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AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: M. Ilayaraja, K. Sowndhariya & A. Muthusamy (2020): Decomposition of product graphs into paths and stars on five vertices, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2019.09.007

To link to this article: https://doi.org/10.1016/j.akcej.2019.09.007

Published online: 04 May 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2019.09.007

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Decomposition of product graphs into paths and stars on five vertices

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ABSTRACT

Let P_k, S_k and K_k respectively denote a path, a star and a complete graph on k vertices. By a $(k;r,s)$-decomposition of a graph G, we mean a decomposition of G into r copies of P_{k+1} and s copies of S_{k+1}. In this paper, it shown that the graph $K_m \times K_n$ admits a $(4;r,s)$-decomposition if and only if $mn(m-1)(n-1) \equiv 0 \pmod 8$, where $K_m \times K_n$ denotes a tensor product of complete graphs. Also we extend the existence of such a decomposition in complete m-partite graphs.

KEYWORDS
Path; star; tensor product; graph decomposition

2010 MSC
05B30; 05C38

1. Introduction

All graphs considered here are finite. By a *decomposition* of G, we mean a list of edge-disjoint subgraphs of G whose union is G. For the graph G, if $E(G)$ can be partitioned into $E_1, ..., E_k$ such that the subgraph induced by E_i is H_i, for all i, $1 \le i \le k$, then we say that $H_1, ..., H_k$ decompose G and we write $G = H_1 \oplus ... \oplus H_k$. For $1 \le i \le k$, if $H_i \cong H$, we say that G has a *H-decomposition*. If G can be decomposed into r copies of H_1 and s copies of H_2, then we say that G has a *$\{rH_1, sH_2\}$-decomposition* or (H_1, H_2)-multidecomposition. If such a decomposition exits for all possible r and s satisfying trivial necessary conditions, then we say that G has a $\{H_1, H_2\}_{\{r,s\}}$-decomposition or *complete $\{H_1, H_2\}$-decomposition*.

Let K_n be a complete graph on n vertices and $K_{k,k}$ be the *complete bipartite* graph with bipartition (X, Y), where $X = \{1_i\}$ and $Y = \{2_i\}, 1 \le i \le k$. For two graphs G and H, their *tensor product* $G \times H$ and *wreath product* $G \otimes H$ have the same vertex set $V(G) \times V(H)$ and their edge sets are defined as follows: $E(G \times H) = \{(g,h)(g',h') \mid gg' \in E(G)$ and $hh' \in E(H)\}, E(G \otimes H) = \{(g,h)(g',h') \mid gg' \in E(G)$ or $g = g', hh' \in E(H)\}$.

The above products are associative and distributive over edge-disjoint union of graphs and the tensor product is commutative. A graph G having partite sets $X_1, ..., X_m$ with $|X_i| = n$, $1 \le i \le m$ and $E(G) = \{xy \mid x \in X_i$ and $y \in X_j, \forall\, i \ne j\}$ is called *complete m-partite* graph and is denoted by $K_m \otimes \overline{K_n}$. Let P_{k+1}, C_k and S_{k+1} respectively denote a path, cycle and star each with having k edges. Also $[x_1 ... x_k x_{k+1}]$ and $(y_1; x_1, ..., x_k)$ respectively denotes a path P_{k+1} and a star $S_{k+1} (\cong K_{1,k})$. If there are t stars with same end vertices $x_1, x_2, ..., x_k$ and different centers $y_1, y_2, ..., y_t$, we denote it by $(y_1, y_2, ..., y_t; x_1, x_2, ..., x_k)$.

The study of (G, H)-multidecomposition has been introduced by Atif Abueida and M. Daven [1]. Moreover, Atif Abueida and Theresa O'Neil [2] have settled the existence of (G, H)-multidecomposition of $K_m(\lambda)$, when $(G, H) = (K_{1,n-1}, C_n)$ for $n = 3, 4, 5$. Priyadharsini and Muthusamy [11] gave necessary and sufficient condition for the existence of (G_n, H_n)-multidecomposition of $\lambda K_{n,n}$, where G_n, $H_n \in \{C_n, P_n, S_n\}$. H.C. Lee [8] established necessary and sufficient condition for the multidecomposition of $K_{m,n}$ into at least one copy of C_k and S_{k+1}. H.C. Lee and J.J. Lin [10] have obtained necessary and sufficient condition for the decomposition of $K_{m,m}$ graph minus a one factor into cycles and stars. Shyu [12, 13] respectively, considered the existence of a decomposition of $K_{m,n}$, K_n into paths and stars, cycles and stars with k edges. Jeevadoss and Muthusamy [5–7] have proved that the necessary and sufficient condition for the existence of a decomposition of $K_{m,n}$, $\lambda K_{m,n}$ into paths and cycles each having k edges, also product graphs into paths and cycles of length four. H.C. Lee [9] established necessary and sufficient conditions for the existence of a decomposition of complete bipartite multigraph into cycles and stars with at least one copy of each. Recently, Shyu [13] has been proved that the necessary and sufficient conditions for the existence of decomposition of $K_{m,n}$ and K_n into paths, satrs and cycles with four edges each. M. Ilayaraja and A. Muthusamy [4] have obtained necessary and sufficient conditions for the existence of decomposition of $K_{m,n}$ into cycles and stars with four edges. By a $(k;r,s)$-decomposition of a graph G, we mean a decomposition of G into r copies of P_{k+1} and s copies of S_{k+1}. In this paper, we prove that the there exists a $(4;r,s)$-decomposition of $K_m \times K_n$ if and only if $mn(m-1)(n-1) \equiv 0 \pmod 8$, where $K_m \times K_n$ denotes a tensor product of complete graphs. Also we extend the existence of such a decomposition in complete regular m-partite graphs.

Remarks.

(1) Let $A + B = \{(x_1 + y_1, x_2 + y_2) \mid (x_1, x_2) \in A,\ (y_1,\ y_2) \in B\}$ and rA is the sum of r copies of A.

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(2) If G_1 and G_2 have a $(4;r,s)$-decomposition, then $G_1 \oplus G_2$ has a such decomposition.

To prove our main results we require the following:

Theorem 1.1. [12] *Let r and s be nonnegative integers, and let n be a positive integer with $n \geq 16$. There exists a $(4;r,s)$-decomposition of K_n if and only if $4(r+s) = e(K_n)$.*

Theorem 1.2. [14] *Let r and s be nonnegative integers, and let n be a positive integer with $n \geq 2$. There exists a $(4;r,s)$-decomposition of $K_{2,2n}$ if and only if $4(r+s) = e(K_{2,2n})$ and s is even.*

Theorem 1.3. [12] *Let r and s be nonnegative integers, and let k and m be positive integers such that $m \geq k$. There exists a $(k;r,s)$-decomposition of $K_{k,m}$ if and only if the following conditions are fulfilled:*

(1) $k(r+s) = e(K_{k,m})$;
(2) $r \leq \lceil \frac{k}{2} \rceil - 1 \Rightarrow (r \equiv 0 (mod\ 2) \wedge m \geq k+r)$;
(3) $(\lceil \frac{k}{2} \rceil \leq r \leq k-1 \wedge k \equiv 1 (mod\ 2) \wedge r \equiv 1 (mod\ 2)) \Rightarrow m \geq k+1$.

Theorem 1.4. [3] *A nontrivial connected graph G has a P_3-decomposition if and only if G has even size.*

Theorem 1.5. [15] *Let k, m and n be positive integers with $m \leq n$. There exists an S_{k+1}-decomposition of $K_{m,n}$ if and only if one of the following holds:*
(i) $k \leq m$ and $mn \equiv 0\ (mod\ k)$; (ii) $m < k \leq n$ and $n \equiv 0\ (mod\ k)$.

2. Constructions

In this section we present some basic constructions which are required to prove our main result.

Lemma 2.1. *There exists a (4; r,s)-decomposition of K_n, when $n = 8, 9$.*

Proof. **Case 1** Let $V(K_9) = \{\{i\} \mid 1 \leq i \leq 8\}$. The required (4; r, s)-decompositions are given below:

(1). $r = 7$ and $s = 0$. The required paths are [71862], [27843], [42165], [32851], [25763], [13546], [14738].
(2). $r = 6$ and $s = 1$. The required paths and stars are [71862], [27843], [56124], [76328], [13546], [14738], (5;1,2,7,8).
(3). $r = 5$ and $s = 2$. The required paths and stars are [71862], [27843], [42165], [32851], [25763], (3;1,5,7,8), (4;1,5,6,7).
(4). $r = 4$ and $s = 3$. The required paths and stars are [71862], [27843], [56124], [76328], (3;1,5,7,8), (4;1,5,6,7).
(5). $r = 3$ and $s = 4$. The required paths and stars are [71862], [56124], [76328], (3;1,4,5,8), (4;1,5,6,8), (5;1,2,7,8), (7;2,3,4,8).
(6). $r = 2$ and $s = 5$. The required paths and stars are [71862], [27843], (2;1,3,4,8), (3;1,5,7,8), (4;1,5,6,7), (5;1,2,7,8).
(7). $r = 1$ and $s = 6$. The required paths and stars are [71862], (2;1,3,4,8), (3;1,4,5,8), (4;1,5,6,8), (5;1,2,7,8), (6;1,3,5,7), (7;2,3,4,8).
(8). $r = 0$ and $s = 7$. The required stars are (1;5,6,7,8), (2;1,3,4,8), (3;1,4,5,8), (4; 1,5,6,8), (5;2,6,7,8), (6;2,3,7,8), (7;2,3,4,8).

Case 2. Let $V(K_9) = \{\{i\} \mid 1 \leq i \leq 9\}$. The required (4; r,s)-decompositions are given below:

(1). $r = 9$ and $s = 0$. The required paths are [51326], [52436], [35846], [37456], [67689], [16879], [21496], [19275], [28395].
(2). $r = 8$ and $s = 1$. The required paths and stars are [51326], [52436], [35846], [37456], [19283], [41275], [67189], [16879],(9;3456).
(3). $r = 7$ and $s = 2$. The required paths and stars are [63715], [57216], [43569], [46259], [39485], [38679], [54789], (1;3489), (2;3489).
(4). $r = 6$ and $s = 3$. The required paths and stars are [51326], [52436], [35846], [37456], [17695], [18275], (1;2469), (8;3679), (9;2347).
(5). $r = 5$ and $s = 4$. The required paths and stars are [51326], [52436], [35967], [37546], [65847], (1;4678), (2;1789), (8;3679), (9;1347).
(6). $r = 4$ and $s = 5$. The required paths and stars are [51326], [52436], [35846], [37456], (1;4689), (2;1789), (7;1569), (8;3679), (9;3456).
(7). $r = 3$ and $s = 6$. The required paths and stars are [41239], [58349], [48765], (1;3579), (2;4679), (5;2349), (6;1349), (7;3459), (8;1269).
(8). $r = 2$ and $s = 7$. The required paths and stars are [12349], [14839], (1;3789), (2;5679), (4;2567), (5;1389), (6;1359), (7;3569), (8;2679).
(9). $r = 1$ and $s = 8$. The required paths and stars are [41237], (3;1456), (4;2567), (5;1279), (6;1259), (7;1269), (8;1234), (8;5679), (9;1279).
(10). $r = 0$ and $s = 9$. The required stars are (1;2345), (2;3456), (3;4567), (4;5678), (5;6789), (6;1789), (7;1289), (8;1239), (9;1234). □

Lemma 2.2. *There exists a $(4;r,s)$-decomposition of $K_{5,5} - I$, where I is a 1-factor of distance zero in $K_{5,5}$, and $r \neq 1$.*

Proof. Let $V(K_{5,5} - I) = \{(\{1_i\}, \{2_i\}) \mid 1 \leq i \leq 5\}$. The required $(4;r,s)$-decompositions are given below:

(1) $r = 5$ and $s = 0$. The required paths are $[2_4 1_5 2_2 1_1 2_5]$, $[2_2 1_4 2_5 1_3 2_4]$, $[2_4 1_1 2_3 1_2 2_5]$, $[2_2 1_3 2_1 1_4 2_3]$, $[2_4 1_2 2_1 1_5 2_3]$.
(2) $r = 4$ and $s = 1$. The required paths and stars are $[2_1 1_2 2_3 1_1 2_4]$, $[2_2 1_1 2_5 1_2 2_4]$, $[2_1 1_3 2_2 1_4 2_3]$, $[2_1 1_4 2_5 1_3 2_4]$, $(1_5; 2_1, 2_2, 2_3, 2_4)$.
(3) $r = 3$ and $s = 2$. The required paths and stars are $[2_1 1_2 2_3 1_1 2_4]$, $[2_2 1_3 2_4 1_2 2_5]$, $[2_1 1_3 2_5 1_1 2_2]$, $(1_4; 2_1, 2_2, 2_3, 2_5)$, $(1_5; 2_1, 2_2, 2_3, 2_4)$.
(4) $r = 2$ and $s = 3$. The required paths and stars are $[2_1 1_2 2_3 1_1 2_4]$, $[2_2 1_1 2_5 1_2 2_4]$, $(1_3; 2_1, 2_2, 2_4, 2_5)$, $(1_4; 2_1, 2_2, 2_3, 2_5)$, $(1_5; 2_1, 2_2, 2_3, 2_4)$.
(5) $r = 1$ and $s = 4$. It is easy to see that $K_{5,5} - I$, can not be decomposed into $1P_5$ and $4S_5$.

(6) $r=0$ and $s=5$. The required stars are $(1_1;2_2, 2_3, 2_4, 2_5)$, $(1_2;2_1, 2_3, 2_4, 2_5)$, $(1_3;2_1, 2_2, 2_4, 2_5)$, $(1_4;2_1, 2_2, 2_3, 2_5)$, $(1_5;2_1, 2_2, 2_3, 2_4)$. □

Lemma 2.3. *There exists a $(4;r,s)$-decomposition of $K_{9,9}-I$, where I is a 1-factor of distance zero in $K_{9,9}$.*

Proof. We can write, $K_{9,9}-I = 2(K_{5,5}-I)\oplus 2K_{4,4}$. By Lemma 2.2 and Theorem 1.3, the graphs $K_{5,5}-I$ and $K_{4,4}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_{9,9}-I$ has the desired decomposition except $\{1P_5, 17S_5\}$.

Finally, $(4;r,s)$-decomposition for the case $\{1P_5, 17S_5\}$ is given as follows: $[1_3 2_5 1_4 2_6 1_5]$, $(1_1;2_2, 2_3, 2_6, 2_8)$, $(1_3;2_2, 2_4, 2_6, 2_7)$, $(1_5;2_3, 2_4, 2_7, 2_8)$, $(1_6;2_2, 2_4, 2_5, 2_9)$, $(1_7;2_4, 2_5, 2_8, 2_9)$, $(1_8;2_2, 2_3, 2_4, 2_5)$, $(1_9;2_3, 2_4, 2_5, 2_8)$, $(2_1;1_2, 1_3, 1_6, 1_8)$, $(2_1;1_4, 1_5, 1_7, 1_9)$, $(2_2;1_4, 1_5, 1_7, 1_9)$, $(2_3;1_2, 1_4, 1_6, 1_7)$, $(2_6;1_2, 1_7, 1_8, 1_9)$, $(2_7;1_4, 1_6, 1_8, 1_9)$, $(2_8;1_2, 1_3, 1_4, 1_6)$, $(2_9;1_3, 1_4, 1_5, 1_8)$, $(1_1, 1_2;2_4, 2_5, 2_7, 2_9)$. □

Lemma 2.4. *There exists a $(4;r,s)$-decomposition of $K_{4x+1,4x+1}-I$, where I is a 1-factor of distance zero in $K_{4x+1,4x+1}, x\geq 1$, and $r\neq 1$ for $x=1$.*

Proof. When $x=1$ and 2, the graphs $K_{5,5}-I$ and $K_{9,9}-I$ have a $(4;r,s)$-decomposition, by Lemmas 2.2 and 2.3. For $x\geq 3$, we can write, $K_{4x+1,4x+1}-I = K_{9,9}-I\oplus(x-2)(K_{5,5}-I)\oplus 2(x-2)K_{4,8}\oplus(x-2)(x-3)K_{4,4}$. By Lemmas 2.2 and 2.3 and Theorem 1.3, the graphs $K_{5,5}-I, K_{9,9}-I$, $K_{4,4}$ and $K_{4,8}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_{4x+1,4x+1}-I$ has the desired decomposition. □

Lemma 2.5. *There exists a $(4;r,s)$-decomposition of $P_3\times K_3$, where $r\neq 1$.*

Proof. Let $V(P_3\times K_3)=\cup_{i=1}^3 X_i$, where $X_i=\{i_j \mid 1\leq j\leq 3\}$. The required $(4;r,s)$-decompositions are given below:

(1) $r=3$ and $s=0$. The required paths are $[1_1 2_2 1_3 2_1 1_2]$, $[1_1 2_3 3_1 2_2 3_3]$, $[1_2 2_3 3_2 2_1 3_3]$.
(2) $r=2$ and $s=1$. The required paths and stars are $[1_1 2_3 1_2 2_1 1_3]$, $[3_1 2_3 3_2 2_1 3_3]$, $(2_2;1_1, 1_3, 3_1, 3_3)$.
(3) $r=1$ and $s=2$. It is easy to see that $P_3\times K_3$, can not be decomposed into $1P_5$ and $2S_5$.
(4) $r=0$ and $s=3$. The required stars are $(2_1;1_2, 1_3, 3_2, 3_3)$, $(2_2;1_1, 1_3, 3_1, 3_3)$, $(2_3;1_1, 1_2, 3_1, 3_3)$. □

Lemma 2.6. *There exists a $(4;r,s)$-decomposition of $P_3\times K_6$.*

Proof. Let $V(P_3\times K_6)=\cup_{i=1}^3 X_i$, where $X_i=\{i_j \mid 1\leq j\leq 6\}$. The required $(4;r,s)$-decompositions are given below:
Now, we decompose the graph $P_3\times K_6$ into $15S_5$'s as follows:

$\{(1_2,3_2;2_3,2_4,2_5,2_6),(1_4,3_4;2_1,2_2,2_3,2_6),(1_5,3_5;2_1,2_2,2_4,2_6),$
$(1_6,3_6;2_1,2_2,2_3,2_4),(2_5,2_6;1_1,1_3,3_1,3_3),(2_2,2_4;1_1,1_3,3_1,3_3),$
$(2_1;1_2,1_3,3_2,3_3),(2_3;1_1,1_5,3_1,3_5),(2_5;1_4,1_6,3_4,3_6)\}$.

First, we decompose the given $2S_5$'s into $\{2P_5\}$ as follows:

(1) $(1_2,3_2;2_3,2_4,2_5,2_6)\Rightarrow\{[2_3,1_2,2_4,3_2,2_5],[2_5,1_2,2_6,3_2,2_3]\}$.
(2) $(1_4,3_4;2_1,2_2,2_3,2_6)\Rightarrow\{[2_1,1_4,2_2,3_4,2_3],[2_3,1_4,2_6,3_4,2_1]\}$.
(3) $(1_5,3_5;2_1,2_2,2_4,2_6)\Rightarrow\{[2_2,1_5,2_1,3_5,2_6],[2_2,3_5,2_4,1_5,2_6]\}$.
(4) $(1_6,3_6;2_1,2_2,2_3,2_4)\Rightarrow\{[2_1,1_6,2_2,3_6,2_3],[2_3,1_6,2_4,3_6,2_1]\}$.
(5) $(2_5,2_6;1_1,1_3,3_1,3_3)\Rightarrow\{[1_3,2_6,1_1,2_5,3_3],[1_3,2_5,3_1,2_6,3_3]\}$.

Now, we consider the $5S_5$'s $\{(2_1;1_2,1_3,3_2,3_3)$, $(2_3;1_1,1_5,3_1,3_5)$, $(2_5;1_4,1_6,3_4,3_6),(2_2,2_4;1_1,1_3,3_1,3_3)\}$ which can be decomposed into either $\{4P_5, 1S_5\}$ or $\{3P_5, 2S_5\}$ or $\{2P_5, 3S_5\}$ as follows: $\{[1_2 2_1 1_3 2_2 3_1], [1_1 2_2 3_3 2_1 3_2], [1_3 2_4 3_1 2_3 1_5], [3_3 2_4 1_1 2_3 3_5], (2_5;1_4,1_6,3_4,3_6)\}$ or $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 1_5], (2_1;1_2,1_3,3_2,3_3), (2_5;1_4,1_6,3_4,3_6)\}$ or $\{[1_1 2_2 3_3 2_1 3_2], [1_2 2_1 1_3 2_2 3_1], (2_3;1_1,1_5,3_1,3_5), (2_4;1_1,1_3,3_1,3_3), (2_5;1_4,1_6,3_4,3_6)\}$.
Further, we consider the $7S_5$'s $\{(2_5,2_6;1_1,1_3,3_1,3_3)$, $(2_2,2_4;1_1,1_3,3_1,3_3)$, $(2_1;1_2,1_3,3_2,3_3)$, $(2_3;1_1,1_5,3_1,3_5)$, $(2_5;1_4,1_6,3_4,3_6)\}$ which can be decomposed into either $\{7P_5\}$ or $\{5P_5, 2S_5\}$ or $\{4P_5, 3S_5\}$ or $\{3P_5, 4S_5\}$ as follows: $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 3_5], [1_2 2_1 1_3 2_5 1_4], [3_2 2_1 3_3 2_5 3_4], [1_3 2_6 1_1 2_5 1_6], [3_3 2_6 3_1 2_5 3_6]\}$ or $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 3_5], [1_2 2_1 1_3 2_5 1_4], [3_2 2_1 3_3 2_5 3_4], (2_5;1_1,1_6,3_1,3_6), (2_6;1_1,1_3,3_1,3_3)\}$ or $\{[1_2 2_1 1_3 2_5 1_4], [3_2 2_1 3_3 2_5 3_4], [1_3 2_6 1_1 2_5 1_6], [3_3 2_6 3_1 2_5 3_6], (2_2;1_1,1_3,3_1,3_3), (2_3;1_1,1_5,3_1,3_5), (2_4;1_1,1_3,3_1,3_3)\}$ or $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 3_5], (2_1;1_2,1_3,3_2,3_3), (2_5;1_3,1_4,3_3,3_4), (2_5;1_1,1_3,3_1,3_3), (2_6;1_1,1_3,3_1,3_3)\}$.

Finally, $(4;r,s)$-decomposition for the case $\{1P_5, 14S_5\}$ is given as follows:

$[1_2 2_5 3_2 2_3 1_4], (2_1;1_2,1_3,3_2,3_3), (2_1;1_4,1_6,3_4,3_6),$
$(2_2;1_1,1_3,1_4,3_4), (2_2;1_6,3_1,3_3,3_6), (2_3;1_1,1_5,3_1,3_5),$
$(2_3;1_2,1_6,3_4,3_6), (2_4;1_1,1_2,1_3,3_2), (2_4;1_6,3_1,3_3,3_6),$
$(2_5;1_4,1_6,3_4,3_6), (2_6;1_2,1_4,3_2,3_4), (2_5,2_6;1_1,1_3,3_1,3_3),$
$(1_5,3_5;2_1,2_2,2_4,2_6)$.

Lemma 2.7. *There exists a $(4;r,s)$-decomposition of $P_3\times K_7$.*

Proof. Let $V(P_3\times K_7)=\cup_{i=1}^3 X_i$, where $X_i=\{i_j \mid 1\leq j\leq 7\}$. The required $(4;r,s)$-decompositions are given below:
Now, we decompose the graph $P_3\times K_7$ into $21S_5$'s as follows:

$\{(2_3,2_4;1_1,1_7,3_1,3_7),(2_4,2_7;1_2,1_3,3_2,3_3),(2_5,2_6;1_1,1_2,3_1,3_2),$
$(2_1;1_3,1_5,3_3,3_5),(2_2;1_3,1_5,3_3,3_5),(2_6;1_3,1_4,3_3,3_4),$
$(2_1;1_4,1_6,3_4,3_6),(2_4;1_5,1_6,3_5,3_6),(2_7;1_4,1_5,3_4,3_5),$
$(2_2;1_6,1_7,3_6,3_7),(2_3;1_4,1_6,3_4,3_6),(2_5;1_4,1_7,3_4,3_7),$
$(2_2;1_1,1_3,3_1,3_3),(2_5;1_3,1_6,3_3,3_6),(2_7;1_1,1_6,3_1,3_6),$
$(2_1;1_2,1_7,3_2,3_7),(2_3;1_2,1_5,3_2,3_5),(2_6;1_5,1_7,3_5,3_7)\}$.

□

First, we decompose the given $2S_5$'s into $\{2P_5\}$ as follows:

(1) $(2_3,2_4;1_1,1_7,3_1,3_7)\Rightarrow\{[1_1 2_3 3_1 2_4 1_7], [1_1 2_4 3_7 2_3 1_7]\}$.
(2) $(2_4,2_7;1_2,1_3,3_2,3_3)\Rightarrow\{[1_2 2_4 3_3 2_7 1_3], [1_2 2_7 3_2 2_4 1_3]\}$.

(3) $(2_5,2_6;1_1,1_2,3_1,3_2) \Rightarrow \{[1_12_53_22_61_2],\ [1_12_63_12_51_2]\}$.
Now the remaining $3S_5$'s can be decomposed into $\{3P_5\}$ as follows:

(4) $\{(2_1;1_3,1_5,3_3,3_5),(2_2;1_3,1_5,3_3,3_5),(2_6;1_3,1_4,3_3,\ 3_4)\} \Rightarrow \{[1_32_13_32_61_4],[1_42_23_52_11_5],\ [1_32_63_42_21_5]\}$.

(5) $\{(2_1;1_4,1_6,3_4,3_6),(2_4;1_5,1_6,3_5,3_6),\ (2_7;1_4,1_5,3_4,3_5)\} \Rightarrow \{[1_42_13_42_71_5],[1_52_43_62_11_6],\ [1_42_73_52_41_6]\}$.

(6) $\{(2_2;1_6,1_7,3_6,3_7),(2_3;1_4,1_6,3_4,3_6),\ (2_5;1_4,1_7,3_4,3_7)\} \Rightarrow \{[1_42_33_62_21_6],[1_42_53_72_21_7],\ [1_62_33_42_51_7]\}$.

(7) $\{(2_2;1_1,1_3,3_1,3_3),(2_5;1_3,1_6,3_3,3_6),\ (2_7;1_1,1_6,3_1,3_6)\} \Rightarrow \{[1_12_23_32_51_6],[1_12_73_62_51_3],\ [1_32_23_12_71_6]\}$.

(8) $\{(2_1;1_2,1_7,3_2,3_7),(2_3;1_2,1_5,3_2,3_5),\ (2_6;1_5,1_7,3_5,3_7)\} \Rightarrow \{[1_22_13_72_61_5],[1_22_33_52_61_7],\ [1_52_33_22_11_7]\}$.

Further, the decomposition of the case $\{20P_5, 1S_5\}$ is given as follows: From (1) and (3), we obtain $\{3P_5, 1S_5\}$ as $\{[1_12_4 1_22_53_1],\ [1_72_31_12_63_2],\ [3_22_53_12_33_7],\ (2_4;1_1,1_7,3_1,3_7)\}$ and together $\{17P_5\}$ given in (2) and (4) to (8) gives the required paths and stars except the case $\{1P_5, 20S_5\}$.
Finally, $(4;r,s)$-decomposition for the case $\{1P_5, 20S_5\}$ is given as follows:

$[1_32_23_72_11_7], (2_1;1_3,1_4,3_3,3_4), (2_2;1_4,1_7,3_3,3_4),$
$(1_1,3_1;2_2,2_3,2_4,2_5), (1_2,3_2;2_1,2_3,2_4,2_5), (1_3,3_3;2_4,2_5,2_6,2_7),$
$(1_4,3_4;2_3,2_5,2_6,2_7), (1_5,3_5;2_3,2_4,2_6,2_7), (1_6,3_6;2_3,2_4,2_5,2_7),$
$(1_7,3_7;2_3,2_4,2_5,2_6), (2_1,2_2;1_5,1_6,3_5,3_6), (2_6,2_7;1_1,1_2,3_1,3_2).$ □

Lemma 2.8. *There exists a $(4;r,s)$-decomposition of $K_{m,6}$ for $m = 2, 4, 6$.*

Proof. The cases $m = 2, 4$ follows by Theorems 1.2 and 1.3. For $m = 6$, let $V(K_{6,6}) = \{(\{1_i\},\{2_i\}) \mid 1 \le i \le 6\}$. We can write, $K_{6,6} = K_{2,6} \oplus K_{4,6}$. By Theorems 1.2 and 1.3, we obtain the required decomposition for the cases $(r,s) \in \{(3,0),\ (1,2)\} + \{(6,0),\ (5,1),\dots,(2,4),\ (0,6)\} = \{(9,0), (8,1),\dots,(3,6),(1,8)\}$. Further, the required decomposition for the case $\{2P_5, 7S_5\}$ is given as follows: $[2_11_12_21_22_3]$, $[2_11_22_51_12_3]$, $(1_3,1_4;2_1,2_2,2_3,2_6)$, $(1_5,1_6;2_1,2_2,2_3,2_4)$, $(2_4;1_1,1_2,1_3,1_4),(2_5;1_3,1_4,1_5,1_6),(2_6;1_1,1_2,1_5,1_6)$. Finally, by Theorem 1.5, we get $(r,s) = (0,9)$. □

Theorem 2.1. *Let r and s be nonnegative integers, and let m be a positive even integer with $m \ge 4$. Then there exists a $(4;r,s)$-decomposition of $K_{m,m}$ if and only if $4(r+s) = e(K_{m,m})$, and $r \ne 1$ for $m = 4$.*

Proof. Necessity. The condition $4(r+s) = e(K_{m,m})$ is trivial by using counting arguments. Sufficiency. The cases $m = 4$, 6 follows by Theorem 1.3 and Lemma 2.8. For $m = 8$, we can write, $K_{8,8} = K_{6,6} \oplus K_{2,6} \oplus 2K_{2,4}$. By Lemma 2.8 and Theorem 1.2, we obtain the required decomposition for the cases $(r,s) \in \{(9,0),(8,1),\dots,(1,8),(0,9)\} + \{(3,0),(1,2)\} + \{(4,0),(2,2),(0,4)\} = \{(16,0),\ (15,1),\dots,(2,14),\ (1,15)\}$. Further, by Theorem 1.5, we get $(r,s) = (0,16)$. For $m \ge 10$, we deal the proof is two cases as follows:

Case 1. $m \equiv 2 \pmod 4 \ge 10$. We can write, $K_{m,m} = K_{6,6} \oplus \left(\frac{m-6}{4}\right)K_{4,4} \oplus \{\oplus_i K_{4,i}\} \oplus \{\oplus_j K_{j,4}\}$, where $6 \le i,\ j \equiv 2 \pmod 4 \le m-4$. Note that $K_{j,4} \cong K_{4,j}$. By Theorem 1.3 and Lemma 2.8, the graphs $K_{4,4}, K_{4,i}, K_{j,4}$ and $K_{6,6}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_{m,m}$ has the desired decomposition.

Case 2. $m \equiv 0 \pmod 4 \ge 12$. We can write, $K_{m,m} = K_{8,8} \oplus \left(\frac{m-12}{4}\right)K_{4,4} \oplus \{\oplus_i K_{i,4}\} \oplus \{\oplus_j K_{4,j}\}$, where $4 \le i \equiv 0 \pmod 4 \le m-4$ and $8 \le j \equiv 0 \pmod 4 \le m-4$. Note that $K_{i,4} \cong K_{4,i}$. By Theorem 1.3 and the first paragraph of the proof, the graphs $K_{4,4}, K_{i,4}, K_{4,j}$ and $K_{8,8}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_{m,m}$ has the desired decomposition. □

3. (4;r,s)-decomposition of $K_m \times K_n$

Lemma 3.1. *There exists a $(4;r,s)$-decomposition of $K_4 \times K_4$.*

Proof. Let $V(K_4 \times K_4) = \cup_{i=1}^4 X_i$ and $X_i = \{i_j \mid 1 \le j \le 4\}$. Then the required $(4;r,s)$-decompositions are as given below:
First, we decompose the given $2S_5$'s into $\{2P_5\}$ as follows:

(1) $\{(1_1;2_2,2_3,2_4,3_2),(1_2;2_1,2_3,2_4,3_3)\} \Rightarrow \{[3_21_12_31_23_3], [2_11_22_41_12_2]\}$.

(2) $\{(1_3;2_1,2_2,2_4,3_4),(1_4;2_1,2_2,2_3,3_1)\} \Rightarrow \{[3_11_42_21_33_4], [2_41_32_11_42_3]\}$.

(3) $\{(2_1;3_2,3_3,3_4,4_4),(2_2;3_1,3_3,3_4,4_1)\} \Rightarrow \{[3_12_23_42_13_2], [4_12_23_32_14_4]\}$.

(4) $\{(3_1;4_2,4_3,4_4,1_3),(3_2;4_1,4_3,4_4,1_3)\} \Rightarrow \{[4_13_21_33_14_3], [4_23_14_43_24_3]\}$.

(5) $\{(3_3;4_1,4_2,4_4,1_1),(3_4;4_1,4_2,4_3,1_1)\} \Rightarrow \{[4_23_31_13_44_3], [4_23_44_13_34_4]\}$.

(6) $\{(4_1;1_2,1_3,1_4,2_3),(4_4;1_1,1_3,2_2,2_3)\} \Rightarrow \{[1_14_41_34_11_2], [1_44_12_34_42_2]\}$.

(7) $\{(4_2;1_1,1_3,2_1,2_4),(4_3;1_1,2_1,2_2,2_4)\} \Rightarrow \{[1_14_22_44_32_2], [1_14_32_14_21_3]\}$.

(8) $\{(1_2;3_1,3_4,4_3,4_4),(2_3;3_1,3_2,3_4,4_2)\} \Rightarrow \{[3_22_33_11_24_3], [4_22_33_41_24_4]\}$.

(9) $\{(1_4;3_2,3_3,4_2,4_3),(2_4;3_1,3_2,3_3,4_1)\} \Rightarrow \{[3_12_43_31_44_3], [4_12_43_21_44_2]\}$.

Now, from (1) and (2), we have $4S_5$'s which can be decomposed into either $\{4P_5\}$ or $\{3P_5, S_5\}$ or $\{2P_5, 2S_5\}$ as follows:

$\{[3_21_12_31_23_3],\ [2_11_22_41_12_2],\ [3_11_42_21_33_4],\ [2_41_32_11_42_3]\}$ or $\{[2_21_32_41_23_3],\ [2_11_22_31_43_1],\ [2_21_42_11_33_4],\ (1_1;2_2,2_3,2_4,3_2)\}$ or $\{[3_21_12_31_23_3],\ [2_11_22_41_12_2],\ (1_3;2_1,2_2,2_4,3_4),(1_4;2_1,2_2,2_3,3_1)\}$.

Finally, $(4;r,s)$-decomposition for the case $\{1P_5, 17S_5\}$ is given as follows:

$[1_13_22_43_32_2]$, $(1_2;2_1,2_3,2_4,3_3)$, $(1_2;3_1,3_4,4_3,4_4)$, $(1_3;2_1,2_2,2_4,3_4)$, $(1_4;2_1,2_2,2_3,3_1)$, $(1_4;3_2,3_3,4_2,4_3)$, $(2_1;3_2,3_3,3_4,4_4)$, $(2_2;1_1,3_1,3_4,4_1)$, $(2_3;1_1,3_1,3_2,3_4)$, $(2_4;1_1,3_1,4_1,4_2)$, $(3_1;1_3,4_2,4_3,4_4)$, $(3_2;1_3,4_1,4_3,4_4)$, $(3_3;1_1,4_1,4_2,4_4)$, $(3_4;1_1,4_1,4_2,4_3)$, $(4_1;1_2,1_3,1_4,2_3)$, $(4_2;1_1,1_3,2_1,2_3)$, $(4_3;1_1,\ 2_1,2_2,2_4)$, $(4_4;1_1,1_3,2_2,2_3)$. □

Lemma 3.2. *There exists a $(4; r, s)$-decomposition of $K_4 \times K_5$.*

Proof. We can write, $K_4 \times K_5 = 6(K_{5,5} - I)$. By Lemma 2.2, the graph $K_{5,5} - I$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_4 \times K_5$ has the desired decomposition except the case $\{1P_5, 29S_5\}$. We can write, $K_4 \times K_5 = K_4 \times K_4 \oplus 12K_{1,4}$, then by Lemma 3.1 and Theorem 1.5, we have $(r, s) \in (1, 17) + (0, 12) = (1, 29)$. □

Lemma 3.3. *There exists a $(4; r, s)$-decomposition of $K_4 \times K_6$.*

Proof. We can write, $K_4 \times K_6 = 3(P_3 \times K_6)$. By Lemma 2.6, the graph $P_3 \times K_6$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_4 \times K_6$ has the desired decomposition. □

Lemma 3.4. *There exists a $(4; r, s)$-decomposition of $K_4 \times K_7$.*

Proof. We can write, $K_4 \times K_7 = 3(P_3 \times K_7)$. By Lemma 2.7, the graph $P_3 \times K_7$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_4 \times K_7$ has the desired decomposition. □

Lemma 3.5. *There exists a $(4; r, s)$-decomposition of $K_5 \times K_3$.*

Proof. We can write, $K_5 \times K_3 = 3(K_{5,5} - I)$. By Lemma 2.2, the graph $K_{5,5} - I$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_5 \times K_3$ has the desired decomposition except the case $\{1P_5, 14S_5\}$.
Further, $(4; r, s)$-decomposition for the case $\{1P_5, 14S_5\}$ is given as follows:

$$[1_3 2_2 1_1 3_2 1_4], (1_1; 2_3, 2_4, 2_5, 3_3), (1_2; 2_3, 2_4, 2_5, 3_4),$$
$$(1_3; 2_1, 2_4, 2_5, 3_2), (1_4; 2_1, 2_2, 2_3, 2_5), (1_5; 2_2, 2_3, 2_4, 3_2),$$
$$(2_1; 1_2, 1_5, 3_2, 3_5), (2_2; 3_1, 3_3, 3_4, 3_5), (2_3; 3_1, 3_2, 3_4, 3_5),$$
$$(2_4; 3_1, 3_2, 3_3, 3_5), (2_5; 3_1, 3_2, 3_3, 3_4), (3_1; 1_2, 1_3, 1_4, 1_5),$$
$$(3_3; 1_2, 1_4, 1_5, 2_1), (3_4; 1_1, 1_3, 1_5, 2_1), (3_5; 1_1, 1_2, 1_3, 1_4).$$

Lemma 3.6. *There exists a $(4; r, s)$-decomposition of $K_5 \times K_5$.*

Proof. We can write, $K_5 \times K_5 = 10(K_{5,5} - I)$. By Lemma 2.2, the graph $K_{5,5} - I$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_5 \times K_5$ has the desired decomposition except the case $\{1P_5, 49S_5\}$. Let $K_5 \times K_5 = K_4 \times K_4 \oplus 3(4)K_{1,4} \oplus 4(5)K_{1,4}$, then by Lemma 3.1 and Theorem 1.5, we have $(r, s) \in (1, 17) + (0, 32) = (1, 49)$. □

Lemma 3.7. *There exists a $(4; r, s)$-decomposition of $K_5 \times K_6$.*

Proof. We can write, $K_5 \times K_6 = 5(P_3 \times K_6)$. By Lemma 2.6, the graph $P_3 \times K_6$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_5 \times K_6$ has the desired decomposition. □

Lemma 3.8. *If $m, n \equiv 0 \pmod 4$, then there exists a $(4; r, s)$-decomposition of $K_m \times K_n$.*

Proof. Let $m = 4x$ and $n = 4y$, where $x, y \geq 1$. Then we can write,

$$\begin{aligned} K_{4x} \times K_{4y} &= xy(K_4 \times K_4) \oplus \frac{xy(y-1)}{2}(K_4 \times K_{4,4}) \\ &\oplus \frac{xy(x-1)}{2}(K_{4,4} \times K_4) \\ &\oplus \frac{xy(x-1)(y-1)}{4}(K_{4,4} \times K_{4,4}) \\ &= xy(K_4 \times K_4) \oplus 2xy(x+y-2)(K_{4,12}) \\ &\oplus 2xy(x-1)(y-1)(K_{4,16}). \end{aligned}$$

By Lemma 3.1 and Theorem 1.3, the graphs $K_4 \times K_4$, $K_{4,12}$ and $K_{4,16}$ have a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_m \times K_n$ has the desired decomposition. □

Lemma 3.9. *If $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, then there exists a $(4; r, s)$-decomposition of $K_m \times K_n$.*
Proof. Let $m = 4x$ and $n = 4y + 2$, where $x, y \geq 1$. Then we can write,

$$\begin{aligned} K_{4x} \times K_{4y+2} &= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_6) \\ &\oplus x(y-1)(K_4 \times K_{4,6}) \\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_4) \\ &\oplus \frac{x(x-1)}{2}(K_{4,4} \times K_6) \\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_{4,6}) \\ &= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_6) \\ &\oplus 4x(y-1)(K_{4,18}) \\ &\oplus 2x(x-1)(y-1)(K_{4,12}) \oplus 3x(x-1)K_{4,20} \\ &\oplus 4x(x-1)(y-1)(K_{4,24}). \end{aligned}$$

By Lemmas 3.1 and 3.3 and Theorem 1.3, the graphs $K_4 \times K_4, K_4 \times K_6, K_{4,12}, K_{4,18}, K_{4,20}$ and $K_{4,24}$ have a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_m \times K_n$ has the desired decomposition. □

Lemma 3.10. *If $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 4$, then there exists a $(4; r, s)$-decomposition of $K_m \times K_n$.*

Proof. We deal the proof in two cases.
Case 1. $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$.
Let $m = 4x$ and $n = 4y + 1$, where $x, y \geq 1$. Then we can write,

$$\begin{aligned}K_{4x}\times K_{4y+1} &= x(y-1)(K_4\times K_4)\oplus x(K_4\times K_5)\\ &\oplus x(y-1)(K_4\times K_{4,5})\\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4}\times K_4)\\ &\oplus \frac{x(x-1)}{2}(K_{4,4}\times K_5)\\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4}\times K_{4,5})\\ &= x(y-1)(K_4\times K_4)\oplus x(K_4\times K_5)\\ &\oplus 4x(y-1)(K_{4,15})\\ &\oplus 2x(x-1)(y-1)(K_{4,18})\oplus \frac{x(x-1)}{2}(5K_{4,16})\\ &\oplus 4x(x-1)(y-1)(K_{4,20}).\end{aligned}$$

By Lemmas 3.1 and 3.2 and Theorem 1.3, the graphs $K_4\times K_4, K_4\times K_5, K_{4,15}, K_{4,16}, K_{4,18}$ and $K_{4,20}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_m\times K_n$ has the desired decomposition.

Case 2. $m\equiv 2 \pmod 4$ and $n\equiv 1 \pmod 4$.
Let $m=4x+2$ and $n=4y+1$, where $x,y\geq 1$. Then we can write,

$$\begin{aligned}K_{4x+2}\times K_{4y+1} &= (x-1)(y-1)(K_4\times K_4)\oplus (x-1)(K_4\times K_5)\\ &\oplus (x-1)(y-1)(K_4\times K_{4,5})\\ &\oplus (y-1)(K_6\times K_4)\oplus (K_6\times K_5)\\ &\oplus (y-1)(K_6\times K_{4,5})\\ &\oplus (x-1)(y-1)(K_{4,6}\times K_4)\\ &\oplus (x-1)(K_{4,6}\times K_5)\\ &\oplus (x-1)(y-1)(K_{4,6}\times K_{4,5})\\ &= (x-1)(y-1)(K_4\times K_4)\oplus (x-1)(K_4\times K_5)\\ &\oplus 4(x-1)(y-1)(K_{4,15})\oplus (y-1)(K_6\times K_4)\\ &\oplus (K_6\times K_5)\oplus 6(y-1)(K_{4,25})\\ &\oplus 4(x-1)(y-1)(K_{4,18})\oplus 5(x-1)(K_{4,24})\\ &\oplus (x-1)(y-1)(5K_{4,24}\oplus 4K_{4,30}).\end{aligned}$$

By Lemmas 3.1 to 3.3 and 3.7 and Theorem 1.3, the graphs $K_4\times K_4, K_4\times K_5, K_4\times K_6, K_5\times K_6, K_{4,15}, K_{4,18}, K_{4,24}, K_{4,25}$ and $K_{4,30}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_m\times K_n$ has the desired decomposition. □

Lemma 3.11. *If $m\equiv 0 \pmod 4$ and $n\equiv 3 \pmod 4$, then there exists a $(4;r,s)$-decomposition of $K_m\times K_n$.*

Proof. Let $m=4x$ and $n=4y+3$, where $x\geq 1$ and $y\geq 0$. When $y=0$, we can write, $K_{4x}\times K_3 = x(4x-1)(P_3\times K_3)$, by Theorem 1.4. Then the graph $P_3\times K_3$ has a $(4;r,s)$-decomposition, by Lemma 2.5. Hence, the graph $K_{4x}\times K_3$ has the desired decomposition. When $y\geq 1$, we can write,

$$\begin{aligned}K_{4x}\times K_{4y+3} &= x(y-1)(K_4\times K_4)\oplus x(K_4\times K_7)\\ &\oplus x(y-1)(K_4\times K_{4,7})\\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4}\times K_4)\\ &\oplus \frac{x(x-1)}{2}(K_{4,4}\times K_7)\\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4}\times K_{4,7})\\ &= x(y-1)(K_4\times K_4)\oplus x(K_4\times K_7)\\ &\oplus 4x(y-1)(K_{4,21})\\ &\oplus 2x(x-1)(y-1)(K_{4,12})\oplus 7\frac{x(x-1)}{2}(K_{4,24})\\ &\oplus 4x(x-1)(y-1)(K_{4,28}).\end{aligned}$$

By Lemmas 3.1 and 3.4 and Theorem 1.3, the graphs $K_4\times K_4, K_4\times K_7$, $K_{4,12}, K_{4,21}, K_{4,24}$ and $K_{4,28}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_m\times K_n$ has the desired decomposition. □

Lemma 3.12. *If $m\equiv 1 \pmod 4$ and $n\equiv 1 \pmod 2$, then there exists a $(4;r,s)$-decomposition of $K_m\times K_n$.*

Proof. We deal the proof in two cases.

Case 1. $m=5$ and $n\equiv 1 \pmod 2$.
Let $m=5$ and $n=2x+1$, where $x\geq 1$. For $x=1$, then the graph $K_5\times K_3$ has a $(4;r,s)$-decomposition, by Lemma 3.5. For $x>1$. Then we can write,

$$\begin{aligned}K_5\times K_{2x+1} &= (x-2)(K_5\times K_2)\oplus K_5\times K_5\\ &\oplus (x-2)(K_5\times K_{2,5})\\ &= (x-2)(K_{5,5}-I)\oplus K_5\times K_5\oplus 5(x-2)(K_{2,20}).\end{aligned}$$

By Lemmas 2.2 and 3.6 and Theorem 1.2, the graphs $K_{5,5}-I, K_5\times K_5$ and $K_{2,20}$ have a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_m\times K_n$ has the desired decomposition.

Case 2. $m\equiv 1 \pmod 4 > 5$ and $n\equiv 1 \pmod 2$.
Let $m=4x+1$ and $n=2y+1$, where $x\geq 2$ and $y\geq 1$. Then $K_{4x+1}\times K_{2y+1} = (2y+1)y(K_{4x+1,4x+1}-I)$, where I is a 1-factor of distance zero in $K_{4x+1,4x+1}$. By Lemma 2.4, the graph $K_{4x+1,4x+1}-I$ has a $(4;r,s)$-decomposition. Hence, by the remark, the graph $K_{4x+1}\times K_{2y+1}$ has the desired decomposition. □

Now, we prove our main result as follows:

Theorem 3.1. *Let r and s be nonnegative integers, and let m and n be positive integers. There exists a $(4;r,s)$-decomposition of $K_m\times K_n$ if and only if $mn(m-1)(n-1)\equiv 0 \pmod 8$, where $K_m\times K_n$ denotes a tensor product of complete graphs.*

Proof. Necessity is trivial by counting the number of edges of the graph $K_m\times K_n$. Sufficiency follows from Lemmas 3.8 to 3.12. □

4. $(4; r, s)$-decomposition of $K_m \otimes \overline{K_n}$

In this section we obtain necessary and sufficient conditions for the existence of a $(4; r, s)$-decomposition of $K_m \otimes \overline{K_n}$.

Lemma 4.1. *If $m \equiv 0$ (or) $1 \pmod 2$ and $n \equiv 0 \pmod 2 \geq 4$, then there exists a $(4; r, s)$-decomposition of $K_m \otimes \overline{K_n}$, and $r \neq 1$ when $m = 2$ and $n = 4$.*

Proof. We can write $K_m \otimes \overline{K_n} = \frac{m(m-1)}{2} K_{n,n}$. By Theorem 2.1, the graph $K_{n,n}$ has a $(4; r, s)$-decomposition except $r = 1$ when $n = 4$. Hence, by the remark, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition except when $(n, r) = (4, 1)$. Further, $(4; 1, s)$-decomposition of $K_m \otimes \overline{K_4}$ is given as follows: For $m = 2$, then $K_2 \otimes \overline{K_4} = K_{4,4}$ can not be decomposed into $\{1P_5, 3S_5\}$, by Theorem 2.1.

Case 1. For $m = 3$, then $K_3 \otimes \overline{K_4}$ can be decomposed into $\{1P_5, 11S_5\}$ is given as follows: $[1_2 2_1 1_1 2_3 1_3]$, $(1_2; 2_2, 2_3, 2_4, 3_1)$, $(1_3; 2_1, 2_2, 2_4, 3_1)$, $(1_4; 2_1, 2_2, 2_3, 2_4)$, $(3_1; 1_1, 1_4, 2_2, 2_4)$, $(2_1, 2_3; 3_1, 3_2, 3_3, 3_4)$, $(2_2, 2_4; 1_1, 3_2, 3_3, 3_4)$, $(3_2, 3_3, 3_4;\ 1_1, 1_2, 1_3, 1_4)$.

Case 2. For $m = 4$, then we can write $K_4 \otimes \overline{K_4} = K_3 \otimes \overline{K_4} \oplus K_{4,12}$. By case 1 and Theorem 1.5, we obtain the required decomposition for the case $(r, s) \in (1, 11) + (0, 12) = (1, 23)$.

Case 3. For $m > 4$, then we can write $K_m \otimes \overline{K_4} = K_4 \otimes \overline{K_4} \oplus \frac{(m-4)(m-5)}{2} K_{4,4} \oplus (m-4) K_{4,16}$. By Case 2, and Theorem 1.5, we obtain the required decomposition for the case $(r, s) \in (1, 23) + (0, 2(m-4)(m-5)) + (0, 16(m-4)) = (1, 23 + 2(m-4)\{8 + (m-5)\})$. □

Lemma 4.2. *If $m \equiv 1 \pmod 8$ and $n \equiv 1 \pmod 2$, then there exists a $(4; r, s)$-decomposition of $K_m \otimes \overline{K_n}$.*

Proof. We can write, $K_m \otimes \overline{K_n} = nK_m \oplus (K_m \times K_n)$. By Theorem 1.1, the graph K_m (If $m = 9$, the graph K_9 has a $(4; r, s)$-decomposition, by Lemma 2.1) has a $(4; r, s)$-decomposition and by Lemma 3.12, the graph $K_m \times K_n$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition. □

Lemma 4.3. *If $m \equiv 0 \pmod 8$ and $n \equiv 1 \pmod 2$, then there exists a $(4; r, s)$-decomposition of $K_m \otimes \overline{K_n}$.*

Proof. We can write, $K_m \otimes \overline{K_n} = nK_m \oplus (K_m \times K_n)$. By Theorem 1.1, the graph K_m (If $m = 8$, the graph K_8 has a $(4; r, s)$-decomposition, by Lemma 2.1) has a $(4; r, s)$-decomposition and by Lemmas 3.10 and 3.11, the graph $K_m \times K_n$ has a $(4; r, s)$-decomposition. Hence, by the remark, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition. □

Theorem 4.1. *Let r and s be nonnegative integers, and let m and n be positive integers. Then there exists a $(4; r, s)$-decomposition of $K_m \otimes \overline{K_n}$ if and only if $mn^2(m-1) \equiv 0 \pmod 8$.*

Proof. Necessity is trivial by counting the number of edges of the graph $K_m \otimes \overline{K_n}$. Sufficiency follows from Lemmas 4.1 to 4.3. □

Acknowledgments

The authors are thankful for the referee's helpful comments, and they also thank the University Grant Commission, Government of India, New Delhi for its support through the Grant No.F.510/7/DRS-I/2016(SAP-I). The second author thanks the Department of Science and Technology, Government of India, New Delhi for its financial support through the Grant No. DST/INSPIRE Fellowship/2015/IF150211.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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