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AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: S. K. Vaidya & Kalpesh M. Popat (2020): Construction of L-equienergetic graphs using some graph operations, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2019.06.012

To link to this article: https://doi.org/10.1016/j.akcej.2019.06.012

Published online: 22 Apr 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2019.06.012

Construction of L-equienergetic graphs using some graph operations

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ABSTRACT
For a graph G with n vertices and m edges, the eigenvalues of its adjacency matrix $A(G)$ are known as eigenvalues of G. The sum of absolute values of eigenvalues of G is called the energy of G. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix with $(i,j)^{th}$ entry is the degree of vertex v_i. The collection of eigenvalues of $L(G)$ with their multiplicities is called spectra of $L(G)$. If $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of $L(G)$ then the Laplacian energy $LE(G)$ of G is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. It is always interesting and challenging as well to investigate the graphs which are L-equienergetic but L-noncopectral as L-cospectral graphs are obviously L-equienergetic. We have devised a method to construct L-equienergetic graphs which are L-noncospectral.

KEYWORDS
Eigenvalue; graph energy; spectrum; equienergetic

2010 MSC
05C50; 05C76

1. Introduction

All the graphs considered here are simple, finite, connected and with n vertices and m edges denoted as $G(n, m)$. We denote the complement of graph G by $\overline{G}$, the complete graph on p vertices by K_p, the null graph by $\overline{K_p}$. The average vertex degree denoted by $\overline{d}$, defined as $\overline{d} = \frac{2m}{n}$. For any undefined term in graph theory we rely upon Balakrishnan and Ranganathan [2] while for terminology related to matrix theory we refer to Horn and Johnson [11].

The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, \cdots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that,

$$\begin{aligned} a_{ij} &= 1, \text{ if } v_i \text{ is adjacent with } v_j \\ &= 0, \text{ otherwise} \end{aligned}$$

The spectra of adjacency matrix of graph G is called spectra of G. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ be eigenvalues of graph G then the *energy of graph* G is $E(G) = \sum_{i=1}^{n} |\lambda_i|$, The concept of graph energy was introduced by Gutman [9] in 1978. A brief account of graph energy can be found in Cvetković [6] and Li [12].

Let $D(G)$ be the diagonal matrix of whose $(i,i)^{th}$ entry is the degree of a vertex v_i. The matrix $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are called the *Laplacian* and *Signless Laplacian* matrices of G and their spectra are called *Laplacian spectra* (L- spectra) and *signless Laplacian spectra* (Q-spectra) of G. Let $0 = \mu_n \leq \mu_n - 1 \leq \cdots \mu_1$ be L-spectra of G. Fiedler [7] have prove that $\mu_n = 0$ with multiplicity equal to the number of connected components of G. It is easy to see that

$$tr(L(G)) = \sum_{i=1}^{n} \mu_i = 2m \quad tr(L^+(G)) = \sum_{i=1}^{n} \mu_i^+ = 2m$$

with tr is the trace of the matrix.

All Laplacian eigenvalues are nonnegative, and therefore their sum is non-zero. On the other hand,

$$\sum_{i=1}^{n} \left(\mu_i - \frac{2m}{n}\right) = 0$$

Gutman and Zhou [10] have pointed out that the equality $LE(G) = E(G)$ holds, if G is regular.

The multiplicity of μ_i is denoted by $m(\mu_i)$. The collection of all eigenvalues μ_i together their multiplicity is known as Laplacian spectra of G denoted by $spec_L(G)$. Hence,

$$spec_L(G) = \begin{pmatrix} \mu_i & \mu_2 & \cdots & \mu_n \\ m(\mu_1) & m(\mu_2) & \cdots & m(\mu_n) \end{pmatrix}$$

The *Laplacian energy* of a graph G is defined by

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

Basic properties and other results on Laplacian energy can be found in Andriantiana [1]. Two graphs G_1 and G_2 of same order are said to be *L-equienergetic* if $LE(G_1) = LE(G_2)$. Two graphs are said to be *L-cospectral* if they have same Laplacian eignevalues. Since L-cospectral graphs are always L-equienergetic, the problem of constructing L-equienergetic graphs is challenging for $L-$noncospectral graphs.

The *join* of G_1 and G_2 is the graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2. The L-spectra of join of graphs is given by the following result.

Proposition 1.1. [5] *If* $G_1(n_1, m_1)$ *and* $G_2(n_2, m_2)$ *are two graphs having L-spectra* $\mu_1, \mu_2, \cdots, \mu_{n_1-1}, \mu_{n_1} = 0$ *and* $\sigma_1, \sigma_2, \cdots, \sigma_{n_2-1}, \sigma_{n_2} = 0$ *respectively then,*

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$$spec_L(G_1 \vee G_2) = \begin{pmatrix} n_1+n_2 & n_1+\sigma_1 & \cdots & n_1+\sigma_{n_2-1} & n_2+\mu_1 & \cdots & n_2+\mu_{n_1-1} & 0 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

The *Kronecker product* of G_1 and G_2 is the graph $G = G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if u_1 is adjacent to u_2 and v_1 is adjacent to v_2 in G_1 and G_2 respectively. The following result gives the L-spectra of the Kronecker product of graphs of $G \otimes K_2$.

Proposition 1.2. [3] *Let G(n, m) be a graph having L-spectra and Q-spectra respectively as $\mu_1, \mu_2, \cdots, \mu_n$ and $\mu_1^+, \mu_2^+, \cdots, \mu_n^+$ then,*

$$spec_L(G \otimes K_2) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n & \mu_1^+ & \mu_2^+ & \cdots & \mu_n^+ \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

The *m-shadow graph* $D_m(G)$ of a connected graph G is graph obtained by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$. For $m = 2$, the graph is known as shadow (double) graph.

Proposition 1.3. [13] *Let G be a graph with n vertices having degrees $d_1, d_2, \cdots, d_n$ and let $\mu_1, \mu_2, \cdots, \mu_n$ be its Laplacian spectra. Then the Laplacian spectra of $D_m(G)$ is $m\mu_i, md_i$ for $1 \leq i \leq n$.*

Proposition 1.4. [8] *Let $D_2(G)$ be the shadow graph of the graph G(n, m). Then, for $p \geq 2n + k$ and $m \leq \frac{k^2+2nk}{8}$, $k \geq 4$ we have*

$$LE(D_2(G) \vee \overline{K_p}) = 4n + (p - 2n)\frac{2m'}{n'} + 8m$$

with $\frac{2m'}{n'} = \frac{4m+2np}{n+2p}$

The extended double cover [4] of the graph $G(n, m)$ with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ is a bipartite graph G^* with bipartition $(X, Y), X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ where two vertices x_i and y_j are adjacent if and only if $i = j$ or v_i adjacent v_j in G. It is easy to see that G^* is connected if and only if G is connected and a vertex v_i is of degree d_i in G if and only if it is of degree $d_i + 1$ in G^*. Following are some results associated with G^*

Proposition 1.5. [8] *Let G(n, m) be a graph with L-spectra and Q-spectra as $\mu_1, \mu_2, \cdots, \mu_n$ and $\mu_1^+, \mu_2^+, \cdots, \mu_n^+$ respectively, then*

$$spec_L(G^*) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n & \mu_1^+ + 2 & \mu_2^+ + 2 & \cdots & \mu_n^+ + 2 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Proposition 1.6. [8] *Let G(n, m) be the graph then for $p \geq 2n + k$ and $m \leq \frac{(k-1)n}{2} + \frac{k^2}{4}$, $k \geq 3$, we have*

$$LE(G^* \vee \overline{K_p}) = 6n + (p - 2n)\frac{2m'}{n'} + 4m$$

with $\frac{2m'}{n'} = \frac{4m+2np+2n}{p+2n}$

Proposition 1.7. [11] *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric block matrix. Then the spectra of A is the union of $A_0 + A_1$ and $A_0 - A_1$.

2. Laplacian energy of extended shadow graph

Definition 2.1. The extended shadow graph $D_2^*(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' and with u'' in G''.

Theorem 2.2. *Let G be a graph with Laplacian eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ and degrees $d_1, d_2, \cdots, d_n$ then the Laplacian spectra of $D_2^*(G)$ is*

$$spec_L(D_2^*(G)) = \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1+1) & 2(d_2+1) & \cdots & 2(d_n+1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the graph G and $A(G), D(G)$ be the adjacency matrix and degree matrix of the graph G respectively.
Then,

$$L(G) = D(G) - A(G)$$

Consider second a copy graph G with vertices $u_1, u_2, u_3, \cdots, u_n$ to obtain $D_2^*(G)$, such that, $N(u_i) = N(v_i) \cup \{u_i\}, i = 1, 2, \cdots n$. Let $G_1 = D_2^*(G)$.
The adjacency matrix and degree matrix of G_1 are respectively given as

$$A(G_1) = \begin{bmatrix} A(G) & A(G)+I \\ A(G)+I & A(G) \end{bmatrix}$$

$$D(G_1) = \begin{bmatrix} 2D(G)+I & 0 \\ 0 & 2D(G)+I \end{bmatrix}$$

Then,

$$\begin{aligned} L(G_1) &= D(G_1) - A(G_1) \\ &= \begin{bmatrix} 2D(G) - A(G) + I & -A(G) - I \\ -A(G) - I & 2D(G) - A(G) + I \end{bmatrix} \\ &= \begin{bmatrix} L(G) + D(G) + I & L(G) - D(G) - I \\ L(G) - D(G) - I & L(G) + D(G) + I \end{bmatrix} \end{aligned}$$

Hence, by Proposition 1.7, spectra of $L(G_1)$ is union of spectra of $2L(G)$ and $2(D(G) + I)$.
Hence,

$$spec_L(D_2^*(G)) = \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1+1) & 2(d_2+1) & \cdots & 2(d_n+1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Lemma 2.3. *Let $G(n, m)$ be a graph then for $p \geq (2n + k)$ and $m \leq \frac{k^2+2nk}{4}$, we have*

$$LE((G \otimes K_2) \vee \overline{K_p}) = 4n + (p - 2n)\frac{2m'}{n'} + 4m$$

Proof. Let $G(n, m)$ be an n-vertex graph having L-spectra and Q-spectra, as $\mu_1, \mu_2, \cdots, \mu_n$ and $\mu_1^+, \mu_2^+, \cdots, \mu_n^+$ respectively, then by Proposition 1.2

$$spec_L(G \otimes K_2) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n & \mu_1^+ & \mu_2^+ & \cdots & \mu_n^+ \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and so by Proposition 1.1

$$spec_L((G \otimes K_2) \vee \overline{K_p}) = \begin{pmatrix} p+2n & p+\mu_1 & \cdots & p+\mu_{n-1} & p+\mu_1^+ & \cdots & p+\mu_n^+ & 2n & 0 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & p-1 & 1 \end{pmatrix}$$

Average vertex degree of $(G \otimes K_2) \vee \overline{K_p}$ is

$$\frac{2m'}{n'} = \frac{4m + 4np}{p + 2n}$$

Therefore,

$$\begin{aligned} LE((G \otimes K_2) \vee \overline{K_p}) = {} & \left|p + 2n - \frac{2m'}{n'}\right| + \sum_{i=1}^{n-1}\left|p + \mu_i - \frac{2m'}{n'}\right| \\ & + \sum_{i=1}^{n}\left|p + \mu_i^+ - \frac{2m'}{n'}\right| \\ & + (p-1)\left|2n - \frac{2m'}{n'}\right| + \left|\frac{2m'}{n'}\right| \end{aligned}$$

Now if, $p \geq 2n + k$ and $m \leq \frac{k^2+2nk}{4}$, we have for $i = 1, 2, \cdots, n$

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &\geq \frac{k(2n + k) - k(2n + k)}{p + 2n} = 0 \end{aligned}$$

Similarly,

$$p + \mu_i^+ - \frac{2m'}{n'} > 0$$

Therefore,

$$\begin{aligned} LE((G \otimes K_2) \vee \overline{K_p}) &= \left|p + 2n - \frac{2m'}{n'}\right| + \sum_{i=1}^{n-1}\left|p + \mu_i - \frac{2m'}{n'}\right| \\ &\quad + \sum_{i=1}^{n}\left|p + \mu_i^+ - \frac{2m'}{n'}\right| \\ &\quad + (p - 1)\left|2n - \frac{2m'}{n'}\right| + \left|\frac{2m'}{n'}\right| \\ &= \left(p + 2n - \frac{2m'}{n'}\right) + \left(\sum_{i=1}^{n-1}\mu_i + 0\right) \\ &\quad + \sum_{i=1}^{n}\mu_i^+ + (n - 1)\left(p - \frac{2m'}{n'}\right) \\ &\quad + n\left(p - \frac{2m'}{n'}\right) + (p - 1)\left(\frac{2m'}{n'} - 2n\right) \\ &\quad + \frac{2m'}{n'} \\ &= 4n + (p - 2n)\frac{2m'}{n'} + 4m \end{aligned}$$

Remark 2.4. We have considered the only case when $p \geq 2n + k$ and $m \leq \frac{k^2+2nk}{4}$. We discard the remaining possibilities for p and m due to following reasons.
Case(I) If, $p < 2n + k$ and $m \leq \frac{k^2+2nk}{4}$,

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &\geq \frac{p(p - 2n) - 4m}{p + 2n} \\ &\geq \frac{p(p - 2n) - k^2 - 2nk}{p + 2n} \\ &= \frac{(p - k)(p + k) - 2n(p + k)}{p + 2n} \\ &= \frac{(p + k)(p - k - 2n)}{p + 2n} \\ &\geq \frac{(p - k - 2n)}{p + 2n} \end{aligned}$$

As, $p < 2n + k$ and $p + 2n > 0$, we have

$$\frac{(p - k - 2n)}{p + 2n} < 0$$

Hence, in this case we are not able to determine the sign of $p + \mu_i - \frac{2m'}{n'}$. In this situation the term on L.H.S. might be either positive or negative.
Case(II) If, $p \geq 2n + k$ and $m > \frac{k^2+2nk}{4}$,

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &\geq \frac{p(p - 2n) - 4m}{p + 2n} \\ &\geq \frac{k(2n + k) - 4m}{p + 2n} \\ &= \frac{k^2 + 2nk - 4m}{p + 2n} \end{aligned}$$

Here, $m > \frac{k^2+2nk}{4}$ and $p+2n > 0$, we have

$$\frac{k^2+2nk-4m}{p+2n} < 0$$

Again, in this case we are not able to determine the sign of $p+\mu_i-\frac{2m'}{n'}$.

Case(III) If, $p < 2n+k$ and $m > \frac{k^2+2nk}{4}$,

$$\begin{aligned} p+\mu_i-\frac{2m'}{n'} &= p+\mu_i-\frac{4m+4np}{p+2n} \\ &= \frac{p(p-2n)+(p+2n)\mu_i-4m}{p+2n} \\ &< \frac{k(2n+k)+(p+2n)\mu_i-4m}{p+2n} \\ &< \frac{k(2n+k)+(p+2n)\mu_i-k^2-2nk}{p+2n} \\ &= \frac{(p+2n)\mu_i}{p+2n} \\ &= \mu_i \end{aligned}$$

In this case also we are not able to decide the sign of $p+\mu_i-\frac{2m'}{n'}$ as $\mu_i \geq 0$.

Thus, in all the cases discussed above, it is not possible to determine the sign of the term $p+\mu_i-\frac{2m'}{n'}$ definitely.

3. Construction of L-equienergetic graphs

Theorem 3.1. *Let $G_1(n,m)$ and $G_2(n,m)$ be two graphs having L-spectra as $\mu_1,\mu_2,\cdots,\mu_n$ and $\gamma_1,\gamma_2,\cdots,\gamma_n$ respectively, then for $p \geq 2n+k$ and $m \leq \frac{k^2+2n(k-1)}{8}$, $k \geq 4$ we have*

$$LE(D_2^*(G_1)\vee\overline{K_p}) = LE(D_2^*(G_2)\vee\overline{K_p})$$

Proof. Let $D_2^*(G_1)$ be the extended shadow graph of G_1. Then by Theorem 2.2,

$$spec_L(D_2^*(G_1)) = \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1+1) & 2(d_2+1) & \cdots & 2(d_n+1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and so by Proposition 1.1,

$$spec_L(D_2^*(G_1)\vee\overline{K_p}) = \begin{pmatrix} p+2n & p+2\mu_1 & \cdots & p+2\mu_{n-1} & p+2(d_1+1) & \cdots & p+2(d_n+1) & 2n & 0 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & p-1 & 1 \end{pmatrix}$$

Average vertex degree of $D_2^*(G_1)\vee\overline{K_p}$ is

$$\frac{2m'_1}{n'} = \frac{8m+2n+4np}{p+2n}$$

Therefore,

$$\begin{aligned} LE(D_2^*(G_1)\vee\overline{K_p}) = & \left|p+2n-\frac{2m'}{n'}\right| \\ & + \sum_{i=1}^{n-1}\left|p+2\mu_i-\frac{2m'}{n'}\right| \\ & + \sum_{i=1}^{n}\left|p+2(d_i+1)-\frac{2m'}{n'}\right| \\ & + (p-1)\left|2n-\frac{2m'}{n'}\right| + \left|\frac{2m'}{n'}\right| \end{aligned}$$

Now if, $p \geq 2n+k$ and $m \leq \frac{k^2+2n(k-1)}{8}$, $k \geq 4$, we have for $i = 1,2,\cdots,n$

$$\begin{aligned} p+2\mu_i-\frac{2m'}{n'} &= p+2\mu_i-\frac{8m+2n+4np}{p+2n} \\ &= \frac{p(p-2n)+2\mu_i(p+2n)-8m-2n}{p+2n} \\ &\geq \frac{k(2n+k)-k(2n+k)+2n-2n}{p+2n} = 0 \end{aligned}$$

Similarly we see that,

$$p+2(d_i+1)-\frac{2m'}{n'} \geq 2 > 0$$

Therefore,

$$\begin{aligned} LE(D_2^*(G)\vee\overline{K_p}) = & \left|p+2n-\frac{2m'}{n'}\right| + \sum_{i=1}^{n-1}\left|p+2\mu_i-\frac{2m'}{n'}\right| \\ & + \sum_{i=1}^{n}\left|p+2(d_i+1)-\frac{2m'}{n'}\right| \\ & + (p-1)\left|2n-\frac{2m'}{n'}\right| + \left|\frac{2m'}{n'}\right| \\ = & \left(p+2n-\frac{2m'}{n'}\right) + 2\left(\sum_{i=1}^{n-1}\mu_i+0\right) \\ & + (n-1)\left(p-\frac{2m'}{n'}\right) \\ & + 2\sum_{i=1}^{n}d_i + n\left(p+2-\frac{2m'}{n'}\right) \\ & + (p-1)\left(\frac{2m'}{n'}-2n\right) + \frac{2m'}{n'} \\ = & 6n+(p-2n)\frac{2m'}{n'}+8m \end{aligned}$$

Remark 3.2. We can prove the remaining cases by similar arguments as discussed in Remark 2.4.

Corollary 3.3. *Let $G_1(n,m_1)$, $G_2(n,m_2)$, $G_3(n,m_3)$ and $G_4(n,m_4)$ be four graphs of order $n \equiv 0 \pmod 4$ with $m_2 =$*

$m_1 + \frac{n}{4}$, $m_3 = 2m_1$ *and* $m_4 = 2m_1 + \frac{n}{2}$. *Then for* $p \geq 2n + k$ *and* $m_1 \leq \frac{k^2+2n(k-1)}{8}$ *we have*

$$LE(D_2^*(G_1) \vee \overline{K_p}) = LE(D_2(G_2) \vee \overline{K_p}) = LE(G_3^* \vee \overline{K_p}) = LE((G \otimes K_2) \vee \overline{K_p})$$

Proof. Let $D_2^*(G_1)$, $D_2(G_2)$ and G_3^* be the extended shadow graph of $G_1(n, m_1)$, shadow graph of $G_2(n, m_2)$ and extended double cover of $G_3(n, m_3)$, respectively. Average degrees of $D_2^*(G_1)$, $D_2(G_2)$, G_3^* and $(G \otimes K_2) \vee \overline{K_p}$ are respectively as,

$$\frac{2m_1'}{n'} = \frac{8m_1 + 4np + 2n}{p + 2n}, \quad \frac{2m_2'}{n'} = \frac{8m_2 + 4np}{p + 2n},$$
$$\frac{2m_3'}{n'} = \frac{4m_3 + 4np + 2n}{p + 2n}, \quad \frac{2m_4'}{n'} = \frac{4m_4 + 4np}{p + 2n}$$

Now, for $p \geq 2n + k$ and $m_1 \leq \frac{k^2+2n(k-1)}{8}$, we have by Theorem 3.1

$$LE(D_2^*(G) \vee \overline{K_p}) = 6n + (p - 2n)\frac{2m_1'}{n'} + 8m_1 \qquad (1)$$

For $p \geq 2n + k$ and $m_2 \leq \frac{k^2+2nk}{8}$ we have by Proposition 1.4

$$LE(D_2(G) \vee \overline{K_p}) = 4n + (p - 2n)\frac{2m_2'}{n'} + 8m_2$$

If $m_2 = m_1 + \frac{n}{4}$ then

$$LE(D_2(G) \vee \overline{K_p}) = 6n + (p - 2n)\frac{2m_1'}{n'} + 8m_1 \qquad (2)$$

For $p \geq 2n + k$ and $m_3 \leq \frac{n(k-1)}{2} + \frac{k^2}{8}$, we have by Proposition 1.6

$$LE(G_3^* \vee \overline{K_p}) = 6n + (p - 2n)\frac{2m_3'}{n'} + 4m_3$$

and if we suppose that $m_3 = 2m_1$ then

$$LE(G_3^* \vee \overline{K_p}) = 6n + (p - 2n)\frac{2m_1'}{n'} + 8m_1 \qquad (3)$$

Also for, For $p \geq 2n + k$ and $m_3 \leq \frac{n(k-1)}{2} + \frac{k^2}{8}$, we have by Lemma 2.3

$$LE((G \otimes K_2) \vee \overline{K_p}) = 4n + (p - 2n)\frac{2m_4'}{n'} + 4m_4$$

and if we suppose that $m_4 = 2m_1 + \frac{n}{2}$ then

$$LE((G \otimes K_2) \vee \overline{K_p}) = 6n + (p - 2n)\frac{2m_1'}{n'} + 8m_1 \qquad (4)$$

Therefore, from (1)–(4) it is clear that

$$LE(D_2^*(G_1) \vee \overline{K_p}) = LE(D_2(G_2) \vee \overline{K_p}) = LE(G_3^* \vee \overline{K_p}) = LE((G \otimes K_2) \vee \overline{K_p})$$

Theorem 3.4. *Let* $G_1(n, m_1)$ *and* $G_2(n, m_2)$ *be two graphs having* $L-$*spectra respectively as* $\mu_1, \mu_2, \cdots, \mu_n$ *and* $\gamma_1, \gamma_2, \cdots, \gamma_n$. *Then with* $n \equiv 0 (\text{mod } 8)$ *and* $m_2 = m_1 + \frac{n}{4}$, *for* $p \geq 4n + k$ *and* $m_2 \leq \frac{k^2+4nk-4n}{32}$ *we have*

$$LE(D_2(D_2^*(G_1)) \vee \overline{K_p}) = LE(D_2^*(D_2(G_2)) \vee \overline{K_p})$$

Proof. Let $D_2^*(G)$ and $D_2(G)$ be the shadow and extended shadow graphs of G, respectively. $D_2(D_2^*(G_1)) \vee \overline{K_p}$ and $D_2^*(D_2(G_2)) \vee \overline{K_p}$ are graphs with $p + 4n$ vertices and average degrees respectively as

$$\frac{2m_1'}{n'} = \frac{32m + 8n + 8np}{p + 4n}, \quad \frac{2m_2'}{n'} = \frac{32m + 4n + 8np}{p + 4n}.$$

By Theorem 2.2,

$$spec_L(D_2^*(G_1)) = \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1+1) & 2(d_2+1) & \cdots & 2(d_n+1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

by Lemma 1.3,

$$spec_L(D_2(D_2^*(G_1))) = \begin{pmatrix} 4\mu_1 & \cdots & 4\mu_n & 4(d_1+1) & \cdots & 4(d_n+1) & 2(2d_1+1) & \cdots & 2(2d_n+1) \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 \end{pmatrix}$$

and so by Proposition 1.1, L-spectra of $D_2(D_2^*(G)) \vee \overline{K_p}$ is $p+4n$, $p+4\mu_i$ $(1 \leq i \leq n-1)$, $p+4(d_i+1)$, $p+2(2d_i+1)$ (2 times) $(1 \leq i \leq n)$, $4n$ ($(p-1)$ times), 0
So if $p \geq 4n + k$ and $m_1 \leq \frac{k(4n+k)-8n}{32}$, $k \leq 4$, we have for $i = 1, 2, \cdots, n$

$$p + 4\mu_i - \frac{2m_1'}{n'} = p + 4\mu_i - \frac{32m + 8n + 8np}{p + 4n} = \frac{p(p-4n) + 4(p+4n)\mu_i - 32m - 8n}{p + 4n} \geq \frac{k(4n+k) - k(4n+k) + 8n - 8n}{p + 4n} = 0$$

Similarly we can show

$$p + 4(d_i + 1) - \frac{2m_1'}{n'}, \quad \geq 0, \quad p + 2(2d_i + 1) \quad \geq 0$$

Therefore,

$$LE(D_2(D_2^*(G)) \vee \overline{K_p}) = \left|p + 4n - \frac{2m_1'}{n'}\right| + \sum_{i=1}^{n-1}\left|p + 4\mu_i - \frac{2m_1'}{n'}\right| + \sum_{i=1}^{n}\left|p + 4(d_i + 1) - \frac{2m_1'}{n'}\right| + 2\sum_{i=1}^{n}\left|p + 2(2d_i + 1) - \frac{2m_1'}{n'}\right| + (p-1)\left|4n - \frac{2m_1'}{n'}\right| + \left|\frac{2m_1'}{n'}\right| = 8n + (p - 4n)\frac{2m_1'}{n'} + 16m_1$$

Similarly,
L-spectra of $D_2^*(D_2(G)) \vee \overline{K_p}$ is $p + 4n$, $p + 4\gamma_i$ $(1 \leq i \leq n-1)$, $p + 4d_i'$, $p + 2(2d_i' + 1)$ (2 times) $(1 \leq i \leq n)$, $4n$ $((p-1)$ times), 0 and

$$LE(D_2(D_2^*(G)) \vee \overline{K_p}) = 4n + (p - 4n)\frac{2m_2'}{n'} + 16m_2$$

Using $m_2 = m_1 + \frac{n}{4}$

$$LE(D_2^*(D_2(G_1)) \vee \overline{K_p}) = LE(D_2(D_2^*(G_2)) \vee \overline{K_p})$$

4. Concluding remarks

In most of the existing results only a pair of graphs are shown to be L-equienergetic while we have investigated four graphs which are simultaneously L-equienergetic. Moreover, we have used the concept of extended shadow graph to construct L-equienergetic graphs from the given graphs.

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