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An explicit construction of optimal dominating and [1, 2]–dominating sets in grid

P. Sharifani[a,b*], M. R. Hooshmandasl[a,b#], and M. Alambardar Meybodi[a,b‡]

[a]Department of Computer Science, Yazd University, Yazd, Iran; [b]The Laboratory of Quantum Information Processing, Yazd University, Yazd, Iran

ABSTRACT

A dominating set in a graph G is a subset of vertices D such that every vertex in $V \setminus D$ is a neighbor of some vertex of D. The domination number of G is the minimum size of a dominating set of G and it is denoted by $\gamma(G)$. A dominating set with cardinality $\gamma(G)$ is called optimal dominating set. Also, a subset D of a graph G is a [1, 2]-set if, each vertex $v \in V \setminus D$ is adjacent to either one or two vertices in D and the minimum cardinality of [1, 2]-dominating set of G, is denoted by $\gamma_{[1,2]}(G)$. Chang's conjecture says that for every $16 \le m \le n, \gamma(G_{m,n}) = \lfloor \frac{(n+2)(m+2)}{5} \rfloor - 4$ and this conjecture has been proven by Goncalves et al. This paper presents an explicit constructing method to find an optimal dominating set for grid graph $G_{m,n}$ where $m, n \ge 16$ in O (size of answer). In addition, we will show that $\gamma(G_{m,n}) = \gamma_{[1,2]}(G_{m,n})$ where $m, n \ge 16$ holds in response to an open question posed by Chellali et al.

KEYWORDS
Grid graph; dominating set; [1, 2]-dominating set; NP-complete; dynamic programming

1. Introduction

The concept of domination and dominating set is a well-studied topic in graph theory and has many extensions and applications. A discussion of some of these can be found in [14, 15]. Many variations of domination problems have shown to be NP-complete [2, 5, 6, 17, 19, 21, 24]. Also, many algorithmic results have studied for these problems in different classes of graphs. A subset S of vertices is a dominating set if every vertex not in S has at least one neighbor in S. A dominating set with minimum cardinality is called an optimal dominating set of a graph G; its cardinality is called the domination number of G and is denoted by $\gamma(G)$. Note that although the domination number of a graph, $\gamma(G)$, is unique, there may be different optimal dominating sets. Grid graphs are a special class of graphs and the dominating set of them have many applications in robotics and sensor networks [3]. Due to the special structure of grids, their domination number can be determined optimally, although this number was known recently by Goncalves et al., [10]. They proved that for $m \times n$ grids, where $m, n \ge 16$,

$$\gamma(G_{m,n}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$

The idea behind their proof is using a dynamic programming method to store all dominating sets that occur in borders of a grid. Various attempts have been made in recent years to find an algorithm for the optimal dominating set. Chang [4], by using diagonalization and projection, constructed a dominating set for grids in polynomial-time, such that

$$\gamma(G_{m,n}) \le \left\lceil \frac{(n+2)(m+2)}{5} \right\rceil.$$

The cardinality of dominating set constructed by Chang's method is at most $\gamma(G_{m,n}) + 5$, when $16 \le m \le n$.

In [1], Alanko et al used brute-force computational technique to find optimal dominating set in grids of size up to $n = m = 29$.

Fata et al. [8] presented a distributed algorithm for finding near optimal dominating sets on grids. The size of the dominating set provided by their algorithm is upper-bounded by $\left\lceil \frac{(n+2)(m+2)}{5} \right\rceil$ for $m \times n$ grids and its difference from the optimal domination number of the grid is upper-bounded by five.

P. Pisantechakool et al. [23] improved upon the distributed algorithm of Fata et al. and presented a new distributed algorithm that computes a dominating set of size $\left\lceil \frac{(n+2)(m+2)}{5} \right\rceil - 3$ on an $m \times n$ grid, $8 \le m, n$ and its difference from the optimal domination number of the grid is upper-bounded by two.

There are numerous intermediate results for minimal dominating set and $\gamma(G_{m,n})$ for small values of n and m by a dynamic programming algorithm [12, 13, 18, 20, 24, 28].

CONTACT M. R. Hooshmandasl hooshmandasl@uma.ac.ir Department of Computer Science, Yazd University, Yazd, Iran.
*P. Sharifani is also now affiliated with the Institute for Research in Fundamental Sciences (IPM), Tehran, Iran.
#M. R. Hooshmandasl is also now affiliated with the Department of Computer Science, University of Mohaghegh Ardabili, Ardabil, Iran.
‡M. Alambardar Meybodi is also now affiliated with the Department of Applied Mathematics and Computer Science, University of Isfahan, Isfahan, Iran.

Most of those algorithms are not efficient in practice when the values n or m be over 20.

One of the interesting issues that can be expressed in different types of domination problems is that under which conditions the domination number of graph is equal to the domination number of that particular domination problem [7, 11, 16].

A dominating set like D of graph $G(V, E)$, called [1, 2]-dominating set if each vertex $v \in V \setminus D$ is adjacent to at most two vertices in D. The concept of [1, 2]-dominating set, as a special case of $[\rho, \sigma]$-dominating set [26], is introduced by Chellali et al [6]. They studied [1, 2]-dominating sets in graphs and posed a number of open problems. Some of those problems are solved in [9, 27]. One of the proposed questions is:

Question: Is it true for grid graphs $G_{m,n}$ that $\gamma(G_{m,n}) = \gamma_{[1,2]}(G_{m,n})$?

We show that the answer to this question is positive by constructing a γ-set for $G_{m,n}$ which is also a $\gamma_{[1,2]}$-set.

The main result of this paper is a construction to find an optimal dominating set for grid $G_{m,n}$ where $m, n \geq 16$. The rest of this paper proceeds as follows. In Section 2 we describe some notations and definitions are needed. In section 3, we present our construction and the correctness argument for it. Also to ease of understanding we illustrate some examples. Finally, in section 4 we prove that $\gamma(G_{m,n}) = \gamma_{[1,2]}(G_{m,n})$ for $m, n \geq 16$.

2. Terminology

In this section, we introduce some definitions and notations that will be needed in the sequel. For all terminologies and notations are not defined here, we refer to [22]. Let $G = (V, E)$ be a simple graph, the neighborhood of a vertex $v \in V$ is the set of all vertices adjacent to v and is denoted by $N(v)$, i.e. $N(v) = \{u \in V | uv \in E\}$. The closed neighborhood of a vertex v is defined $N[v] = N(v) \cup \{v\}$. A set S is called a dominating set of G if every vertex is either in S or adjacent to a vertex in S. The size of the smallest dominating sets of a graph G is denoted by $\gamma(G)$. Any such set is called a γ-set or minimal dominating set of G.

A set $S \subseteq V$ is called a [1, 2]-set of G if for each $v \in V \setminus S$ we have $1 \leq |N(v) \cap S| \leq 2$, i.e. v is adjacent to at least one but not more than two vertices in S. The size of the smallest [1, 2]-sets of G is denoted by $\gamma_{[1,2]}(G)$. Any such set is called a $\gamma_{[1,2]}$-set of G. We know that for every graph G, $\gamma(G) \leq \gamma_{[1,2]}(G) \leq n$ [6]. In some classes of graphs, the domination number and [1, 2]-domination number are equal. This equality holds for cycles, caterpillars, claw-free graphs, P_4-free graphs, and nontrivial graph G with $\Delta(G) \geq |V(G)| - 3$, are proved in [6].

An $m \times n$ grid graph $G_{m,n} = (V; E)$ has vertex set $V = \{v_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and edge set $E = \{(v_{i,j}, v_{i,j'}) : |j - j'| = 1\} \cup \{(v_{i,j}, v_{i',j}) : |i - i'| = 1\}$. For ease of exposition, we will fix an orientation and labeling of the vertices, so that vertex $v_{1,1}$ is the upper-left vertex and vertex $v_{m,n}$ is the lower-right vertex of the grid.

We also require the following definitions.

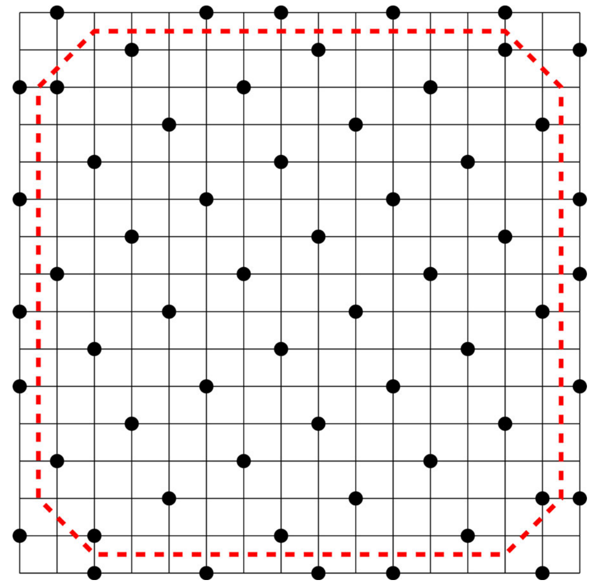


Figure 1. Example of dominating set of size 60 for grid $G_{16,16}$ and its sub-grid is highlighted by a red dashed line. The intersection of each row and column is a vertex of grid.

Definition 2.1. The boundary of grid $G_{m,n}$, denoted by $B(G)$, is the set of vertices like $v \in V$ such that $|N(v)| < 4$.

Definition 2.2. A sub-grid of $G_{m,n}$ is induced graph by vertices $V = \{v_{i,j} : 2 \leq i \leq m-1,\ 2 \leq j \leq n-1\} \setminus \{v_{2,2}, v_{2,n-1}, v_{m-1,2}, v_{m-1,n-1}\}$ (see Figure 1).

3. Construction of dominating set in grid

The idea behind our method is choosing a proper pattern that dominates every vertex in sub-grid exactly once. Our purpose is selecting most of the vertices of dominating set from sub-grid as soon it is possible because every vertex in sub-grid dominates at most five vertices (its four neighbors and itself). These selected vertices are indicated by black disks. Since, just by selecting vertices of sub-grid, the boundary vertices of the grid may be not dominated, then we have to add some vertices of the boundary, indicated by white squares, to the dominating set. To do so, we identify vertices of the optimal dominating set by two following steps:

Step 1: Identifying domination points (black disks) to dominate the vertices of sub-grid and some of the vertices of the boundary.

Step 2: Identifying domination points (white squares) to dominate the boundary vertices which are not dominated by black disks.

In the step 1, according to the number of columns, we select the index of first appropriate column that a black disk must be located in row p, denoted a_p. In other words, the vertex v_{p,a_p} is the first position in row p that a domination point is located.

For the first row, a_p is defined as

$$a_1 = \begin{cases} 2 & \text{if } \ n \equiv 0 \ (mod\ 5), \\ n \mod 5 & \text{otherwise}, \end{cases}$$

and for other rows, $a_p \equiv a_1 + 3(p-1) \ (mod\ 5)$.

Then, we construct the set D_d as union of the following sets

$$\begin{aligned} D_F &= \{v_{1,5k+a_1} : 3 \le 5k + a_1 \le n-2 \text{ for some } k\}, \\ D_M &= \{v_{p,5k+a_p} : 2 \le p \le n-1 \text{ and } 1 \le 5k + a_1 \le n \text{ for some } k\}, \\ D_L &= \{v_{m,5k+a_n} : 3 \le 5k + a_1 \le n-2 \text{ for some } k\}, \end{aligned}$$

where D_F, D_M and D_L are the sets of all black disks in first row, middle rows and last row, respectively.

In the step 2, at first we define the set $A_k^{(i,j)}$ as

$$A_k^{(i,j)} = \{\ 5t + k \mid i \le t \le j\ \}.$$

Also, we set $S = \lfloor \frac{n}{5} \rfloor$ and $T = \lfloor \frac{m}{5} \rfloor$.

In order to cover vertices in the borders which are not dominated, the following vertices are added to the dominating set. These vertices are indicated by white squares.

First Row: Selecting the white squares in first row is just depend on n and is independent from m. The set of all white squares of first row is defined by D_s^{FR}. This set is selected as follow:

$$D_s^{FR} = \begin{cases} \left\{v_{1,p} :\ p \in A_4^{(1,S-1)}\right\} \cup \{v_{1,3}\} & \text{if} \quad n \equiv 0 \ (mod\ 5), \\ \left\{v_{1,p} :\ p \in A_3^{(1,S-2)}\right\} \cup \{v_{1,2}, v_{1,n-2}\} & \text{if} \quad n \equiv 1 \ (mod\ 5), \\ \left\{v_{1,p} :\ p \in A_4^{(1,S-2)}\right\} \cup \{v_{1,3}, v_{1,n-2}\} & \text{if} \quad n \equiv 2 \ (mod\ 5), \\ \left\{v_{1,p} :\ p \in A_0^{(1,S-1)}\right\} \cup \{v_{1,n-2}\} & \text{if} \quad n \equiv 3 \ (mod\ 5), \\ \left\{v_{1,p} :\ p \in A_1^{(1,S-1)}\right\} \cup \{v_{1,n-2}\} & \text{if} \quad n \equiv 4 \ (mod\ 5), \end{cases}$$

Left-Right Columns and Last Row: Selecting the white squares in these borders, aside from the first row, depend on both m and n. The sets of all white squares in the first, last columns and last row are denoted by D_s^{FC}, D_s^{LC} and D_s^{LR}, respectively. These points are selected as follow:

Case 1: $n \equiv 0 \ (mod\ 5)$

$$D_s^{FC} = \begin{cases} \left\{v_{p,1} : p \in A_2^{(0,T-2)}\right\} \cup \{v_{m-3,1}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_2^{(0,T-1)}\right\} & \text{if} \quad m \equiv 1,2 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_2^{(0,T)}\right\} & \text{if} \quad m \equiv 3 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_2^{(0,T-1)}\right\} \cup \{v_{m-1,1}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

$$D_s^{LC} = \begin{cases} \left\{v_{p,n} : p \in A_4^{(1,T-1)}\right\} \cup \{v_{3,n}\} & \text{if} \quad m \equiv 0,3,4 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_4^{(1,T-2)}\right\} \cup \{v_{3,n}, v_{m-1,n}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_4^{(1,T-2)}\right\} \cup \{v_{3,n}, v_{m-2,n}\} & \text{if} \quad m \equiv 2 \ (mod\ 5). \end{cases}$$

$$D_s^{LR} = \begin{cases} \left\{v_{m,p} : p \in A_2^{(0,S-2)}\right\} \cup \{v_{m,n-2}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_0^{(1,S-1)}\right\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_1^{(1,S-1)}\right\} & \text{if} \quad m \equiv 3 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_3^{(1,S-2)}\right\} \cup \{v_{m,2}, v_{m,n-1}\} & \text{if} \quad m \equiv 2 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_4^{(1,S)}\right\} \cup \{v_{m,3}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

Case 2: $n \equiv 1 \ (mod\ 5)$

$$D_s^{FC} = \begin{cases} \left\{v_{p,1} : p \in A_4^{(1,T-1)}\right\} \cup \{v_{3,1}\} & \text{if} \quad m \equiv 0,3,4 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_4^{(1,T-2)}\right\} \cup \{v_{3,1}, v_{m-1,1}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_4^{(1,T-2)}\right\} \cup \{v_{3,1}, v_{m-2,1}\} & \text{if} \quad m \equiv 2 \ (mod\ 5). \end{cases}$$

$$D_s^{LC} = \begin{cases} \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}, v_{m-1,n}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}\} & \text{if} \quad m \equiv 2,3 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}, v_{m-2,n}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-1)}\right\} \cup \{v_{2,n}, v_{m-2,n}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

$$D_s^{LR} = \begin{cases} \left\{v_{m,p} : p \in A_1^{(1,S-1)}\right\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_4^{(1,S-1)}\right\} \cup \{v_{m,3}, v_{m,n-1}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_2^{(0,S-1)}\right\} & \text{if} \quad m \equiv 2 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_0^{(1,S)}\right\} & \text{if} \quad m \equiv 3 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_3^{(1,S-2)}\right\} \cup \{v_{m,2}, v_{m,n-2}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

Case 3: $n \equiv 2 \ (mod\ 5)$

$$D_s^{FC} = \begin{cases} \left\{v_{p,1} : p \in A_2^{(0,T-2)}\right\} \cup \{v_{m-2,1}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_2^{(0,T-1)}\right\} & \text{if} \quad m \equiv 1,2,3 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_2^{(0,T-1)}\right\} \cup \{v_{m-1,1}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

$$D_s^{LC} = \begin{cases} \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}v_{m-1,n}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}v_{m-2,n}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-1)}\right\} \cup \{v_{2,n}\} & \text{if} \quad m \equiv 2,3 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T)}\right\} \cup \{v_{2,n}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

$$D_s^{LR} = \begin{cases} \left\{v_{m,p} : p \in A_2^{(0,S-1)}\right\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_0^{(1,S-1)}\right\} \cup \{v_{m,n-1}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_3^{(1,S-1)}\right\} \cup \{v_{m,2}\} & \text{if} \quad m \equiv 2 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_1^{(1,S)}\right\} & \text{if} \quad m \equiv 3 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_4^{(1,S-2)}\right\} \cup \{v_{m,3}, v_{m,n-2}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

Case 4: $n \equiv 3 \ (mod\ 5)$

$$D_s^{FC} = \begin{cases} \left\{v_{p,1} : p \in A_0^{(1,T-1)}\right\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_0^{(1,T)}\right\} & \text{if} \quad m \equiv 1,4 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_0^{(1,T-1)}\right\} \cup \{v_{m-1,1}\} & \text{if} \quad m \equiv 2 \ (mod\ 5), \\ \left\{v_{p,1} : p \in A_0^{(1,T-1)}\right\} \cup \{v_{m-2,1}\} & \text{if} \quad m \equiv 3 \ (mod\ 5). \end{cases}$$

$$D_s^{LC} = \begin{cases} \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}, v_{m-1,n}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-2)}\right\} \cup \{v_{2,n}, v_{m-2,n}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T-1)}\right\} \cup \{v_{2,n}\} & \text{if} \quad m \equiv 2,3 \ (mod\ 5), \\ \left\{v_{p,n} : p \in A_3^{(1,T)}\right\} \cup \{v_{2,n}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

$$D_s^{LR} = \begin{cases} \left\{v_{m,p} : p \in A_3^{(1,S-1)}\right\} \cup \{v_{m,2}\} & \text{if} \quad m \equiv 0 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_1^{(1,S-1)}\right\} \cup \{v_{m,n-1}\} & \text{if} \quad m \equiv 1 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_4^{(1,S-1)}\right\} \cup \{v_{m,3}\} & \text{if} \quad m \equiv 2 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_2^{(0,S)}\right\} & \text{if} \quad m \equiv 3 \ (mod\ 5), \\ \left\{v_{m,p} : p \in A_0^{(1,S-1)}\right\} \cup \{v_{m,n-2}\} & \text{if} \quad m \equiv 4 \ (mod\ 5). \end{cases}$$

Table 1. Number of black disks in blocks.

	m, n	$n=5k$	$n=5k+1$	$n=5k+2$	$n=5k+3$	$n=5k+4$
First block	$m\geq 1$	$5S-2$	$5S-1$	$5S$	$5S+2$	$5S+3$
Middle blocks	$m\geq 1$	$5S$	$5S+11$	$5S+2$	$5S+3$	$5S+4$
	$m=5l$	$5S-2$	$5S+1$	$5S+1$	$5S+2$	$5S+3$
	$m=5l+1$	S	$S-1$	S	S	S
Last block	$m=5l+2$	$2S-1$	$2S$	$2S+1$	$2S+1$	$2S+1$
	$m=5l+3$	$3S$	$3S+1$	$3S+1$	$3S+1$	$3S+2$
	$m=5l+4$	$4S-1$	$4S$	$4S$	$4S+2$	$4S+2$

Table 2. Number of white squares in boundary.

$n\backslash m$	$5k$	$5k+1$	$5k+2$	$5k+3$	$5k+4$
$n=5l$	$2T+2S$	$2T+2S-1$	$2T+2S$	$2T+2S-1$	$2T+2S+2$
$n=5l+1$	$2T+2S-1$	$2T+2S+1$	$2T+2S-1$	$2T+2S-1$	$2T+2S+1$
$n=5l+2$	$2T+2S+1$	$2T+2S$	$2T+2S$	$2T+2S$	$2T+2S+2$
$n=5l+3$	$2T+2S-1$	$2T+2S$	$2T+2S$	$2T+2S+1$	$2T+2S+1$
$n=5l+4$	$2T+2S+1$	$2T+2S+1$	$2T+2S$	$2T+2S+1$	$2T+2S$

Case 5: $n\equiv 4 \pmod 5$

$$D_s^{FC}=\begin{cases}\left\{v_{p,1}: p\in A_3^{(1,T-2)}\right\}\cup\{v_{2,1},v_{m-1,1}\} & \text{if} \quad m\equiv 0\,(mod\,5),\\ \left\{v_{p,1}: p\in A_3^{(1,T-2)}\right\}\cup\{v_{2,1},v_{m-2,1}\} & \text{if} \quad m\equiv 1\,(mod\,5),\\ \left\{v_{p,1}: p\in A_3^{(1,T-1)}\right\}\cup\{v_{2,1}\} & \text{if} \quad m\equiv 2,3,4\,(mod\,5),\end{cases}$$

$$D_s^{LC}=\begin{cases}\left\{v_{p,n}: p\in A_3^{(1,T-2)}\right\}\cup\{v_{2,n},v_{m-1,n}\} & \text{if} \quad m\equiv 0\,(mod\,5),\\ \left\{v_{p,n}: p\in A_3^{(1,T-2)}\right\}\cup\{v_{2,n},v_{m-2,n}\} & \text{if} \quad m\equiv 1\,(mod\,5),\\ \left\{v_{p,n}: p\in A_3^{(1,T-1)}\right\}\cup\{v_{2,n}\} & \text{if} \quad m\equiv 2,3,4\,(mod\,5).\end{cases}$$

$$D_s^{LR}=\begin{cases}\left\{v_{m,p}: p\in A_4^{(1,S-1)}\right\}\cup\{v_{m,3}\} & \text{if} \quad m\equiv 0\,(mod\,5),\\ \left\{v_{m,p}: p\in A_2^{(0,S-1)}\right\}\cup\{v_{m,n-1}\} & \text{if} \quad m\equiv 1\,(mod\,5),\\ \left\{v_{m,p}: p\in A_0^{(1,S)}\right\}, & \text{if} \quad m\equiv 2\,(mod\,5),\\ \left\{v_{m,p}: p\in A_3^{(1,S)}\right\}\cup\{v_{m,2}\} & \text{if} \quad m\equiv 3\,(mod\,5),\\ \left\{v_{m,p}: p\in A_1^{(1,S-1)}\right\}\cup\{v_{m,n-2}\} & \text{if} \quad m\equiv 4\,(mod\,5).\end{cases}$$

Now, we show that the union of constructed sets builds a dominating set for $G_{m,n}$. We define D_s and D^{out} as follows

$$D_s=D_s^{FR}\cup D_s^{FC}\cup D_s^{LR}\cup D_s^{LC} \tag{1}$$

and

$$D^{out}=D_d\cup D_s. \tag{2}$$

In Theorem 3.5, we will prove that the set D^{out} is an optimal dominating set for $G_{m,n}$.

Example 3.1. The resulting dominating sets for grids $G_{24,20}, G_{24,21}, G_{24,22}, G_{24,23}$ and $G_{24,24}$ are illustrated in Figure 2(a–e).

3.1. Correctness and time complexity

We consider two case when the remainder m by five be zero or not. In the first case, we partition the grid $G_{m,n}$ into $B_1, B_2, \ldots B_T$ such that every block B_i, $1\leq i\leq T$ be a grid $P_5\times P_n$. In the second case, we divide the grid $G_{m,n}$ into $B_1, B_2, \ldots B_{T+1}$ such that every block B_i, $1\leq i\leq T$ be a grid $P_5\times P_n$ and the last block, B_{T+1}, is a grid $P_{m-5T}\times P_n$ that is denoted by B_l. We remember that $T=\lfloor\frac{m}{5}\rfloor, S=\lfloor\frac{n}{5}\rfloor$ and the blocks are distinct. The number of black disks in the blocks is summarized in Table 1.

Also the sum of all white square that locate on boundary of grid are summarized in Table 2.

Lemma 3.2. *The set D^{out} is a dominating set for $G_{m,n}$.*

Proof. Let $v_{p,q}\in V$ and $a_p\equiv_5(n+3(p-1))$, where $2\leq p\leq m-1$, be the first column in row p such that a black disk is appeared. If $q=5k+a_p$, then $v_{p,q}\in D^{out}$.

Let $p\in\{1,m\}$ and $q\in\{1,2,n,n-1\}$. If $a_p\in\{1,2\}$, then $v_{p,q}$ is not added to D^{out} in step 1. Therefore, there are at most four 4–degree vertices $\{v_{2,2},v_{2,n-1},v_{m-1,2},v_{m-1,n-1}\}$ which may be not dominated by black disks. If one of them is not dominated in step 1, then it is dominated in step 2.

We consider other four cases:

Case 1: If $q\equiv a_p+1 \pmod 5$. Since that all $5k'+a_p$ in row p was added to D^{out} then there exist a k'' such that $q-1=5k''+a_p$, therefor $v_{p,q-1}\in D^{out}$. Also $v_{p-1,q}, v_{p+1,q}, v_{p,q+1}$ are not selected in step 1, because $q\neq a_{p-1},\ a_{p+1}, a_p+4 \pmod 5$.

Case 2: If $q\equiv a_p+2 \pmod 5$. In this case, if $p\geq 1$ then $v_{p-1,q}\in D^{out}$ because

$$q\equiv a_p+2\equiv a_{p-1} \pmod 5.$$

Also similar to previous case $v_{p,q-1}, v_{p,q+1}, v_{p+1,q}$ are not selected in step 1, because $q\neq a_p+1, a_p+5,\ a_{p+1} \pmod 5$.

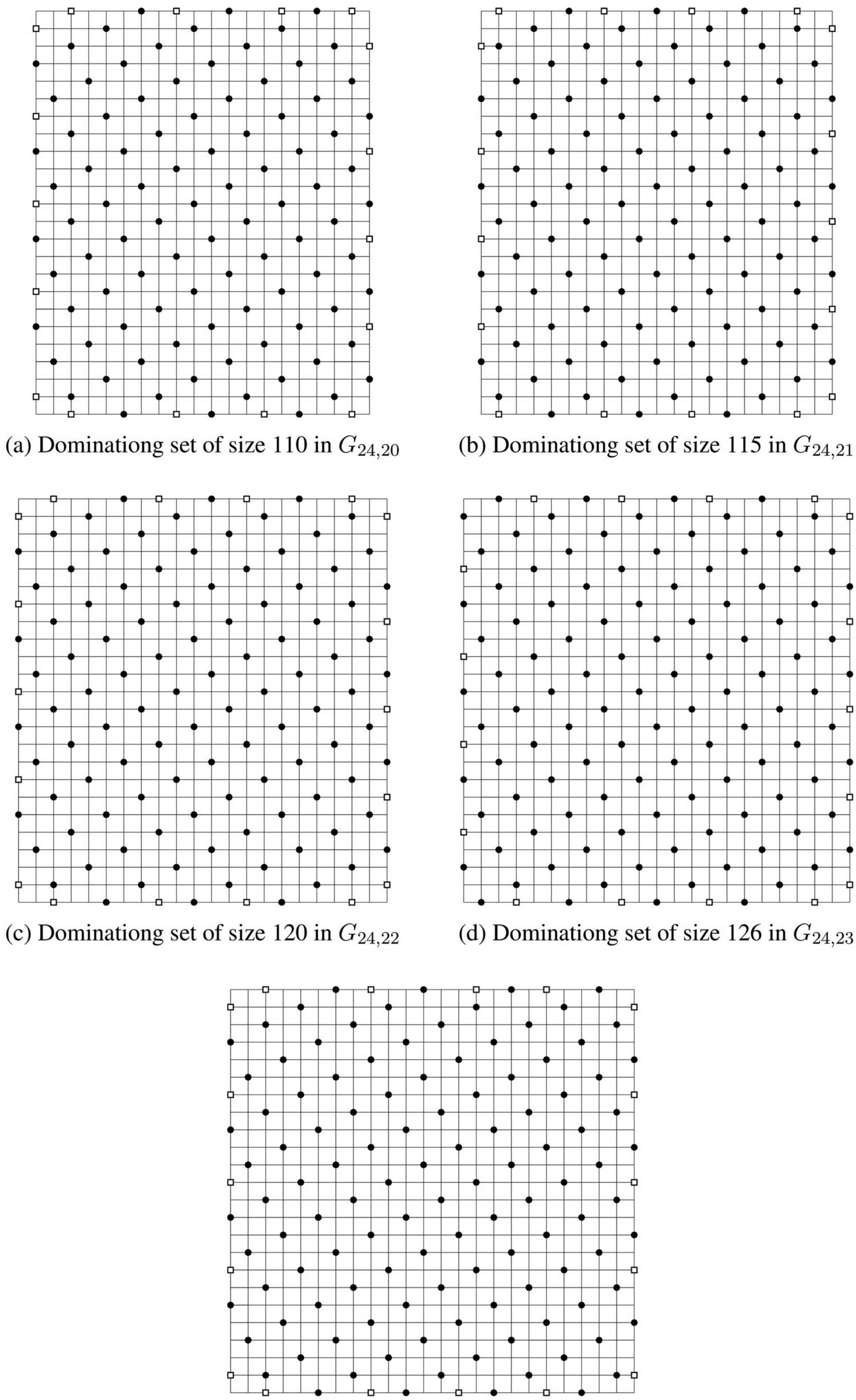
Case 3: If $q\equiv a_p+3 \pmod 5$. In this case, if $p+1\leq m$ then $v_{p+1,q}\in D^{out}$ because

$$q\equiv a_p+3\equiv a_{p+1} \pmod 5,$$

and $v_{p,q-1}, v_{p,q+1}, v_{p-1,q}$ are not selected in step 1, because $q\neq a_p+2, a_p+4, a_{p-1} \pmod 5$.

Case 4: If $q\equiv a_p+4 \pmod 5$. Since that all $5k'+a_p$ in the row p were added to D^{out}, then there exist a k'' such that $q+1\equiv 5k''+a_p \pmod 5$ and $v_{p,q+1}$ was added to D^{out} and $v_{p,q-1}, v_{p-1,q}, v_{p+1,q}$ are not selected in step 1, because $q\neq a_p+3, a_{p-1}, a_{p+1} \pmod 5$.

It is clear that every vertex of degree four is dominated by at most one black disk.



(a) Dominationg set of size 110 in $G_{24,20}$

(b) Dominationg set of size 115 in $G_{24,21}$

(c) Dominationg set of size 120 in $G_{24,22}$

(d) Dominationg set of size 126 in $G_{24,23}$

(e) Dominating set of size 131 for grid $G_{24,24}$

Figure 2. Examples of some grids.

For the boundary vertices, we discuss just the correctness of first row and the argument for the other vertices in boundary is the same manner. In the first row, selection of dominating vertices only depend on n and independent from m. In Table 3, we summarize black disks and white squares that are selected according to different n.

For instance, let $n \equiv 0 \ (mod\ 5)$. Then

$$D^{FR} = D_d^{FR} \cup D_s^{FR} = \left\{ v_{1,p} : p \in A_2^{1,S-1} \cup A_4^{1,S-1} \right\} \cup \{v_{1,3}\}.$$

Therefore, every vertex in the first row except $v_{1,1}$, is either in the set D^{FR} or it is dominated by a vertex in D^{FR}. Since

Table 3. Dominating vertices in first row (in the case 1 and 3, $v_{1,1}$ is not dominated and in the case 4 and 5 both of $v_{1,1}$ and $v_{1,n}$ are not dominated).

Case	n	Black disks	White squares
1	$n = 5l$	$\{v_{1,p}:\ p \in A_2^{(1,S-1)}\}$	$\{v_{1,p}:\ p \in A_4^{(1,S-1)}\} \cup \{v_{1,3}\}$
2	$n = 5l+1$	$\{v_{1,p}:\ p \in A_1^{(1,S-1)}\}$	$\{v_{1,p}:\ p \in A_3^{(1,S-2)}\} \cup \{v_{1,a}, v_{1,n-2}\}$
3	$n = 5l+2$	$\{v_{1,p}:\ p \in A_2^{(1,S-1)}\}$	$\{v_{1,p}:\ p \in A_4^{(1,S-2)}\} \cup \{v_{1,a+1}, v_{1,n-2}\}$
4	$n = 5l+3$	$\{v_{1,p}:\ p \in A_3^{(1,S-1)}\}$	$\{v_{1,p}:\ p \in A_0^{(1,S-1)}\} \cup \{v_{1,n-2}\}$
5	$n = 5l+4$	$\{v_{1,p}:\ p \in A_4^{(1,S-1)}\}$	$\{v_{1,p}:\ p \in A_1^{(1,S-1)}\} \cup \{v_{1,n-2}\}$

Table 4. White squares that dominate $\{v_{2,2}, v_{2,n-1}, v_{n-1,2}, v_{n-1,n-1}\}$.

$n\backslash m$	$5k$	$5k+1$	$5k+2$	$5k+3$	$5k+4$
$5l$	$\{v_{2,1}, v_{1,n-1}, v_{n,2}, v_{n-1,n}\}$	$\{v_{2,1}, v_{1,n-1}, v_{n-1,n}\}$	$\{v_{2,1}, v_{1,n-1}, v_{n,2}, v_{n,n-1}\}$	$\{v_{2,1}, v_{1,n-1}, v_{n-1,1}\}$	$\{v_{2,1}, v_{1,n-1}, v_{n-1,1}, v_{n,n-1}\}$
$5l+1$	$\{v_{1,2}, v_{n-1,n}\}$	$\{v_{1,2}, v_{n,n-1}\}$	$\{v_{1,2}, v_{n,2}\}$	$\{v_{1,2}, v_{n,n-1}\}$	$\{v_{1,2}, v_{n-1,n}\}$
$5l+2$	$\{v_{2,1}, v_{n,2}, v_{n-1,n}\}$	$\{v_{2,1}\}$	$\{v_{2,1}, v_{n,2}\}$	$\{v_{2,1}, v_{n-1,1}, v_{n,n-1}\}$	$\{v_{2,1}, v_{n-1,1}, v_{n-1,n}\}$
$5l+3$	$\{v_{n,2}, v_{n-1,n}\}$	$\{v_{n-1,1}, v_{n,n-1}\}$	$\{v_{n-1,1}\}$	$\{v_{n,2}, v_{n,n-1}\}$	$\{v_{n-1,n}\}$
$5l+4$	$\{v_{1,2}, v_{n-1,1}, v_{n-1,n}\}$	$\{v_{1,2}, v_{n,2}, v_{n,n-1}\}$	$\{v_{1,2}\}$	$\{v_{1,2}, v_{n,2}, v_{n,n-1}\}$	$\{v_{1,2}, v_{n-1,1}, v_{n-1,n}\}$

in all cases, the vertex $v_{2,1}$ is selected as a white square, so vertex $v_{1,1}$ also is dominated.

In other cases, $n \not\equiv 0\ (mod\ 5)$, all vertices in the first row are dominated except maybe $v_{1,1}$ and $v_{1,n}$. In the case 1 and 3, the vertex $v_{2,1}$ in the first column and in the case 4 and 5, the vertices $v_{2,1}$ and $v_{2,n}$ always are selected in step 2. Hence every vertex in the first row is dominated. □

So, every vertex of degree four is dominated by at most one black disk.

Lemma 3.3. *The cardinality of the set D^{out} is $\left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$.*

Proof. It is straightforward to investigate $|D^{out}| = \gamma(G_{m,n})$ as follows.

The black disks of the set D^{out} are divided into three part: black disks in the first block, middle blocks and last block. Hence

$$D_d = \begin{cases} D_d(B_1) + (T-2)D_d(B_i) + D_d(B_l), & \text{if } m = 5k, \\ D_d(B_1) + (T-1)D_d(B_i) + D_d(B_l), & \text{otherwise.} \end{cases}$$

For instance, we show for m, n if they are multiples of five, $D^{out} = D_d \cup D_s$, where D_s is defined in Eq. (1). The other cases are similar.

$$D^{out} = D_d \cup D_s = 5S - 2 + (T-2)(5S) + 5S - 2 + 2T + 2S = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$

□

Lemma 3.4. *The set D^{out} can be computed in time O (size of answer).*

Proof. By lemma 3.3, $|D^{out}|$ is equal to $\gamma(G_{m,n})$. □

According to Lemmas 3.2, 3.3 and 3.4, we have the following theorem.

Theorem 3.5. *The set D^{out} is an optimal dominating set for $G_{m,n}$ and is computed in O (size of answer) time.*

4. [1, 2]-Domination number of grid

In this section, we follow the construction is proposed in the Section 3 to obtain a [1, 2]-dominating set for grid $G_{m,n}$, where $16 \le m \le n$. It is not hard to investigate that $\gamma_{[1,2]}(G_{m,n})$ for $n, m \le 16$ by constructions are presented in [4]. We also show that $\gamma_{[1,2]}(G_{m,n}) = \gamma(G_{m,n})$ where $m, n \ge 16$ which is a positive answer to open question is posed in [6].

Theorem 4.1. *Let $m, n \ge 16$ and $G_{m,n}$ be a $m \times n$ grid, then*

$$\gamma_{[1,2]}(G_{m,n}) = \gamma(G_{m,n}).$$

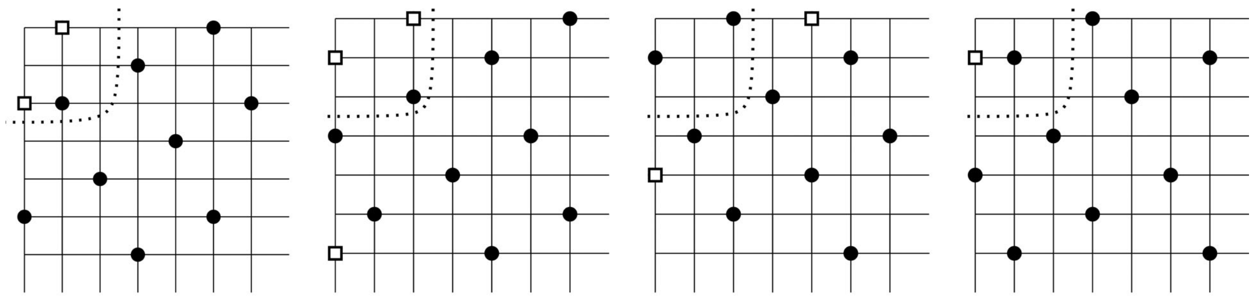
Proof. We show that every vertex $v_{p,q} \in V$ is dominated by at most two vertices in D^{out} and according to the $\gamma_{[1,2]}(G) \ge \gamma(G)$, the result is obtained.

In proof of Lemma 3.3, we show that every vertex of sub-grid is dominated exactly by one black disk. Also, the distance between every two white squares is at least 5. Then every vertex $v_{p,q}$ is dominated at most twice.

In Table 4, the white squares in D^{out} that dominate vertices $v_{2,2}, v_{2,n-1}, v_{m-1,2}$ and $v_{m-1,n-1}$ are shown.

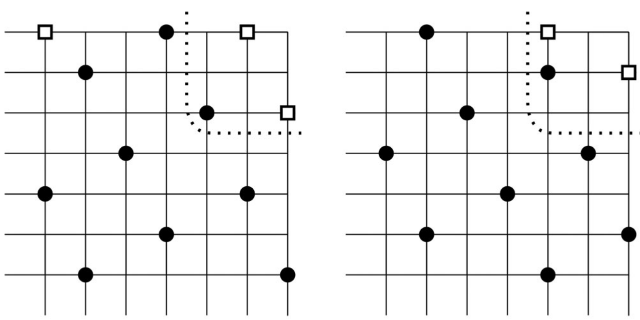
By Table 4, it can be seen that none of white square pairs $\{v_{1,2}, v_{2,1}\}$, $\{v_{2,n}, v_{1,n-1}\}$, $\{v_{n-1,1}, v_{n,2}\}$ and $\{v_{n-1,n}, v_{n,n-1}\}$ appears in any cell. So the vertices $\{v_{2,2}, v_{2,n-1}, v_{n-1,2}, v_{n-1,n-1}\}$ are not dominated more than two times.

These claims are appeared in Figures 3 and 4. In fact, Figure 3(a–d), show that the vertex $v_{2,2}$ is dominated at most twice according to different a_1. Since selecting the domination vertices at the right-up corner depend on a_1 and n. Therefor two cases occur, as can be seen in Figure 4(a,b). These figures show that the vertex $v_{2,n-1}$ is



(a) $a_1 = 1$ (b) $a_1 = 2$ (c) $a_1 = 3$ (d) $a_1 = 4$

Figure 3. Left-up corner of grid.



(a) $n \equiv 0 \ (mod\ 5)$ (b) $n \not\equiv 0 \ (mod\ 5)$

Figure 4. Right-up corner of grid.

dominated at most twice. For the other corners, we have a similar argument.

So, every vertex of $G_{m,n}$ is dominated at least one and at most twice by vertices of D^{out}. □

Disclosure statement

No potential conflict of interest was reported by the authors.

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