

AKCE International Journal of Graphs and Combinatorics



ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Christian Barrientos & Sarah Minion (2020): Folding trees gracefully, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2019.10.001

To link to this article: https://doi.org/10.1016/j.akcej.2019.10.001

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Published online: 04 Jun 2020.



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Folding trees gracefully

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ABSTRACT

When a graceful labeling of a bipartite graph assigns the smaller labels to the vertices of one of the stable sets of the graph, the assignment is called an α -labeling. Any graph that admits such a labeling is an α -graph. In this work we extend the concept of vertex amalgamation to generate a new class of α -graphs obtained by a sequence of *k*-vertex amalgamations of *t* copies of an α -tree. This procedure is also applied to any collection of α -trees such that any pair of trees in this collection have stable sets with the same cardinalities. We also use this idea on other types of α -graphs. In addition, we present a family of α -trees of even diameter formed with four caterpillars of the same size.

1. Introduction

A difference vertex labeling of a graph G of size n is an injective mapping f from V(G) into a set N of nonnegative integers, such that every edge uv of G has assigned a weight defined by |f(u) - f(v)|. The labeling f is called graceful when $N = \{0, 1, ..., n\}$ and the set of induced weights is $\{1, 2, ..., n\}$. In this case, G is called a graceful graph. Let G be a bipartite graph and $\{A, B\}$ be the natural bipartition of V(G), we refer to A and B as the stable sets of G and assume that |A| = a and |B| = b. A bipartite labeling of G is an injection $f: V(G) \rightarrow \{0, 1, ..., s\}$ for which there is an integer λ , named the boundary value of f, such that $f(u) \leq \lambda < \lambda$ f(v) for every $(u, v) \in A \times B$, that induces *n* different weights. This is an extension of the definition given by Rosa and Širáň [7]. From the definition we may conclude that $s \ge |E(G)|$; furthermore, the labels assigned by f on the vertices of A and B are in the integer interval $[0, \lambda]$ and $[\lambda +$ 1, s], respectively. If s = n, the function f is an α -labeling and *G* is an α -graph. If *f* is an α -labeling of a tree and $f^{-1}(0) \in$ A, then its boundary value is $\lambda = a - 1$.

Suppose that $f: V(G) \rightarrow \{0, 1, ..., n\}$ is a graceful labeling of a graph G of size n:

- g: V(G) → {c, c + 1, ..., c + n}, defined for every v ∈ V(G) and c ∈ N as g(v) = c + f(v), is the shifting of f in c units. Note that this labeling preserves the weights induced by f.
- $\hat{f}: V(G) \to \{0, 1, ..., n\}$, defined for every $v \in V(G)$ as $\hat{f}(v) = \lambda f(v)$ if $f(v) \le \lambda$, and $\hat{f}(v) = n + \lambda + 1 f(v)$ if $f(v) > \lambda$, is the *reverse* labeling of f. Thus, \hat{f} is also an α -labeling with boundary value λ .

 $f : V(G) → \{0, 1, ..., n + d - 1\},$ defined for every v ∈ V(G) and $d ∈ \mathbb{N}$ as g(v) = f(v) if $f(v) ≤ \lambda$ and g(v) = f(v) + d - 1 if $f(v) > \lambda$, is the *d*-graceful labeling of G obtained from f. The labels assigned by g on the stable sets of V(G) are in the intervals $[0, \lambda]$ and $[\lambda + d, n + d - 1]$ and the set of induced weights is $\{d, d + 1, ..., n + d - 1\}.$

For example, let f be an α -labeling of a tree T of size n with boundary value λ . Suppose that f is transformed into a d-graceful labeling shifted c units. Then the elements of A are labeled with the integers in $[c, \lambda + c]$, the elements of B are labeled with the integers in $[c + \lambda + d, c + n + d - 1]$, and the induced weights form the interval [d, n + d - 1].

In Section 2, we study α -labelings for graphs that result of *k*-vertex amalgamations of smaller α -graphs. Section 3 is devoted to α -labelings of trees, there we prove the existence of an α -labeling for trees obtained by amalgamating caterpillars. The reader interested in graph labelings is referred to Gallian' survey [4] for more information about the subject. In this paper, we follow the notation and terminology used in [3] and [4].

2. Folding α -trees

For i = 1, 2, let G_i be a graph of order n_i and size m_i . A graph G, of order $n_1 + n_2 - k$ and size $m_1 + m_2$, is said to be a *k*-vertex amalgamation (or strong vertex amalgamation) of G_1 and G_2 if it is obtained identifying k independent vertices of G_1 with k independent vertices of G_2 . We use here strong vertex amalgamation of α -graphs to construct new classes of α -graphs.

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS https://doi.org/10.1016/j.akcej.2019.10.001



KEYWORDS

2010 MSC

05C78

 α -labeling; graceful labeling;

k-vertex amalgamation

Check for updates



Figure 1. α -labeling of the 3-fold of Q_3 .

Suppose That G is a bipartite graph of order n and size m, with stable sets $A = \{u_1, u_2, ..., u_a\}$ and $B = \{v_1, v_2, ..., v_b\}$. Let $G_1, G_2, ..., G_t$ be disjoint copies of G with $A_i = \{u_1^i, u_2^i, ..., u_a^i\}$ and $B_i = \{v_1^i, v_2^i, ..., v_b^i\}$. If for every even value of $i \in \{1, 2, ..., t\}$, the vertices of B_i are identified with the vertices of B_{i-1} , in such a way that $v_j^i = v_j^{i-1}$ for every $1 \le j \le b$, and the vertices of A_i are identified with the vertices of A_{i+1} , in such a way that $u_j^i = u_j^{i+1}$ for every $1 \le j \le a$, then we obtain a bipartite graph of size tm and order $\frac{tn}{2} + a$ when t is even or $\frac{(t+1)n}{2}$ when t is odd. We call this graph the t-fold of G. We claim that the t-fold of G is an α -graph.

Notice that when t is odd, there is only one t-fold of G; that is, the t-fold of G is independent of the stable set of G chosen to be B. On the other hand, when t is even, there are, in general, two non-isomorphic t-folds of G, depending on the stable set chosen to be B. When G is vertex transitive, this selection is irrelevant and there is only one t-fold of G for any value of t.

Theorem 1. If G is an α -graph, then any t-fold of G is an α -graph.

Proof. Suppose that G is an α -graph of order n and size m with stable sets $A = \{u_1, u_2, ..., u_a\}$ and $B = \{v_1, v_2, ..., v_b\}$. Let f be an α -labeling of G with boundary value λ such that $f^{-1}(0) \in A$. Consider t copies of G, each labeled using f. For every value of $i \in \{1, 2, ..., t\}$, let f_i be the d_i -graceful labeling of G_i , obtained from f, shifted c_i units, where $d_i = 1 + m(t - i)$ and $c_i = m |\frac{i}{2}|$.

Thus, the set of weights induced by f_i on the edges of G_i is $[d_i, d_i + m - 1]$. This implies that the set of weights on the edges of the *t*-fold of *G* is



Figure 2. α -labeling of the 7-fold of C_8 .

$$\sum_{i=1}^{t} [d_i, d_i + m - 1] = \bigcup_{i=1}^{t} [1 + m(t - i), 1 + m(t - i) + m - 1]$$
$$= \bigcup_{i=1}^{t} [1 + m(t - i), m(t + 1 - i)]$$
$$= [1, mt]$$

Suppose that $r_1 < r_2 < ... < r_a$ and $\rho_1 < \rho_2 < ... < \rho_b$ are the labels assigned by f on the vertices of A and B, respectively. Thus, f_i assigns the labels $r_1 + c_i, r_2 + c_i, ..., r_a + c_i$ to the elements of A_i and the labels $\rho_1 + d_i - 1 + c_i, \rho_2 + d_i - 1 + c_i, ..., \rho_b + d_i - 1 + c_i$ to the elements of B_i . Note that when iis even, i = 2s, the labels on the vertices of B_i are $\rho_j + m(t - s), 1 \le j \le b$, and the labels on the vertices of B_{i-1} are $\rho_j + m(t - s)$, as well. Similarly, the labels on the vertices of A_i and A_{i+1} are $r_1 + ms, r_2 + ms, ..., r_a + ms$. Therefore, the vertices of the *t*-fold of G are labeled with integers from [0, mt] that induce the weights 1, 2, ..., mt. Since the labelings of the G_i are bipartite, the final labeling is an α -labeling with boundary value $m + \lfloor \frac{t}{2} \rfloor + \lambda$. Hence, the *t*-fold of G is an α -graph. \Box

In Figure 1, we show an α -labeling of the 3-fold of the hypercube Q_3 .

When the graph *G* in Theorem 1 is the path P_n , with $n \ge 4$, the *t*-fold of *G* results in a graph that contains as a subgraph a convex Eulerian polyomino, the induced labeling of this polyomino can be transformed into the α -labeling used by Acharya [1] to prove that all convex Eulerian polyominoes are arbitrarily graceful. Another nice family of α -graphs that can be obtained using Theorem 1 is the one containing the fractal type structure constructed by *t*-folding the cycle C_{4m} , $m \ge 2$. In Figure 2 we show an example of one of these structures constructed using C_8 folded 7 times.

Note that the α -labeling of the *t*-fold of an α -graph *G* can be also obtained using the labeling scheme of the weak tensor product of *G* and P_{t+1} , introduced by Snevily [8] and extended by López and Muntaner-Batle [5]. The labeling technique used in the proof of Theorem 1 is applied to



Figure 3. A 3-vertex amalgamation of an α -graph of size 7.

other types of folded graphs that cannot be explained using the weak tensor product.

In the next proposition we prove that for any $1 \le k \le b$, there is an α -graph *G* that results of a *k*-vertex amalgamation of two copies of an α -tree. Note that when k = 1, *G* is the well-known vertex amalgamation (or one-point union) of two copies of *T*. The gracefulness of this type of graph was proven in [2]. When k = b, an α -labeling of *G* can be obtained using Theorem 1 with t = 2.

Proposition 1. Let T be an α -tree of size m with one stable set of cardinality b. If $k \leq b$ is a positive integer, then there is a k-vertex amalgamation of two copies of T that is an α -graph.

Proof. For i = 1, 2, let T_i be a copy of T with stable sets A_i and B_i , where $|B_i| = b$. Suppose that f is an α -labeling of T such that $f^{-1}(0) \in A$; so, its boundary value is $\lambda = |A| - 1 = a - 1$. Let f_i be the d_i -graceful labeling of T_i , obtained from f, shifted c_i units, where

$$(d_i, c_i) = \begin{cases} (m+1, 0) & \text{if } i = 1, \\ (1, \lambda + k) & \text{if } i = 2. \end{cases}$$

Thus, the vertices in A_1 are labeled with the integers in $\{0, 1, ..., \lambda\}$, the vertices in B_1 are labeled with the integers in $\{\lambda + m + 1, \lambda + m + 2, ..., 2m\}$, the vertices in A_2 are labeled with the integers in $\{\lambda + k, \lambda + k + 1, ..., 2\lambda + k\}$, and the vertices in B_2 are labeled with the integers in $\{2\lambda + k + 1, 2\lambda + k + 2, ..., m + \lambda + k\}$.

Identifying the vertices of B_1 and B_2 labeled $2\lambda + k + 1$, $2\lambda + k + 2$, ..., $m + \lambda + k$, we obtain a graph G that is a k-vertex amalgamation of T_1 and T_2 . Since the weights on T_1 are m + 1, m + 2, ..., 2m and on T_2 are 1, 2, ..., m, we have that all edges of G have different weights; in addition, the vertices of T_1 and T_2 , with the same label, were amalgamated, hence there is no repetition of labels. Observe that the labels assigned to the elements of $A_1 \cup A_2$ are smaller that those assigned to the elements of $B_1 \cup B_2$, so the number $2\lambda + k$ is the boundary value of the α -labeling of G, therefore G is an α -graph (Figure 3).

Suppose that *T* is an α -labeled tree of size *m*, where $A = \{0, 1, ..., \lambda\}$ and $B = \{\lambda + 1, \lambda + 2, ..., m\}$ are considered ordered sets. By a *generalized t-fold* of *T* we mean a graph *G* obtained using *t* α -labeled copies of *T*, where for every even value of *i*, the copy T_i is merged with the copies T_{i-1} and T_{i+1} in such a way that the last k_i vertices in B_i are

amalgamated with the first k_i vertices in B_{i-1} , and the last k'_i vertices in A_i are amalgamated with the first k'_i vertices in A_{i+1} . These amalgamations must be done in ascending order; for example, suppose that x is amalgamated with y and x' is amalgamated with y', if x < x', then y < y'.

Theorem 2. If T is an α -tree, then any generalized t-fold of T is an α -graph.

Proof. Suppose that T is an α -tree of size m with stable sets A and B. Let $T_1, T_2, ..., T_t$ be disjoint copies of T. Assume that f is an α -labeling of T with boundary value λ . Without loss of generality, suppose that $f^{-1}(0) \in A$. Thus, the labels assigned to the vertices of A and B form the intervals $L_A = [0, \lambda]$ and $L_B = [\lambda + 1, m]$, respectively. For each $i \in \{1, 2, ..., t\}$, suppose that f is the initial labeling of T_i . The final labeling of T_i denoted by f_i , is obtained by transforming f into a d_i -graceful labeling shifted c_i units, where $d_i = (t - i)m + 1$ and $c_i = \sum_{j=1}^{i} \xi_j$ with

$$\xi_i \in \left\{ \begin{array}{ll} [\lambda+1,m] & \text{ if } n \text{ is even,} \\ [0,\lambda] & \text{ if } n \text{ is odd.} \end{array} \right.$$

Consequently, the labels assigned by f_i to the vertices of A_i and B_i are $L_{A_i} = [c_i, c_i + \lambda]$ and $L_{B_i} = [\lambda + c_i + d_i, m + c_i + d_i - 1]$. In addition, the weights induced by f_i on the edges of T_i form the interval [(t-i)m+1, (t-i)m+m]. Therefore, $\cup_{i=1}^t [(t-i)m+1, (t-i)m+m] = [1, tm]$.

Since $L_{A_i} = [c_i, c_i + \lambda]$ where $c_i = \sum_{j=1}^{i} \xi_j$, we have that for every feasible even value of i, $L_{A_i} \cap L_{A_{i+1}} = [c_{i+1}, c_i + \lambda]$ and $|L_{A_i} \cap L_{A_{i+1}}| = c_i + \lambda - c_{i+1} + 1 = \lambda + 1 - \xi_{i+1} = k'_i$. Thus, the number k'_i , of vertices shared by A_i and A_{i+1} is bounded by $1 \le k'_i \le \lambda + 1 = a$. Similarly, $L_{B_{i-1}} \cap L_{B_i} = [\lambda + c_{i-1} + d_{i-1}, m + c_i + d_i - 1]$ and $|L_{B_{i-1}} \cap L_{B_i}| = \xi_i - \lambda = k_i$. Since $\xi_i \in [\lambda + 1, m] = [\lambda + 1, a + b - 1]$, the number k_i of vertices shared by B_{i-1} and B_i is bounded by $1 \le k_i \le b$.

Identifying the vertices with the same label, we form the graph *G* that is a generalized *t*-fold of *T*. Since the labelings used on the T_i are bipartite, the boundary value of the labeling of *G* is the largest number in L_{A_i} , that is, $\lambda + c_t$. Therefore, *G* is an α -graph.

In Figure 4, we show an example of an α -labeling for a generalized 5-fold of an α -tree of size 10, where $k_2 = 4, k'_2 = 2, k_4 = 5, \xi'_4 = 5, \xi_1 = 0, \xi_2 = 8, \xi_3 = 3, x_4 = 9$, and $\xi_5 = 0$.

Let T_1 and T_2 be two α -trees of size *m*. We say that T_1 and T_2 are *analogous* if $|A_1| = |A_2|$ and $|B_1| = |B_2|$. We use this concept to extend the result of Theorem 2 by replacing any number of copies of *T* with analogous trees.

Theorem 3. If G is a generalized t-fold of an α -tree T, then any of the copies of T, used to construct G, can be replaced by any tree T' analogous to T, and the resulting graph is an α -graph.

Proof. Since T and T' are analogous, there exist α -labelings f and g, of T and T', respectively, such that $f^{-1}(0) \in A$ and $g^{-1}(0) \in A'$. Let G be a generalized t-fold of T, suppose that the α -labeling of G has been obtained using the procedure in Theorem 2 and T_i is a copy of T in G. By transforming g in the same way that the labeling of T_i was transformed before, we obtain a labeling of T' that assigns the same



Figure 4. α -labeling of a generalized 5-fold of a tree.



Figure 5. α -labeling of a modified 5-fold of a tree.

labels on the corresponding stable sets of T_i and T'. Thus, we can replace the edges of T_i in G with the edges of T' and the resulting graph is still an α -graph. This procedure can be applied as many times as necessary to obtain the desired α -labeling of the aimed graph.

In Figure 5, we show this substitution of edges on the graph shown in Figure 4, where all the copies of T, except T_2 , were replaced with analogous trees.

3. α-trees of even diameter

A caterpillar is a tree with a single path containing at least one endpoint of every edge. Suppose that T is a caterpillar of size $2m \ge 4$ such that m = |A| = |B| - 1. We say that $T \in \mathbf{F}_k$ if T has diameter 2k or 2k + 1, for some positive integer $k \ge 2$. For each $i \in \{1, 2, 3, 4\}$, let $T_i \in F_k$ and T_5 be the tree consisting of one central vertex, denoted by w, which is attached to $t \ge 0$ pendant vertices, that is, $T_5 \cong K_1$ or $T_5 \cong K_{1,t}$. Recall that the *eccentricity* of a vertex in a graph is the maximum distance to other vertices. For each $i \in \{1, 2, 3, 4\}$, let $v_i \in V(T_i)$ such that its eccentricity equals 2k. Thus, v_i is a leaf of T_i when diam $T_i = 2k$ or v_i is adjacent to a vertex of maximum eccentricity when diam $T_i =$ 2k+1. Consider the tree T of size 4(2m+1)+t obtained by connecting, with an edge, all the vertices v_i to the vertex w of T₅. By $T_{m,t}$ we understand the family of all trees of size 4(2m+1) + t obtained in the form described above. We claim that all the elements of $T_{m,t}$ are α -trees.

Proposition 2. If $T \in T_{m,t}$, then T is an α -tree.

Proof. Suppose that for every $i \in \{1, 2, 3, 4\}$, v_i is in the largest stable set of T_i ; let f_i be an α -labeling of T_i such that $f_i(v_i) = 2m$. The existence of this labeling was proven by Rosa [6]. The labeling f_5 of T_5 is the α -labeling, also given by Rosa in the same work, that assigns the label 0 to the vertex *w*. For i = 2, 4, the initial labeling of T_i is the reverse of f_i , that is, \hat{f}_i .

For each $i \in \{1, 2, 3, 4, 5\}$, the initial α -labeling of T_i is transformed into a d_i -graceful labeling shifted c_i units, where

$$(d_i, c_i) = \begin{cases} (6m + t + 5, 0) & \text{if } i = 1, \\ (4m + t + 4, m) & \text{if } i = 2, \\ (2m + 2, 2m + 1) & \text{if } i = 3, \\ (1, 3m + 1) & \text{if } i = 4, \\ (4m + 3, 2m) & \text{if } i = 5. \end{cases}$$

Hence, the labels assigned to the vertices of T_i form the set $[0, m-1] \cup [7m+t+4, 8m+t+4]$ when i=1, $[m, 2m-1] \cup [6m+t+3, 7m+t+3]$ when i=2, $[2m+1, 3m] \cup [5m+2, 6m+2]$ when i=3, $[3m+1, 4m] \cup [4m+1, 5m+1]$ when i=4, and $\{2m\} \cup [6m+3, 6m+t+2]$ when i=5. It follows that the labels assigned on the vertices of T form the interval [0, 8m+t+4] = [0, 4(2m+1)+t].

The weights induced on the edges of T_i form the interval [6m + t + 5, 8m + t + 4] when i = 1, [4m + t + 4, 6m + t + 3] when i = 2, [2m + 2, 4m + 1] when i = 3, [1, 2m] when i = 4, and [4m + 3, 4m + t + 2] when i = 5.

Notice that the labels assigned to v_i are 8m + t + 4 when i = 1, 6m + t + 3 when i = 2, 6m + 2 when i = 3, and 4m + 1 when i = 4. Since the label of w is 2m, the edges v_iw have weights 6m + t + 4, 4m + t + 3, 4m + 2, and 2m + 1, respectively. Thus, the weighst induced on the edges of T form the



Figure 6. α -labeling of a tree in $\mathcal{T}_{5,4}$.

interval [1, 8m + t + 4] = [1, 4(2m + 1) + t]. The form in which the initial α -labelings are combined guarantees that the final labeling is also an α -labeling; its boundary value is $\lambda = 4m$.

We must observe that the caterpillars used above can be replaced by α -trees for which there exist α -labelings that place the label 0 on vertices u_i satisfying the same conditions that the v_i . In Figure 6, we show an example of this labeling on a tree with four branches of length 5.

A *rooted tree* is a tree with a distinguished vertex r, called the *root*. The last proposition tells us that any rooted tree T is an α -tree if T - r consists of four caterpillars of equal size and diameter 2k or 2k + 1 for certain positive integer k. Is it possible to extend this result to trees T such that T - r result in any number of caterpillars of equal size and similar diameters?

Disclosure statement

No potential conflict of interest was reported by the authors.

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