let $\{(x,y),(z,w)\} \subseteq U$. Since $\langle U\rangle$ is semicomplete we can suppose, without loss of generality, that $((x,y),(z,w)) \in A(D\square H)$. So, there are only two cases respect these vertices.

If $x = z$ and $(y,w) \in A(H)$, then we claim that $U \subseteq H_x$. Otherwise, there exists a vertex (u, v) in U such that $u \neq x$, and then (u, v) and (x, y) are adjacent, concluding that $y = v$. In the same way (u, v) and (z, w) are adjacent, then $v = w$, and this implies that $y = w$, this contradicts that $(y,w) \in A(H)$.

In the same way, if $y = w$ and $(x,z) \in A(D)$, the we claim that $U \subseteq D_w$. Then some level of $D\square H$ contains U. □

Proposition 5.1. *If D and H are two nontrivial digraphs, then $D\square H$ has a semicomplete absorbent set if and only if either D is a semicomplete digraph and $\gamma_{sas}(H) = 1$ or H is a semicomplete digraph and $\gamma_{sas}(D) = 1$.*

Proof. Let D and H be two digraphs and suppose that $D\square H$ has a semicomplete absorbent set, say S. We claim that either $S = H_{x_0}$ for some $x_0 \in V(D)$ or $S = D_{z_0}$ for some $z_0 \in V(H)$. According to Lemma 5.1, S is contained either in H_{x_0} for some $x_0 \in V(D)$ or S is contained in D_{z_0} for some $z_0 \in V(H)$. We can suppose without loss of generality that $S \subseteq H_{x_0}$ for some $x_0 \in V(D)$. For the other inclusion, let (x_0,z) be a vertex in H_{x_0} and consider $(x_1,z) \in V(D\square H) \setminus H_{x_0}$. Since S is an absorbent set, there exists $(x_0,w) \in S$ that dominates (x_1,z) and by definition of $D\square H$ we conclude that $w = z$ and then $(x_0,z) \in S$. It follows that $S = H_{x_0}$ and, since $(D\square H)\langle H_{x_0}\rangle \cong H$, we conclude that H is a semicomplete digraph. On the other hand, we claim that x_0 is an absorbent vertex in D. If $y \in V(D) \setminus \{x_0\}$, then $(y,u) \in V(D\square H) \setminus S$ for some $u \in V(H)$. Since $S = H_{x_0}$ is an absorbent set of $D\square H$, then $((y,u),(x_0,u)) \in A(D\square H)$ concluding that $(y,x_0) \in A(D)$. Hence, $\gamma_{sas}(D) = 1$ and H is a semicomplete digraph. A similar proof shows that if $S \subseteq D_{y_0}$ for some $y_0 \in V(H)$, then $\gamma_{sas}(H) = 1$ and D is a semicomplete digraph.

On the other hand, suppose that H is a semicomplete digraph and $\gamma_{sas}(D) = 1$, let x_0 an absorbent vertex in D. Since $(D\square H)\langle H_{x_0}\rangle \cong H$, it follows that $(D\square H)\langle H_{x_0}\rangle$ is a semicomplete digraph. Moreover, by choice of x_0, if $(u,v) \in V(D\square H) \setminus H_{x_0}$, then $((u,v),(x_0,v)) \in A(D\square H)$, concluding that H_{x_0} is a semicomplete absorbent set of $D\square H$. A similar proof shows that the result holds if D is a semicomplete digraph and $\gamma_{sas}(H) = 1$. □

Proposition 5.1 is a characterization of semicomplete absorbent sets in the Cartesian product and it can be used to determinate the upper and lower semicomplete absorbent number in that operation.

Corollary 5.1. *If D and H are two digraphs such that $D \square H$ has a semicomplete absorbent set, then $\Gamma_{sas}(D\square H) \in \{p_1,p_2\}$ and $\gamma_{sas}(D\square H) \in \{p_1,p_2\}$, where p_1 is the order of D and p_2 is the order of H.*

Proof. If D and H are two digraphs and S is a $\Gamma_{sas}-$set of $D\square H$, by Proposition 5.1, S is some level of $D\square H$, so $\Gamma_{sas}(D\square H) \in \{p_1,p_2\}$. Analogously, $\gamma_{sas}(D\square H) \in \{p_1,p_2\}$. □

Now we will show an upper bound for the upper semicomplete absorbent number in the composition of digraphs.

Proposition 5.2. *If D is a digraph and $\alpha = (D_v)_{v\in V(D)}$ is a sequence of digraphs such that $D[\alpha]$ has a semicomplete absorbent set, then D has a semicomplete absorbent set.*

Proof. Let S' be a semicomplete absorbent set of $D[\alpha]$, we claim that $S = \{v \in V(D) : S' \cap V(D_v) \neq \emptyset\}$ is a semicomplete absorbent set of D. Let $\{u,v\} \subseteq S$, notice that if $x \in V(D_v) \cap S'$ and $z \in V(D_u) \cap S'$, then x and z are adjacent in $D[\alpha]$ because $D[\alpha]\langle S'\rangle$ is a semicomplete digraph, and then u and v are adjacent in D. In other case, if $w \in V(D) \setminus S$ then $D_w \cap S = \emptyset$, it follows that if $x \in V(D_w)$ there exists $z \in S'$ such that $(x,z) \in A(D[\alpha])$, in particular $z \in V(D_v)$ for some $v \in V(D)$. Therefore $v \in S$ and $(w,v) \in A(D)$. □

Proposition 5.3. *Let D be a digraph, $\alpha = (D_v)_{v\in V(D)}$ a sequence of digraphs and S a semicomplete absorbent set without sinks in D. If R_v is a semicomplete set in D_v for every $v \in S$, then $S' = \cup_{v\in S} R_v$ is a semicomplete absorbent set in $D[\alpha]$.*

Proof. Let $\{x,z\} \subseteq S'$ and consider two cases for x and z. If $\{x,z\} \subseteq R_v$ for some $v \in S$, by choice of R_v there exists an arc between x and z in $D[\alpha]$. In other case, if $x \in R_v$ and $z \in R_u$ for some $\{u,v\} \subseteq S$ with $u \neq v$, since $D\langle S\rangle$ is a semicomplete subdigraph of D, then there exists an arc between u and v in D, which implies that x and z are adjacent in $D[\alpha]$, concluding that $D[\alpha]\langle S'\rangle$ is a semicomplete subdigraph of $D[\alpha]$.

On the other hand, if $x \in V(D[\alpha]) \setminus S'$, in particular $x \in V(D_v)$ for some $v \in V(D)$. If $v \in V(D) \setminus S$, then there exist u in S such that $(v,u) \in A(D)$, concluding that $(x,z) \in A(D[\alpha])$ for some z in R_u and therefore x is absorbed by S'. In other case, since S has no sinks, there exist u in S such that $(v,u) \in A(D)$ and by definition of $D[\alpha]$ we have that every vertex in D_v is dominated by D_u, concluding that $(x,z) \in A(D[\alpha])$ for some z in R_u. Therefore, S' is a semicomplete absorbent set in $D[\alpha]$.

Corollary 5.2. *Let D be a digraph and $\alpha = (D_v)_{v\in V(D)}$ a sequence of digraphs. If S is a semicomplete absorbent set*

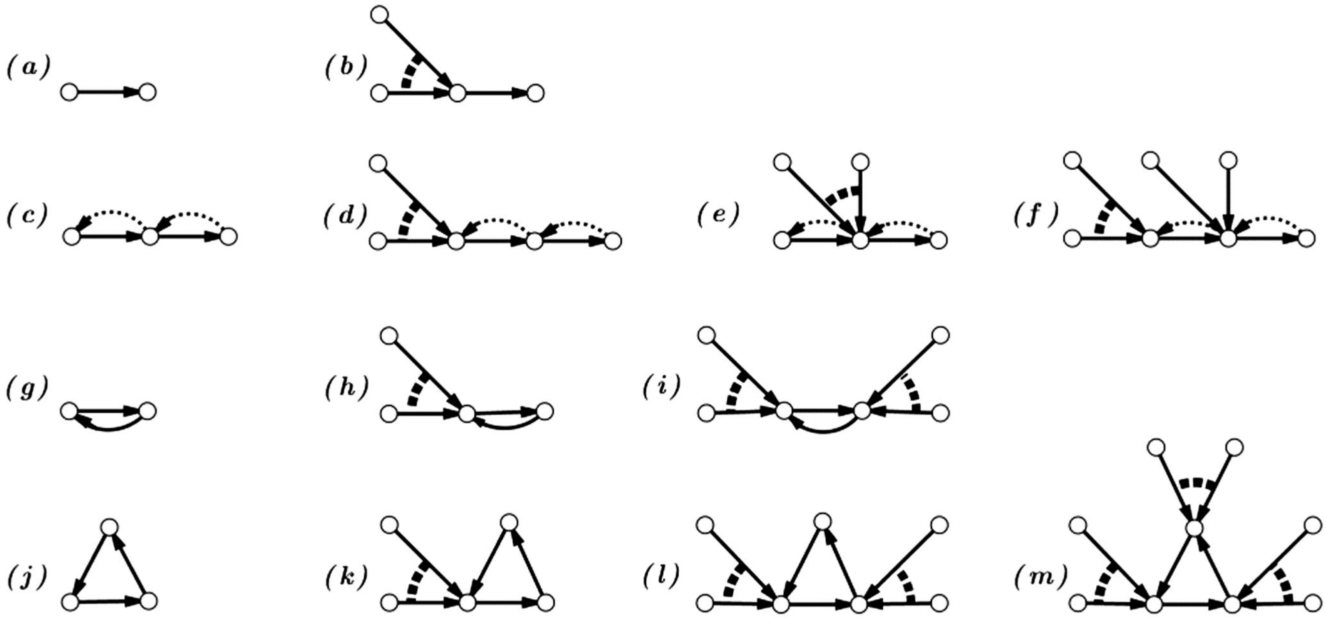

Figure 2. Digraphs whose line digraph contains a semicomplete absorbent set. Dotted arcs indicate that can there exist symmetric arcs.

without sinks in D, then there exists a semicomplete absorbent set in $D[\alpha]$ *of order* $\sum_{v\in S}\omega^*(D_v)$.

Proof. Let R_v be a semicomplete set in D_v with $\omega^*(D_v)$ vertices for every v in S. By Proposition 5.3, $S' = \cup_{v\in S} R_v$ is a semicomplete absorbent set in $D[\alpha]$, it follows that $|S'| = \sum_{v\in S}\omega^*(D_v)$. □

Corollary 5.3. *Let D be a digraph and* $\alpha = (D_v)_{v\in V(D)}$ *a sequence of digraphs such that* $D[\alpha]$ *has a semicomplete absorbent set without sinks. If* $\mathfrak{S}$ *is the family of semicomplete absorbent sets without sinks in D, then*

$$\max_{S\in\mathfrak{S}}\left\{\sum_{v\in S}\omega^*(D_v)\right\} \le \Gamma_{sas}(D[\alpha]).$$

Proof. By Corollary 5.1, for every S in $\mathfrak{S}$ there exists a semicomplete absorbent set in $D[\alpha]$, say S', such that $\sum_{v\in S}\omega^*(D_v) = |S'|$, in particular $\sum_{v\in S}\omega^*(D_v) \le \Gamma_{sas}(D[\alpha])$ for every S in $\mathfrak{S}$, concluding that

$$\max_{S\in\mathfrak{S}}\left\{\sum_{v\in S}\omega^*(D_v)\right\} \le \Gamma_{sas}(D[\alpha]).$$

□

For the line digraph, we will give an upper bound for the upper absorbent semicomplete number in this digraph.

Proposition 5.4. *Let D be a digraph and L(D) its line digraph. If L(D) has a semicomplete absorbent set, then* $\Gamma_{sas}(\ L(D)\ ) \le 3$. *Moreover*

1. $\Gamma_{sas}(L(D)) = 1$ *if and only if D is either (a) or (b) as Figure 2 shows.*
2. $\Gamma_{sas}(L(D)) = 2$ *if and only if D is either (c), (d), (e), (f), (g), (h) or (i) as Figure 2 shows.*
3. $\Gamma_{sas}(L(D)) = 3$ *if and only if D is either (j), (k), (l) or (m) as Figure 2 shows.*

Proof. Let S be a $\Gamma_{sas}-$set of $L(D)$. Suppose that $|S| \ge 4$ and let $S_1 = \{w, x, y, z\}$ be a subset of S. Since $L(D)\langle S\rangle$ is semicomplete, then by Corollary 1.1 we can suppose that $\{(w, x), (x, z), (z, w)\} \subseteq A(L(D))$ (because in particular $L(D)$ has no transitive subtournaments of orden 3). In the same way, for the set $\{x, y, w\}$ we have that $\{(x, y), (y, w)\} \subseteq A(\ L(D)\ )$ because $(w, x) \in A(\ L(D)\ )$. Therefore, $L(D)\langle\{w, z, y\}\rangle$ has a transitive subtournament of orden 3 which is no possible. Therefore, $\Gamma_{sas}(\ L(D)\ ) \le 3$.

Let S be a $\Gamma_{sas}-$set in $L(D)$. Consider the following cases.

- **Case 1.** $|S| = 1$.
  In this case we have that $S = \{(u, v)\}$ for some $\{u, v\} \subseteq V(D)$. Since S is an absorbent set in $L(D)$ we have that for every arc $(x,\ y)$ in $A(D)\setminus S, ((x, y), (u, v)) \in A(L(D))$, which implies that $y = u$. Hence, since S is a $\Gamma_{sas}-$set, D is either (a) or (b) as Figure 2 shows.
- **Case 2.** $|S| = 2$.
  In this case we have that $S = \{(u, v), (s, w)\}$ for some $\{u, v, w, s\} \subseteq V(D)$. Since S is a semicomplete set in $L(D)$, we may assume that $v = s$. On the other hand, since S is an absorbent set in $L(D)$, for every arc $(x,\ y)$ in $A(D)\setminus S$ we have that $y \in \{u, v\}$. Hence, since S is a $\Gamma_{sas}-$set, if $w \ne u$, then D is one of (c), (d), (e) or (f) showed in Figure 2. If $w = u$ then D is one of (g), (h) or (i) as Figure 2 shows.
- **Case 3.** $|S| = 3$.
  In this case, $S = \{(s, t), (u, v), (w, z)\}$ for some $\{s, t, u, v, w, z\} \subseteq V(D)$. According to Corollary 1.1 we have that $L(D)\langle S\rangle$ contains no transitive tournament of order three, which implies that $L(D)\langle S\rangle$ is a cycle. Suppose that $t = u$, $v = w$ and $z = s$. Since S is an absorbent set in $L(D)$, we have that for every arc $(x,\ y)$ in $A(D)\setminus S$ we have that $y \in \{z, u, v\}$. Therefore, D is one of (j), (k), (l) or (m) as Figure 2 shows. □

Note

A pseudograph is a graph that may contains parallel edges or loops.

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Conflict of interest

No conflicts of interest have been reported by the authors.

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