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# Folding trees gracefully 

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#### Abstract

When a graceful labeling of a bipartite graph assigns the smaller labels to the vertices of one of the stable sets of the graph, the assignment is called an $\alpha$-labeling. Any graph that admits such a labeling is an $\alpha$-graph. In this work we extend the concept of vertex amalgamation to generate a new class of $\alpha$-graphs obtained by a sequence of $k$-vertex amalgamations of $t$ copies of an $\alpha$-tree. This procedure is also applied to any collection of $\alpha$-trees such that any pair of trees in this collection have stable sets with the same cardinalities. We also use this idea on other types of $\alpha$-graphs. In addition, we present a family of $\alpha$-trees of even diameter formed with four caterpillars of the same size.


## KEYWORDS

$\alpha$-labeling; graceful labeling;
k-vertex amalgamation

## 2010 MSC

05C78

## 1. Introduction

A difference vertex labeling of a graph $G$ of size $n$ is an injective mapping $f$ from $V(G)$ into a set $N$ of nonnegative integers, such that every edge $u v$ of $G$ has assigned a weight defined by $|f(u)-f(v)|$. The labeling $f$ is called graceful when $N=\{0,1, \ldots, n\}$ and the set of induced weights is $\{1,2, \ldots, n\}$. In this case, $G$ is called a graceful graph. Let $G$ be a bipartite graph and $\{A, B\}$ be the natural bipartition of $V(G)$, we refer to $A$ and $B$ as the stable sets of $G$ and assume that $|A|=a$ and $|B|=b$. A bipartite labeling of $G$ is an injection $f: V(G) \rightarrow\{0,1, \ldots, s\}$ for which there is an integer $\lambda$, named the boundary value of $f$, such that $f(u) \leq \lambda<$ $f(v)$ for every $(u, v) \in A \times B$, that induces $n$ different weights. This is an extension of the definition given by Rosa and Širán [7]. From the definition we may conclude that $s \geq|E(G)|$; furthermore, the labels assigned by $f$ on the vertices of $A$ and $B$ are in the integer interval $[0, \lambda]$ and $[\lambda+$ $1, s]$, respectively. If $s=n$, the function $f$ is an $\alpha$-labeling and $G$ is an $\alpha$-graph. If $f$ is an $\alpha$-labeling of a tree and $f^{-1}(0) \in$ $A$, then its boundary value is $\lambda=a-1$.

Suppose that $f: V(G) \rightarrow\{0,1, \ldots, n\}$ is a graceful labeling of a graph $G$ of size $n$ :

- $g: V(G) \rightarrow\{c, c+1, \ldots, c+n\}$, defined for every $v \in$ $V(G)$ and $c \in \mathbb{N}$ as $g(v)=c+f(v)$, is the shifting of $f$ in $c$ units. Note that this labeling preserves the weights induced by $f$.
- $\hat{f}: V(G) \rightarrow\{0,1, \ldots, n\}$, defined for every $v \in V(G)$ as $\hat{f}(v)=\lambda-f(v)$ if $f(v) \leq \lambda$, and $\hat{f}(v)=n+\lambda+1-f(v)$ if $f(v)>\lambda$, is the reverse labeling of $f$. Thus, $\hat{f}$ is also an $\alpha$-labeling with boundary value $\lambda$.
- $g: V(G) \rightarrow\{0,1, \ldots, n+d-1\}$, defined for every $v \in$ $V(G)$ and $d \in \mathbb{N}$ as $g(v)=f(v)$ if $f(v) \leq \lambda$ and $g(v)=$ $f(v)+d-1$ if $f(v)>\lambda$, is the $d$-graceful labeling of $G$ obtained from $f$. The labels assigned by $g$ on the stable sets of $V(G)$ are in the intervals $[0, \lambda]$ and $[\lambda+d, n+d-$ 1] and the set of induced weights is $\{d, d+$ $1, \ldots, n+d-1\}$.

For example, let $f$ be an $\alpha$-labeling of a tree $T$ of size $n$ with boundary value $\lambda$. Suppose that $f$ is transformed into a $d$-graceful labeling shifted $c$ units. Then the elements of $A$ are labeled with the integers in $[c, \lambda+c]$, the elements of $B$ are labeled with the integers in $[c+\lambda+d, c+n++d-1]$, and the induced weights form the interval $[d, n+d-1]$.

In Section 2, we study $\alpha$-labelings for graphs that result of $k$-vertex amalgamations of smaller $\alpha$-graphs. Section 3 is devoted to $\alpha$-labelings of trees, there we prove the existence of an $\alpha$-labeling for trees obatined by amalgamating caterpillars. The reader interested in graph labelings is refered to Gallian' survey [4] for more information about the subject. In this paper, we follow the notation and terminology used in [3] and [4].

## 2. Folding $\alpha$-trees

For $i=1,2$, let $G_{i}$ be a graph of order $n_{i}$ and size $m_{i}$. A graph $G$, of order $n_{1}+n_{2}-k$ and size $m_{1}+m_{2}$, is said to be a $k$-vertex amalgamation (or strong vertex amalgamation) of $G_{1}$ and $G_{2}$ if it is obtained identifying $k$ independent vertices of $G_{1}$ with $k$ independent vertices of $G_{2}$. We use here strong vertex amalgamation of $\alpha$-graphs to construct new classes of $\alpha$-graphs.


Figure 1. $\alpha$-labeling of the 3-fold of $Q_{3}$.

Suppose That $G$ is a bipartite graph of order $n$ and size $m$, with stable sets $A=\left\{u_{1}, u_{2}, \ldots, u_{a}\right\}$ and $B=$ $\left\{v_{1}, v_{2}, \ldots, v_{b}\right\}$. Let $G_{1}, G_{2}, \ldots, G_{t}$ be disjoint copies of $G$ with $A_{i}=\left\{u_{1}^{i}, u_{2}^{i}, \ldots, u_{a}^{i}\right\}$ and $B_{i}=\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{b}^{i}\right\}$. If for every even value of $i \in\{1,2, \ldots, t\}$, the vertices of $B_{i}$ are identified with the vertices of $B_{i-1}$, in such a way that $v_{j}^{i}=v_{j}^{i-1}$ for every $1 \leq j \leq b$, and the vertices of $A_{i}$ are identified with the vertices of $A_{i+1}$, in such a way that $u_{j}^{i}=u_{j}^{i+1}$ for every $1 \leq j \leq a$, then we obtain a bipartite graph of size $t m$ and order $\frac{t n}{2}+a$ when $t$ is even or $\frac{(t+1) n}{2}$ when $t$ is odd. We call this graph the $t$-fold of $G$. We claim that the $t$-fold of $G$ is an $\alpha$-graph when $G$ is an $\alpha$-graph.

Notice that when $t$ is odd, there is only one $t$-fold of $G$; that is, the $t$-fold of $G$ is independent of the stable set of $G$ chosen to be $B$. On the other hand, when $t$ is even, there are, in general, two non-isomorphic $t$-folds of $G$, depending on the stable set chosen to be $B$. When $G$ is vertex transitive, this selection is irrelevant and there is only one $t$-fold of $G$ for any value of $t$.
Theorem 1. If $G$ is an $\alpha$-graph, then any $t$-fold of $G$ is an $\alpha$-graph.
Proof. Suppose that $G$ is an $\alpha$-graph of order $n$ and size $m$ with stable sets $A=\left\{u_{1}, u_{2}, \ldots, u_{a}\right\}$ and $B=\left\{v_{1}, v_{2}, \ldots, v_{b}\right\}$. Let $f$ be an $\alpha$-labeling of $G$ with boundary value $\lambda$ such that $f^{-1}(0) \in A$. Consider $t$ copies of $G$, each labeled using $f$. For every value of $i \in\{1,2, \ldots, t\}$, let $f_{i}$ be the $d_{i}$-graceful labeling of $G_{i}$, obtained from $f$, shifted $c_{i}$ units, where $d_{i}=$ $1+m(t-i)$ and $c_{i}=m\left\lfloor\frac{i}{2}\right\rfloor$.

Thus, the set of weights induced by $f_{i}$ on the edges of $G_{i}$ is $\left[d_{i}, d_{i}+m-1\right]$. This implies that the set of weights on the edges of the $t$-fold of $G$ is


Figure 2. $\alpha$-labeling of the 7-fold of $C_{8}$.

$$
\begin{aligned}
\bigcup_{i=1}^{t}\left[d_{i}, d_{i}+m-1\right] & =\bigcup_{i=1}^{t}[1+m(t-i), 1+m(t-i)+m-1] \\
& =\bigcup_{i=1}^{t}[1+m(t-i), m(t+1-i)] \\
& =[1, m t]
\end{aligned}
$$

Suppose that $r_{1}<r_{2}<\ldots<r_{a}$ and $\rho_{1}<\rho_{2}<\ldots<\rho_{b}$ are the labels assigned by $f$ on the vertices of $A$ and $B$, respectively. Thus, $f_{i}$ assigns the labels $r_{1}+c_{i}, r_{2}+c_{i}, \ldots, r_{a}+c_{i}$ to the elements of $A_{i}$ and the labels $\rho_{1}+d_{i}-1+c_{i}, \rho_{2}+d_{i}-1+$ $c_{i}, \ldots, \rho_{b}+d_{i}-1+c_{i}$ to the elements of $B_{i}$. Note that when $i$ is even, $i=2 s$, the labels on the vertices of $B_{i}$ are $\rho_{j}+m(t-$ $s), 1 \leq j \leq b$, and the labels on the vertices of $B_{i-1}$ are $\rho_{j}+$ $m(t-s)$, as well. Similarly, the labels on the vertices of $A_{i}$ and $A_{i+1}$ are $r_{1}+m s, r_{2}+m s, \ldots, r_{a}+m s$. Therefore, the vertices of the $t$-fold of $G$ are labeled with integers from $[0, m t]$ that induce the weights $1,2, \ldots, m t$. Since the labelings of the $G_{i}$ are bipartite, the final labeling is an $\alpha$-labeling with boundary value $m+\left\lfloor\frac{t}{2}\right\rfloor+\lambda$. Hence, the $t$-fold of $G$ is an $\alpha$-graph.

In Figure 1, we show an $\alpha$-labeling of the 3 -fold of the hypercube $Q_{3}$.

When the graph $G$ in Theorem 1 is the path $P_{n}$, with $n \geq 4$, the $t$-fold of $G$ results in a graph that contains as a subgraph a convex Eulerian polyomino, the induced labeling of this polyomino can be transformed into the $\alpha$-labeling used by Acharya [1] to prove that all convex Eulerian polyominoes are arbitrarily graceful. Another nice family of $\alpha$-graphs that can be obtained using Theorem 1 is the one containing the fractal type structure constructed by $t$-folding the cycle $C_{4 m}, m \geq 2$. In Figure 2 we show an example of one of these structures constructed using $C_{8}$ folded 7 times.

Note that the $\alpha$-labeling of the $t$-fold of an $\alpha$-graph $G$ can be also obtained using the labeling scheme of the weak tensor product of $G$ and $P_{t+1}$, introduced by Snevily [8] and extended by López and Muntaner-Batle [5]. The labeling technique used in the proof of Theorem 1 is applied to


Figure 3. A 3 -vertex amalgamation of an $\alpha$-graph of size 7.
other types of folded graphs that cannot be explained using the weak tensor product.

In the next proposition we prove that for any $1 \leq k \leq b$, there is an $\alpha$-graph $G$ that results of a $k$-vertex amalgamation of two copies of an $\alpha$-tree. Note that when $k=1, G$ is the well-known vertex amalgamation (or one-point union) of two copies of $T$. The gracefulness of this type of graph was proven in [2]. When $k=b$, an $\alpha$-labeling of $G$ can be obtained using Theorem 1 with $t=2$.

Proposition 1. Let $T$ be an $\alpha$-tree of size $m$ with one stable set of cardinality $b$. If $k \leq b$ is a positive integer, then there is a $k$ vertex amalgamation of two copies of $T$ that is an $\alpha$-graph.

Proof. For $i=1,2$, let $T_{i}$ be a copy of $T$ with stable sets $A_{i}$ and $B_{i}$, where $\left|B_{i}\right|=b$. Suppose that $f$ is an $\alpha$-labeling of $T$ such that $f^{-1}(0) \in A$; so, its boundary value is $\lambda=$ $|A|-1=a-1$. Let $f_{i}$ be the $d_{i}$-graceful labeling of $T_{i}$, obtained from $f$, shifted $c_{i}$ units, where

$$
\left(d_{i}, c_{i}\right)= \begin{cases}(m+1,0) & \text { if } i=1 \\ (1, \lambda+k) & \text { if } i=2\end{cases}
$$

Thus, the vertices in $A_{1}$ are labeled with the integers in $\{0,1, \ldots, \lambda\}$, the vertices in $B_{1}$ are labeled with the integers in $\{\lambda+m+1, \lambda+m+2, \ldots, 2 m\}$, the vertices in $A_{2}$ are labeled with the integers in $\{\lambda+k, \lambda+k+1, \ldots, 2 \lambda+k\}$, and the vertices in $B_{2}$ are labeled with the integers in $\{2 \lambda+k+1,2 \lambda+k+2, \ldots, m+\lambda+k\}$.

Identifying the vertices of $B_{1}$ and $B_{2}$ labeled $2 \lambda+k+$ $1,2 \lambda+k+2, \ldots, m+\lambda+k$, we obtain a graph $G$ that is a $k$ vertex amalgamation of $T_{1}$ and $T_{2}$. Since the weights on $T_{1}$ are $m+1, m+2, \ldots, 2 m$ and on $T_{2}$ are $1,2, \ldots, m$, we have that all edges of $G$ have different weights; in addition, the vertices of $T_{1}$ and $T_{2}$, with the same label, were amalgamated, hence there is no repetition of labels. Observe that the labels assigned to the elements of $A_{1} \cup A_{2}$ are smaller that those assigned to the elements of $B_{1} \cup B_{2}$, so the number $2 \lambda+k$ is the boundary value of the $\alpha$-labeling of $G$, therefore $G$ is an $\alpha$-graph (Figure 3).

Suppose that $T$ is an $\alpha$-labeled tree of size $m$, where $A=$ $\{0,1, \ldots, \lambda\}$ and $B=\{\lambda+1, \lambda+2, \ldots, m\}$ are considered ordered sets. By a generalized $t$-fold of $T$ we mean a graph $G$ obtained using $t \alpha$-labeled copies of $T$, where for every even value of $i$, the copy $T_{i}$ is merged with the copies $T_{i-1}$ and $T_{i+1}$ in such a way that the last $k_{i}$ vertices in $B_{i}$ are
amalgamated with the first $k_{i}$ vertices in $B_{i-1}$, and the last $k_{i}^{\prime}$ vertices in $A_{i}$ are amalgamated with the first $k_{i}^{\prime}$ vertices in $A_{i+1}$. These amalgamations must be done in ascending order; for example, suppose that $x$ is amalgamated with $y$ and $x^{\prime}$ is amalgamated with $y^{\prime}$, if $x<x^{\prime}$, then $y<y^{\prime}$.

Theorem 2. If $T$ is an $\alpha$-tree, then any generalized $t$-fold of $T$ is an $\alpha$-graph.

Proof. Suppose that $T$ is an $\alpha$-tree of size $m$ with stable sets $A$ and $B$. Let $T_{1}, T_{2}, \ldots, T_{t}$ be disjoint copies of $T$. Assume that $f$ is an $\alpha$-labeling of $T$ with boundary value $\lambda$. Without loss of generality, suppose that $f^{-1}(0) \in A$. Thus, the labels assigned to the vertices of $A$ and $B$ form the intervals $L_{A}=$ $[0, \lambda]$ and $L_{B}=[\lambda+1, m]$, respectively. For each $i \in$ $\{1,2, \ldots, t\}$, suppose that $f$ is the initial labeling of $T_{i}$. The final labeling of $T_{i}$, denoted by $f_{i}$, is obtained by transforming $f$ into a $d_{i}$-graceful labeling shifted $c_{i}$ units, where $d_{i}=$ $(t-i) m+1$ and $c_{i}=\sum_{j=1}^{i} \xi_{j}$ with

$$
\xi_{i} \in \begin{cases}{[\lambda+1, m]} & \text { if } n \text { is even } \\ {[0, \lambda]} & \text { if } n \text { is odd }\end{cases}
$$

Consequently, the labels assigned by $f_{i}$ to the vertices of $A_{i}$ and $B_{i}$ are $L_{A_{i}}=\left[c_{i}, c_{i}+\lambda\right]$ and $L_{B_{i}}=\left[\lambda+c_{i}+d_{i}, m+\right.$ $\left.c_{i}+d_{i}-1\right]$. In addition, the weights induced by $f_{i}$ on the edges of $T_{i}$ form the interval $[(t-i) m+1,(t-i) m+m]$. Therefore, $\cup_{i=1}^{t}[(t-i) m+1,(t-i) m+m]=[1, t m]$.

Since $L_{A_{i}}=\left[c_{i}, c_{i}+\lambda\right]$ where $c_{i}=\sum_{j=1}^{i} \xi_{j}$, we have that for every feasible even value of $i, L_{A_{i}} \cap L_{A_{i+1}}=\left[c_{i+1}, c_{i}+\lambda\right]$ and $\quad\left|L_{A_{i}} \cap L_{A_{i+1}}\right|=c_{i}+\lambda-c_{i+1}+1=\lambda+1-\xi_{i+1}=k_{i}^{\prime}$. Thus, the number $k_{i}^{\prime}$, of vertices shared by $A_{i}$ and $A_{i+1}$ is bounded by $1 \leq k_{i}^{\prime} \leq \lambda+1=a$. Similarly, $L_{B_{i-1}} \cap L_{B_{i}}=$ $\left[\lambda+c_{i-1}+d_{i-1}, m+c_{i}+d_{i}-1\right]$ and $\left|L_{B_{i-1}} \cap L_{B_{i}}\right|=\xi_{i}-\lambda=$ $k_{i}$. Since $\xi_{i} \in[\lambda+1, m]=[\lambda+1, a+b-1]$, the number $k_{i}$ of vertices shared by $B_{i-1}$ and $B_{i}$ is bounded by $1 \leq k_{i} \leq b$.

Identifying the vertices with the same label, we form the graph $G$ that is a generalized $t$-fold of $T$. Since the labelings used on the $T_{i}$ are bipartite, the boundary value of the labeling of $G$ is the largest number in $L_{A_{i}}$, that is, $\lambda+c_{t}$. Therefore, $G$ is an $\alpha$-graph.

In Figure 4, we show an example of an $\alpha$-labeling for a generalized 5 -fold of an $\alpha$-tree of size 10 , where $k_{2}=4, k_{2}^{\prime}=$ $2, k_{4}=5, k_{4}^{\prime}=5, \xi_{1}=0, \xi_{2}=8, \xi_{3}=3, x_{4}=9$, and $\xi_{5}=0$.

Let $T_{1}$ and $T_{2}$ be two $\alpha$-trees of size $m$. We say that $T_{1}$ and $T_{2}$ are analogous if $\left|A_{1}\right|=\left|A_{2}\right|$ and $\left|B_{1}\right|=\left|B_{2}\right|$. We use this concept to extend the result of Theorem 2 by replacing any number of copies of $T$ with analogous trees.

Theorem 3. If $G$ is a generalized $t$-fold of an $\alpha$-tree $T$, then any of the copies of $T$, used to construct $G$, can be replaced by any tree $T^{\prime}$ analogous to $T$, and the resulting graph is an $\alpha$-graph.

Proof. Since $T$ and $T^{\prime}$ are analogous, there exist $\alpha$-labelings $f$ and $g$, of $T$ and $T^{\prime}$, respectively, such that $f^{-1}(0) \in A$ and $g^{-1}(0) \in A^{\prime}$. Let $G$ be a generalized $t$-fold of $T$, suppose that the $\alpha$-labeling of $G$ has been obtained using the procedure in Theorem 2 and $T_{i}$ is a copy of $T$ in $G$. By transforming $g$ in the same way that the labeling of $T_{i}$ was transformed before, we obtain a labeling of $T^{\prime}$ that assigns the same


Figure 4. $\alpha$-labeling of a generalized 5 -fold of a tree.


Figure 5. $\alpha$-labeling of a modified 5 -fold of a tree.
labels on the corresponding stable sets of $T_{i}$ and $T^{\prime}$. Thus, we can replace the edges of $T_{i}$ in $G$ with the edges of $T^{\prime}$ and the resulting graph is still an $\alpha$-graph. This procedure can be applied as many times as necessary to obtain the desired $\alpha$-labeling of the aimed graph.

In Figure 5, we show this substitution of edges on the graph shown in Figure 4, where all the copies of $T$, except $T_{2}$, were replaced with analogous trees.

## 3. $\alpha$-trees of even diameter

A caterpillar is a tree with a single path containing at least one endpoint of every edge. Suppose that $T$ is a caterpillar of size $2 m \geq 4$ such that $m=|A|=|B|-1$. We say that $T \in \boldsymbol{F}_{k}$ if $T$ has diameter $2 k$ or $2 k+1$, for some positive integer $k \geq 2$. For each $i \in\{1,2,3,4\}$, let $T_{i} \in \boldsymbol{F}_{k}$ and $T_{5}$ be the tree consisting of one central vertex, denoted by $w$, which is attached to $t \geq 0$ pendant vertices, that is, $T_{5} \cong K_{1}$ or $T_{5} \cong K_{1, t}$. Recall that the eccentricity of a vertex in a graph is the maximum distance to other vertices. For each $i \in\{1,2,3,4\}$, let $v_{i} \in V\left(T_{i}\right)$ such that its eccentricity equals $2 k$. Thus, $v_{i}$ is a leaf of $T_{i}$ when diam $T_{i}=2 k$ or $v_{i}$ is adjacent to a vertex of maximum eccentricity when diam $T_{i}=$ $2 k+1$. Consider the tree $T$ of size $4(2 m+1)+t$ obtained by connecting, with an edge, all the vertices $v_{i}$ to the vertex $w$ of $T_{5}$. By $\boldsymbol{T}_{m, t}$ we understand the family of all trees of size $4(2 m+1)+t$ obtained in the form described above. We claim that all the elements of $\boldsymbol{T}_{m, t}$ are $\alpha$-trees.

Proposition 2. If $T \in \boldsymbol{T}_{m, t}$, then $T$ is an $\alpha$-tree.

Proof. Suppose that for every $i \in\{1,2,3,4\}, v_{i}$ is in the largest stable set of $T_{i}$; let $f_{i}$ be an $\alpha$-labeling of $T_{i}$ such that $f_{i}\left(v_{i}\right)=$ $2 m$. The existence of this labeling was proven by Rosa [6]. The labeling $f_{5}$ of $T_{5}$ is the $\alpha$-labeling, also given by Rosa in the same work, that assigns the label 0 to the vertex $w$. For $i=2$, 4 , the initial labeling of $T_{i}$ is the reverse of $f_{i}$, that is, $\hat{f}_{i}$.

For each $i \in\{1,2,3,4,5\}$, the initial $\alpha$-labeling of $T_{i}$ is transformed into a $d_{i}$-graceful labeling shifted $c_{i}$ units, where

$$
\left(d_{i}, c_{i}\right)= \begin{cases}(6 m+t+5,0) & \text { if } i=1 \\ (4 m+t+4, m) & \text { if } i=2 \\ (2 m+2,2 m+1) & \text { if } i=3 \\ (1,3 m+1) & \text { if } i=4 \\ (4 m+3,2 m) & \text { if } i=5\end{cases}
$$

Hence, the labels assigned to the vertices of $T_{i}$ form the set $[0, m-1] \cup[7 m+t+4,8 m+t+4] \quad$ when $\quad i=1$, $[m, 2 m-1] \cup[6 m+t+3,7 m+t+3]$ when $i=2, \quad[2 m+$ $1,3 m] \cup[5 m+2,6 m+2]$ when $i=3, \quad[3 m+1,4 m] \cup[4 m+$ $1,5 m+1]$ when $i=4$, and $\{2 m\} \cup[6 m+3,6 m+t+2]$ when $i=5$. It follows that the labels assigned on the vertices of $T$ form the interval $[0,8 m+t+4]=[0,4(2 m+1)+t]$.

The weights induced on the edges of $T_{i}$ form the interval $[6 m+t+5,8 m+t+4]$ when $i=1,[4 m+t+4,6 m+t+$ 3] when $i=2,[2 m+2,4 m+1]$ when $i=3,[1,2 m]$ when $i=4$, and $[4 m+3,4 m+t+2]$ when $i=5$.

Notice that the labels assigned to $v_{i}$ are $8 m+t+4$ when $i=1,6 m+t+3$ when $i=2,6 m+2$ when $i=3$, and $4 m+1$ when $i=4$. Since the label of $w$ is $2 m$, the edges $v_{i} w$ have weights $6 m+t+4,4 m+t+3,4 m+2$, and $2 m+1$, respectively. Thus, the weighst induced on the edges of $T$ form the


Figure 6. $\alpha$-labeling of a tree in $\mathscr{T}_{5,4}$.
interval $[1,8 m+t+4]=[1,4(2 m+1)+t]$. The form in which the initial $\alpha$-labelings are combined guarantees that the final labeling is also an $\alpha$-labeling; its boundary value is $\lambda=4 m$.

We must observe that the caterpillars used above can be replaced by $\alpha$-trees for which there exist $\alpha$-labelings that place the label 0 on vertices $u_{i}$ satisfying the same conditions that the $v_{i}$. In Figure 6, we show an example of this labeling on a tree with four branches of length 5 .

A rooted tree is a tree with a distinguished vertex $r$, called the root. The last proposition tells us that any rooted tree $T$ is an $\alpha$-tree if $T-r$ consists of four caterpillars of equal size and diameter $2 k$ or $2 k+1$ for certain positive integer $k$. Is it possible to extend this result to trees $T$ such that $T-r$ result in any number of caterpillars of equal size and similar diameters?

## Disclosure statement

No potential conflict of interest was reported by the authors.

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