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Infinite families of asymmetric graphs

Alejandra Brewer[a], Adam Gregory[b], Quindel Jones[c], Luke Rodriguez[d], Rigoberto Flórez[e], and Darren Narayan[f]

[a]Department of Mathematics and Statistics, Auburn University, Auburn, AL, USA; [b]Department of Mathematics, University of Florida, Gainesville, FL, USA; [c]Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA, USA; [d]Department of Mathematics, University of Washington, Seattle, WA, USA; [e]Department of Mathematics and Computer Science, The Citadel, Charleston, SC, USA; [f]School of Mathematical Sciences, Rochester Institute of Technology, Rochester, NY, USA

ABSTRACT

A graph G is *asymmetric* if its automorphism group of vertices is trivial. Asymmetric graphs were introduced by Erdős and Rényi in 1963. They showed that the probability of a graph on n vertices being asymmetric tends to 1 as n tends to infinity. In this paper, we first consider the enumeration of asymmetric trees, a question posed by Erdős and Rényi. We show that the number of asymmetric subdivided stars is approximately $q(n-1) - \lfloor \frac{n-1}{2} \rfloor$, where $q(n)$ is the number of ways to sum to n using distinct positive integers found by Hardy and Ramanujan in 1918. We also investigate cubic Hamiltonian graphs where asymmetry, where at least for small values of n, seem to be rare. It is known that none of the cubic Hamiltonian graphs on $4 \leq n \leq 10$ vertices are asymmetric, and of the 80 cubic Hamiltonian graphs on 12 vertices only 5 are asymmetric. We give a construction of an infinite family of cubic Hamiltonian graphs that are asymmetric. Then we present an infinite family of quartic Hamiltonian graphs that are asymmetric. We use both of the above results for cubic and quartic asymmetric Hamiltonian graphs to establish the existence of k-regular asymmetric Hamiltonian graphs for all $k \geq 3$.

KEYWORDS

Graph automorphisms; asymmetric graphs; Hamiltonian graphs

1. Introduction

We consider undirected graphs without multiple edges or loops. A graph G is *asymmetric* if its automorphism group of vertices is trivial. Asymmetric graphs were introduced by Erdős and Rényi in 1963. They showed that the probability of a graph on n vertices being asymmetric tends to 1 as n tends to infinity. In this paper, we investigate cubic Hamiltonian graphs where asymmetry, at least for small values of n, seems to be rare. It is known that none of the cubic Hamiltonian graphs on $4 \leq n \leq 10$ vertices are asymmetric, and of the 80 cubic Hamiltonian graphs on 12 vertices only 5 are asymmetric [5].

We will use n to denote the number of vertices in a graph. For a graph G we will use $V(G)$ to denote the set of vertices in G, and $E(G)$ to denote the set of edges in G. The edge between vertices u and v will be denoted uv. Two graphs G and H are *isomorphic* if there is a bijection $f : G \to H$ where $uv \in E(G) \iff f(u)f(v) \in E(H)$. Recall that f is an *automorphism* if it is an isomorphism from a graph to itself, and the set of all automorphisms of a graph form an algebraic group under function composition. We will use $\text{Aut}(H)$ to denote the automorphism group of vertices in a graph H. The *complement* of a graph G will be denoted $\bar{G}$. We will use C_n to denote a cycle on n vertices; $K_{s,t}$ to denote a complete bipartite graph where one part has s vertices and the other part has t vertices; and P_n to denote a path on n vertices. The *degree* of a vertex v is the number of edges incident to v. A graph is *k-regular* if each vertex has degree k. An edge-subdivision of an edge uv is performed by replacing the edge uv by two edges uw and wv, where w is a new vertex. A subdivided star is a graph that can be obtained by subdividing one or more edges of the star $K_{1,n-1}$. The *distance* between two vertices u and v is the number of edges in a shortest path between u and v. The distance between an edge and a subgraph will be the minimum vertex distance between a vertex incident to the edge and a vertex in the subgraph. For any undefined notation, please see the text [4] by West.

In this paper we first investigate asymmetric trees. We show that the number of asymmetric subdivided stars on n vertices equals the number of partitions of an integer into at least three distinct parts, a problem studied by Hardy and Ramanujan [2] in 1918. Furthermore, we identify the smallest asymmetric tree that is not a subdivided star.

We also give a construction of an infinite family of cubic Hamiltonian graphs that are asymmetric. Then we present an infinite family of quartic Hamiltonian graphs that are asymmetric. We use both of the above results to establish the existence of k-regular asymmetric Hamiltonian graphs for all $k \geq 3$.

A vertex $v \in V(G)$ is *unique* if there is a property P such that for each vertex $w \in V(G) - \{v\}$, we have that v satisfies P and w does not satisfy P. A graph G can be shown to have a trivial automorphism group if each vertex is unique. For any graph G, two edges $e_1, e_2 \in E(G)$ are said to be

CONTACT Darren Narayan dansma@rit.edu Rochester Institute of Technology, 1 Lomb Memorial Dr, Rochester, NY 14623, USA.

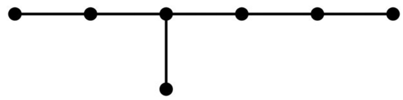

Figure 1. The smallest asymmetric tree.

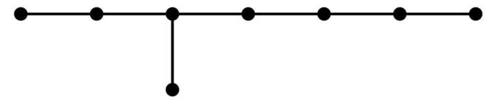

Figure 2. The only asymmetric tree on eight vertices.

different if and only if $G - \{e_1\} \ncong G - \{e_2\}$. It is known that if G has no component isomorphic to P_2, then $\mathrm{Aut}(V(G)) \cong \mathrm{Aut}(E(G))$. In this way, one could alternatively show that each edge in G is different. In the paper, we will show that two edges are different by showing that one edge has a property (e.g. included in a particular type of subgraph, or is incident to a vertex that is unique) that another edge does not. We next state an elementary property relating the automorphism group of a graph and its complement.

Proposition 1.1. *Given any graph G, $\mathrm{Aut}(G) = \mathrm{Aut}(\bar{G})$.*

Theorem 1.1. *Every asymmetric graph on n vertices can be extended to an asymmetric graph on $n+1$ vertices.*

Proof. Let G be an asymmetric graph. If G has no vertex of degree one, then we can add a new vertex w along with the edge wv, where v is a vertex of largest degree. Note that all of the vertices that were unique in $G - v$ will remain unique. The vertex v will be unique since it has the largest degree, and the vertex w is unique since it has the smallest degree.

If G has a vertex of degree one, let u be a vertex with degree one and the greatest distance from a vertex of degree greater than two. Then we can add a new vertex x along with an edge xu. Note that all of the vertices that were unique in $G - u$ will remain unique. The vertex u will be unique since it is the only vertex that is adjacent to a vertex of degree one, and vertex x is unique since it is the only vertex of degree one. □

In Section 2, we give results for the enumeration of a class of asymmetric trees. In Section 3, we investigate asymmetric cubic Hamiltonian graphs and present an infinite family of these graphs. In Section 4, we investigate asymmetric quartic Hamiltonian graphs and again provide an infinite family. Lastly, Section 5 summarizes the results of the paper and states open problems.

2. Trees and subdivided stars

The smallest asymmetric tree was given by Erdős and Rényi in [1] and is depicted in Figure 1.

Notice that the smallest asymmetric tree is a subdivided star with $n=7$. We use $T_{n_1,n_2,\dots,n_k}$ to denote a subdivided star with a vertex of degree k, which we refer to as the center vertex, and pendant paths $P_{n_1}, P_{n_2}, \dots, P_{n_k}$ extending from the center vertex. Hence, the graph shown in Figure 1 will be denoted $T_{1,2,3}$. If we continue to extend any pendant path of the subdivided star on seven vertices, such that no two paths are the same length, we build an asymmetric subdivided star on n vertices. Figure 2 depicts the asymmetric subdivided star $T_{1,2,4}$ when $n=8$. Notice that each pendant path has a distinct length.

Theorem 2.1. *If $G = T_{n_1,n_2,\dots,n_k}$ where $k \geq 3$ and $n_1, n_2, \dots, n_k$ are all distinct, then G is asymmetric.*

Proof. Let $G = T_{n_1,n_2,\dots,n_k}$ where $k \geq 3$ and $n_1, n_2, \dots, n_k$ are all distinct. Then any two vertices on the same pendant path have a different distance to the center vertex v, which is unique as it has degree k. In this way, any two vertices on the same pendant path are unique. For vertices on different pendant paths, if we consider $G - v$ we have k non-isomorphic paths. Thus, any two vertices on different pendant paths are in components of different sizes, and are therefore unique. □

We count the number of asymmetric subdivided stars using integer partitions. A partition of $n - 1$ into three or more distinct parts ensures a sub-divided star where no two pendant paths have the same length. In [2], Hardy and Ramanujan partition integers into distinct parts and give a formula for counting them. We adopt the notation $q(n)$ for this formula, as used in [3]. However, because a partition on less than three parts yields a path, which is symmetric, we are only interested in distinct partitions of three or more parts. Moreover, for a subdivided star on n vertices we consider the integer partitions of $n - 1$. Thus, removing partitions of integers with only two parts from $q(n-1)$ results in our formula gives the number of asymmetric sub-divided stars, $|ASDS_n| = q(n-1) - \lfloor \frac{n-1}{2} \rfloor$.

The smallest asymmetric tree has seven vertices. This graph can be extended to the only asymmetric tree on eight vertices, $T_{1,2,4}$. This graph can be extended to two non-isomorphic asymmetric trees on nine vertices: $T_{1,2,5}$ and $T_{1,3,4}$. However, a vertex sub-division could be performed on the central vertex of $T_{1,2,4}$ to obtain the graph found in Figure 3. By Theorem 1.1, it is possible to extend the longest pendant path of the graph in Figure 3 to create an infinite family of asymmetric trees.

We note by Theorem 1.1, every asymmetric tree on n vertices can be extended to an asymmetric tree on $n+1$ vertices. Hence all asymmetric trees can be extended to infinite families.

3. Asymmetric cubic Hamiltonian graphs

In this section we provide a construction for an infinite family of asymmetric cubic Hamiltonian graphs.

3.1. Construction

We begin by detailing a procedure for constructing the graph:

(1) Construct C_n where n is even and $n \geq 12$ and label the vertices from v_1 to v_n, clockwise.
(2) Add the edge $v_n v_{\frac{n}{2}-1}$ which creates two unequal sets of vertices.
(3) Add the edge $v_1 v_{n-2}$.
(4) Add the edge $v_{n-1} v_{n-3}$.

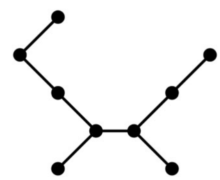

Figure 3. The smallest asymmetric tree that is not a subdivided star.

(5) For each $v_k \in G$, if $2 \le k \le \frac{n}{2} - 2$, then add the edge $v_k v_{n-2-k}$.

This yields a cubic Hamiltonian graph on n vertices, and examples of this graph on 12 and 18 vertices are depicted in Figure 4. Note that there are exactly two C_3 subgraphs, which remain fixed as n increases. We will now show this graph has no symmetry.

3.2. Ensuring a trivial automorphism group

In order to show that this cubic Hamiltonian graph on n vertices has a trivial automorphism group, we will show that every edge is different from all other edges in the graph. We begin by partitioning edges into sets based on various properties, such that edges in one set are different from edges in all other sets.

We partition $E(G)$ as follows:

- $E(\text{I}) = \{v_1v_2\}$
  Edges incident to two C_4 subgraphs and are not contained in a C_4.
- $E(\text{II}) = \{v_nv_{n-1}, v_nv_1, v_1v_{n-2}, v_2v_{n-4}, v_{\frac{n}{2}-3}v_{\frac{n}{2}+1}\}$
  Edges that are in a C_4 and in a C_5.
- $E(\text{III}) = \{v_{n-2}v_{n-1}, v_{n-1}v_{n-3}, v_{\frac{n}{2}-1}v_{\frac{n}{2}}, v_{\frac{n}{2}-1}v_{\frac{n}{2}-2}\}$
  Edges that are in a C_3 and incident to a vertex that is distance 2 from a vertex in the other C_3.
- $E(\text{IV}) = \{v_{n-3}v_{n-2}, v_{\frac{n}{2}-2}v_{\frac{n}{2}}\}$
  Edges that are in a C_3 but not incident to a vertex that is distance 2 from a vertex in the other C_3.
- $E(\text{V}) = \{v_nv_{\frac{n}{2}-1}, v_{n-3}v_{n-4}\}$
  Edges that are incident to a C_3 (and are not in a C_4 nor a C_5).
- $E(\text{VI}) = \{v_kv_{n-2-k} \mid 2 \le k \le \frac{n}{2} - 4\}$
  Edges that are in two C_4s, but no C_5s.
- $E(\text{VII}) = \{v_kv_{k+1} \mid 2 \le k \le \frac{n}{2} - 3 \text{ and } \frac{n}{2} \le k \le n-5\}$
  Edges that are in a C_4 and incident to a C_4.

In this way, edges in each set are different from edges in all other sets. Now we must also show that every edge in each set is different from all other edges within the same set.

- $E(\text{I})$:
  - This set has only one edge.
- $E(\text{II})$:
  - Edge v_nv_{n-1} has three distinct paths of length one to a C_3 subgraph.
  - Edge v_nv_1 is distance one from the C_3 subgraph with vertices $v_{\frac{n}{2}}, v_{\frac{n}{2}-1}, v_{\frac{n}{2}-2}$, while the edge v_1v_{n-2} is not.

  In this way, the edges v_nv_{n-1}, v_nv_1 and v_1v_{n-2} are different.
  - The edges v_2v_{n-4} and $v_{\frac{n}{2}-3}v_{\frac{n}{2}+1}$ are different from the edges v_nv_{n+1}, v_nv_1 and v_1v_{n-2}, as the latter edges are distance one or less from the C_3 with vertices $v_{n-1}, v_{n-2}, v_{n-3}$, whereas the edges v_2v_{n-4} and $v_{\frac{n}{2}-3}v_{\frac{n}{2}+1}$ are not.
  - The edge $v_{\frac{n}{2}-3}v_{\frac{n}{2}+1}$ has two distinct paths of length one to a C_3, while the edge v_2v_{n-4} only has one.

  Thus, the edges v_2v_{n-4} and $v_{\frac{n}{2}-3}v_{\frac{n}{2}+1}$ are different. In this way, all the edges in $E(\text{II})$ are different.
- $E(\text{III})$:
  - Edges $v_{\frac{n}{2}-1}v_{\frac{n}{2}}$ and $v_{\frac{n}{2}-1}v_{\frac{n}{2}-2}$ are in neither a C_4 nor a C_6, while the edge $v_{n-1}v_{n-3}$ is in a C_6 (and not in a C_4), and the edge $v_{n-2}v_{n-1}$ is in a C_4 (and not in a C_6). In this way, the edges $v_{n-1}v_{n-3}$ and $v_{n-2}v_{n-1}$ are different.
  - In order to differentiate between the edges $v_{\frac{n}{2}-1}v_{\frac{n}{2}}$ and $v_{\frac{n}{2}-1}v_{\frac{n}{2}-2}$, we compare the distance from each edge to the C_3 subgraph with vertices $v_{n-1}, v_{n-2}, v_{n-3}$. When comparing distances to the C_3 subgraph between any edge v_kv_t, where $2 \le k \le n-5$ and $t = k+1$, we do not consider paths that use the edge $v_nv_{\frac{n}{2}-1}$, as two edges can reach the C_3 subgraph in the same distance using this edge (due to the symmetry below the edge v_2v_{n-4}). Therefore, when we compare the distance from each edge to the C_3 subgraph, we use paths that only use the exterior edges below the edge v_2v_{n-4} in order to differentiate between edges.
    Using paths that only use the exterior edges below the edge v_2v_{n-4}, we find that the edge $v_{\frac{n}{2}-1}v_{\frac{n}{2}}$ is distance four away from the C_3 subgraph, while the edge $v_{\frac{n}{2}-1}v_{\frac{n}{2}-2}$ is distance three. Therefore, the edges $v_{\frac{n}{2}-1}v_{\frac{n}{2}}$ and $v_{\frac{n}{2}-1}v_{\frac{n}{2}-2}$ are different. Thus, all the edges in $E(\text{III})$ are different.
- $E(\text{IV})$:
  - Edge $v_{\frac{n}{2}-2}v_{\frac{n}{2}}$ is part of a C_4, while the edge $v_{n-3}v_{n-2}$ is not. In this way, the edges in $E(\text{IV})$ are different.
- $E(\text{V})$:
  - Edge $v_nv_{\frac{n}{2}-1}$ has a path of length one to a C_3 subgraph, while the edge $v_{n-3}v_{n-4}$ does not. Thus, the edges in $E(\text{V})$ are different.
- $E(\text{VI})$:
  - We differentiate between edges within this group by comparing distances from each edge to both C_3 subgraphs. Recall that edges below the edge v_2v_{n-4} have symmetry about the edge $v_nv_{\frac{n}{2}-1}$. Thus, when comparing distances from edges to each C_3 subgraph, we use paths that do not use the edge $v_nv_{\frac{n}{2}-1}$ and only use the exterior edges below the edge v_2v_{n-4} to ensure distinct path lengths.
    For each edge in set VI, when we list the shortest path distance to each C_3 subgraph, we find that each edge has a unique pair of distances. This is because the edge $v_nv_{\frac{n}{2}-1}$ splits C_n into two unequal subgraphs so that the path along the exterior of one side of the edge $v_nv_{\frac{n}{2}-1}$ to the top C_3 subgraph is different to the path along the exterior of the other side of the edge $v_nv_{\frac{n}{2}-1}$. This ensures that all edges in $E(\text{VI})$ are different.
- $E(\text{VII})$:
  - We differentiate between edges within this group by comparing distances from each edge to both C_3

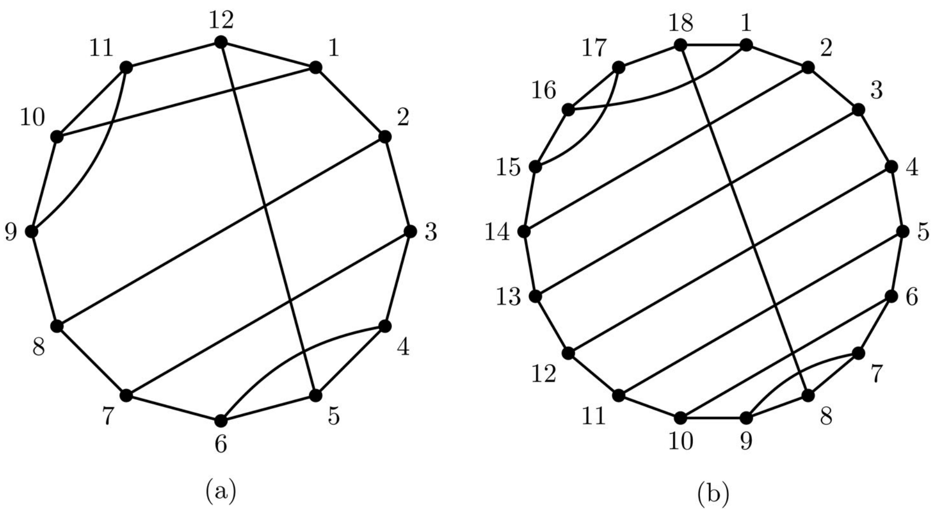

Figure 4. Asymmetric cubic Hamiltonian graphs.

subgraphs. Recall that edges below the edge v_2v_{n-4} have symmetry about the edge $v_nv_{\frac{n}{2}-1}$. Thus, when comparing distances from edges to each C_3 subgraph, we use paths that do not use the edge $v_nv_{\frac{n}{2}-1}$ and only use the exterior edges below the edge v_2v_{n-4} to ensure distinct path lengths. For each edge in Group VII, when we list the shortest path distance to each C_3 subgraph, we find that each edge has a distinct pair of distances. This is because the edge $v_nv_{\frac{n}{2}-1}$ splits C_n into two unequal subgraphs so that the path along the exterior of one side of the edge $v_nv_{\frac{n}{2}-1}$ to the top C_3 subgraph is different to the path along the exterior of the other side of the edge $v_nv_{\frac{n}{2}-1}$. This ensures that all edges in $E(\text{VII})$ are different.

In this way, the edges within each group are different from each other. Thus, every edge is different compared to all other edges in the graph. Consequently, the graph has a trivial automorphism group, and therefore, has no symmetry. In this way, we have shown that this infinite family of cubic Hamiltonian graphs on even $n \geq 12$ vertices has no symmetry.

4. Asymmetric quartic Hamiltonian graphs

We now present constructions of asymmetric quartic Hamiltoninan graphs on even $n \geq 12$ vertices. These constructions depend on congruency of n modulo 4.

Constructing the graph on $n \equiv 0 \mod 4$ vertices

(1) Construct the cubic Hamiltonian graph described in Figure 1(a) starting at C_n and label the vertices from 1 to n consecutively, traversing clockwise.

(2) Add the following edges:

$$v_{\frac{n}{4}}v_{\frac{3n}{4}} \quad \text{and} \quad v_{\frac{n}{4}+i}v_{(\frac{n}{4}-i) \bmod n} \quad \text{where } 1 \leq i \leq \frac{n}{2}-1$$

Constructing the graph on $n \equiv 2 \mod 4$ vertices

(1) Construct the cubic Hamiltonian graph described in Figure 4(b) starting at C_n and label the vertices from 1 to n consecutively, traversing clockwise.

(2) Add the following edges:

$$v_1v_{\frac{n}{2}+1} \quad \text{and} \quad v_{1+i}v_{(1-i) \bmod n} \quad \text{where } 1 \leq i \leq \frac{n}{2}-1$$

Examples of these graphs on $n = 12$ and $n = 18$ vertices are depicted in Figure 5.

We first show asymmetry for the cases of small n; that is, $n = 12, 14, 16, \text{and} 18$.

4.1. Quartic Hamiltonian graph on 12 vertices

We divide $V(G)$ into the following subsets:

- $V(\text{I}) = \{v_2, v_3, v_4\}$
  These vertices form a C_3 that is one away from two C_3s and shares a vertex with another C_3.
- $V(\text{II}) = \{v_1, v_{12}, v_5\}$
  These vertices form a C_3 that is one away from two C_3s and shares an edge with another C_3.
- $V(\text{III}) = \{v_{12}, v_5, v_6\}$
  These vertices form a C_3 that shares a common edge with two C_3s.
- $V(\text{IV}) = \{v_4, v_5, v_6\}$
  These vertices form a C_3 that shares vertices with three distinct C_3s.
- $V(\text{V}) = \{v_8, v_9, v_{10}\}$
  These vertices form a C_3 that is one away from two C_3s.
- $V(\text{VI}) = \{v_7, v_{11}\}$
  These vertices are not in any C_3s.

In this way, vertices in any given subset are different from vertices that are not in that same subset. In order to show that every vertex in $V(G)$ is unique, we must also show that each vertex in every subset is different from all other vertices within the same subset.

- $V(\text{I})$:
  - v_2 is distance one away from $V(\text{II})$.
  - v_3 is not distance one away from $V(\text{II})$
  - v_4 is distance one away from $V(\text{II})$ and in $V(\text{IV})$.

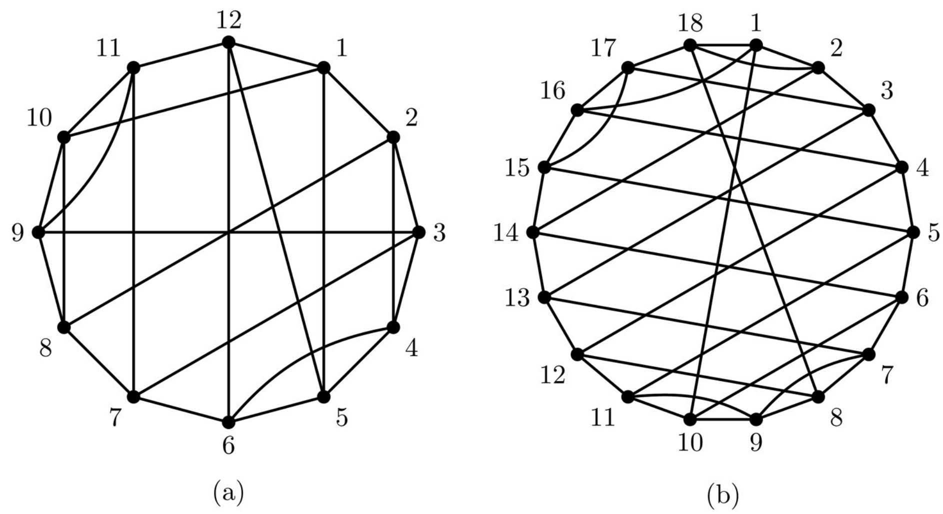

Figure 5. Asymmetric quartic Hamiltonian graphs.

- $V(\text{II})$:
  - v_{12} is in both $V(\text{III})$ and $V(\text{II})$.
  - v_5 is in $V(\text{III}), V(\text{IV})$ and $V(\text{II})$.
  - v_1 is in only $V(\text{II})$.
- $V(\text{III})$:
  - v_{12} is in $V(\text{II})$ and $V(\text{III})$.
  - v_5 is in $V(\text{II}), V(\text{IV})$ and $V(\text{III})$.
  - v_6 is in $V(\text{IV})$ and $V(\text{III})$.
- $V(\text{IV})$:
  - v_4 is distance one away from $V(\text{II})$ and in $V(\text{IV})$.
  - v_5 is in $V(\text{II}), V(\text{IV})$ and $V(\text{III})$.
  - v_6 is in $V(\text{IV})$ and $V(\text{III})$.
- $V(\text{V})$:
  - v_8 is distance two away from v_3, which we now know is unique.
  - v_9 is distance one away from v_3, which is unique.
  - v_{10} is distance one away from $V(\text{II})$.
- $V(\text{VI})$:
  - v_7 is distance one away from $V(\text{IV})$
  - v_{11} is distance two away from $V(\text{IV})$

Since we have shown that each subset of $V(\text{G})$ is different from all other subsets, and since we have described how vertices within each subset are unique, all vertices in $V(\text{G})$ are unique and the graph is asymmetric.

4.2. Quartic Hamiltonian graph on 16 vertices

We divide $V(G)$ into the following subsets:

- $V(\text{I}) = \{v_3, v_4, v_5\}$
  These vertices form a C_3 that is one away from two C_3s.
- $V(\text{II}) = \{v_{11}, v_{12}, v_{13}\}$
  These vertices form a C_3 that is distance one away from two C_3s and is in another C_3.
- $V(\text{III}) = \{v_7, v_8, v_{16}\}$
  These vertices form a C_3 that shares a common edge with two C_3s.
- $V(\text{IV}) = \{v_1, v_7, v_{16}\}$
  These vertices form a C_3 that is two away from two distinct C_3s.
- $V(\text{V}) = \{v_{13}, v_{14}, v_{15}\}$
  These vertices form a C_3 that shares a vertex, but does not share an edge with another C_3.
- $V(\text{VI}) = \{v_6, v_7, v_8\}$
  These vertices form a C_3 that shares a vertex with two distinct C_3s, shares an edge with one C_3, and is one away from a C_3 whose vertices are all in only one C_3.
- $V(\text{VII}) = \{v_2, v_9, v_{10}\}$
  These vertices are not in any C_3s.

In this way, vertices in any given subset are different from vertices that are not in that same subset. In order to show that every vertex in $V(\text{G})$ is unique, we must also show that each vertex in every subset is different from all other vertices within the same subset.

- $V(\text{III})$:
  - v_7 is in $V(\text{VI}), V(\text{IV}),$ and $V(\text{III})$.
  - v_8 is in both $V(\text{VI})$ and $V(\text{III})$.
  - v_{16} is not in $V(\text{VI})$.
- $V(\text{IV})$:
  - v_{16} is in $V(\text{III})$ and $V(\text{IV})$.
  - v_7 is in $V(\text{IV}), V(\text{VI}),$ and $V(\text{III})$.
  - v_1 is not in $V(\text{III})$ nor $V(\text{IV})$.
- $V(\text{V})$:
  - v_{13} is in both $V(\text{II})$ and $V(\text{V})$.
  - v_{14} is distance two away from $V(\text{III})$.
  - v_{15} is distance one away from $V(\text{III})$.
- $V(\text{VI})$:
  - v_8 is in both $V(\text{III})$ and $V(\text{VI})$.
  - v_7 is in $V(\text{IV}), V(\text{III}),$ and $V(\text{VI})$.
  - v_6 is not in $V(\text{IV})$ nor $V(\text{III})$.
- $V(\text{VII})$:
  - v_2 is distance one away from $V(\text{VI})$ and one away from $V(\text{IV})$.
  - v_9 is distance one away from $V(\text{VI})$ and one away from $V(\text{V})$.
  - v_{10} is distance two away from $V(\text{VI})$.
- $V(\text{I})$:
  - v_3 is distance one away from v_2, which we now know is unique.

- v_4 is distance two away from v_2, which we know is unique.
- v_5 is distance one away from $V(\text{VI})$.
- $V(\text{II})$:
  - v_{11} is distance two away from v_2, which we now know is unique.
  - v_{12} is distance one away from v_2, which we know is unique.
  - v_{13} is in both $V(\text{V})$ and $V(\text{II})$.

Since we have shown that each subset of $V(\text{G})$ is different from all other subsets, and since we have described how vertices within each subset are unique, all vertices in $V(\text{G})$ are unique and the graph is asymmetric.

4.3. Quartic Hamiltonian graph on 14 vertices

We distinguish between vertices as follows:

- The vertices $v_1, v_2, v_5, v_6, v_7, v_8, v_9, v_{11}, v_{12}, v_{13}, v_{14}$ are the only vertices contained in a C_3 subgraph:
  - v_7 is unique because it is the only vertex contained in two distinct C_3s.
  - v_2 and v_{11} are the only vertices contained in a C_3 that is not distance one away from another C_3.
    - v_2 is distance three away from the vertex v_7, while the vertex v_{11} is not.
- The vertices $v_1, v_6, v_8, v_{12}, v_{13}, v_5, v_9 \text{and} v_{14}$ are all contained in a C_3 and are one away from a C_3:
  - v_1 and v_{14} are adjacent to v_2.
    - v_1 is only contained in one C_4 whereas v_{14} is contained in two C_4s.
  - v_6, v_8, v_5, and v_9 are adjacent to the vertex v_7.
    - v_6 is adjacent to v_{14}, while v_8, v_5, and v_9 are not.
    - v_8 is adjacent to v_1, while v_6, v_5, and v_9 are not.
    - v_5 is distance two away from v_{14}, while v_9 is not.
  - v_{13} and v_{12} are not adjacent to v_7 nor v_2.
    - v_{13} is adjacent to v_{14}, while the vertex v_{12} is not.
- The vertices v_3, v_4 and v_{10} are not in a C_3:
  - v_3 is adjacent to v_{13}.
  - v_4 is adjacent to v_8.
  - v_{10} is adjacent to v_{11}.

In this way all vertices are unique and the graph is asymmetric.

4.4. Quartic Hamiltonian graph on 18 vertices

We construct a quartic Hamiltonian graph (as described in section 1.1). We distinguish between vertices as follows:

- Vertices $v_1, v_2, v_7, v_8, v_9, v_{10}, v_{11}, v_{15}, v_{16}, v_{17}$, and v_{18} are the only vertices contained in any C_3 subgraph:
  - The vertex v_9 is unique because it is the only vertex contained in two distinct C_3s.
  - The vertices v_2 and v_{15} are the only vertices contained in a C_3 that is not one away from another C_3.
    - v_2 is distance three away from the vertex v_9, while the vertex v_{15} is not.
- The vertices $v_1, v_8, v_{10}, v_{16}, v_{17}, v_7, v_{11} \text{and} v_{18}$ are all contained in a C_3 and are one away from a C_3:
  - v_1 and v_{18} are adjacent to v_2.
    - v_1 is only contained in one C_4 whereas v_{18} is contained in two C_4s.
  - v_8, v_{10}, v_7, and v_{11} are adjacent to the vertex v_9.
    - v_8 is adjacent to v_{18}, while v_{10}, v_7, and v_{11} are not.
    - v_{10} is adjacent to v_1, while v_8, v_7, and v_{11} are not.
    - v_7 is distance two away from v_{18}, while v_{11} is not.
  - v_{17} and v_{16} are not adjacent to v_9 nor v_2.
    - v_{17} is adjacent to v_{18}, while the vertex v_{16} is not.
- The vertices v_3, v_4, v_5, v_6, v_{12}, v_{13} and v_{14} are not in any C_3s and we distinguish between them as follows:
  - v_3: is adjacent to v_{17}, v_2, v_4 and v_{13}. We know v_{17} and v_2 are unique. v_4 is adjacent to v_{16} and v_{13} is not. Thus, v_{16} and v_{13} are unique, and we deduce that v_3 is unique.
  - v_4: is adjacent to v_{16}, v_3, v_5 and v_{12}. We know v_{16} and v_3 are unique. v_5 is adjacent to v_{15} and v_{12} is not. Thus, v_{15} and v_{12} are unique, and we deduce that v_4 is unique.
  - v_5: is adjacent to v_{15}, v_4, v_6 and v_{11}. We know v_{15}, v_4, and v_{11} are unique. Thus, we deduce that v_6 is unique. Therefore, v_5 is unique.
  - v_6: is adjacent to v_{14}, v_5, v_{10} and v_7, which are all unique. Thus, v_6 is unique.

In this way the vertices v_3, v_4, v_5, v_6, v_{12}, v_{13} and v_{14} are all unique. Therefore, all vertices are unique and the graph is asymmetric.

4.5. Quartic Hamiltonian for large n

We now show asymmetry for all $n \geq 20$.

4.5.1. When $n \equiv 0 \bmod 4$

In order to show that a quartic Hamiltonian graph on $n \geq 20, n \equiv 0 \bmod 4$ vertices has a trivial automorphism group (no symmetry), we must show that every vertex in the graph is unique. We do this by comparing every vertex in the graph to every other vertex. We begin by grouping vertices based on differentiating properties, so that vertices in each group are different from vertices in all other groups.

Vertices that are always contained in a 3-cycle

We find that the graph will always have six distinct C_3s, no matter how large n is. These six C_3s will always have the same distinct properties, making each C_3 different from every other C_3. The vertex sets for these C_3s and their distinguishing characteristics are listed below. It is important to note that the edge $v_{\frac{n}{4}} v_{\frac{3n}{4}}$ is unique because it is the only edge that connects two C_3s whose vertices are all in only one C_3, and all have a distance greater than one to another C_3.

- $V(\text{I}) = \{v_{\frac{n}{2}}, v_{\frac{n}{2}-1}, v_{\frac{n}{2}-2}\}$
  These vertices form a C_3 that shares a common vertex with another triangle and shares a common edge with

another triangle. It is also at least distance two from any other triangle.

- $V(\mathrm{II}) = \{v_n, v_1, v_{\frac{n}{2}-1}\}$
  These vertices form a C_3 that is distance one away from a C_3 whose vertices are all in only one C_3.
- $V(\mathrm{III}) = \{v_n, v_{\frac{n}{2}}, v_{\frac{n}{2}-1}\}$
  These vertices form a C_3 that shares a common edge with two other triangles.
- $V(\mathrm{IV}) = \{v_{n-1}, v_{n-2}, v_{n-3}\}$
  These vertices form a C_3 that are distance one from vertices in two triangles.
- $V(\mathrm{V}) = \{v_{\frac{n}{4}}, v_{\frac{n}{4}-1}, v_{\frac{n}{4}+1}\}$
  These vertices form a C_3 that, compared to $V(\mathrm{VI}), V(\mathrm{V})$, has a greater number of distinct shortest paths to vertices in $V(\mathrm{I})$, using the special edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$ first.
- $V(\mathrm{VI}) = \{v_{\frac{3n}{4}}, v_{\frac{3n}{4}-1}, v_{\frac{3n}{4}+1}\}$
  These vertices form a C_3 that, compared to $V(\mathrm{V}), V(\mathrm{VI})$, has a smaller number of distinct shortest paths to vertices in $V(\mathrm{I})$, using the special edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$ first.

In this way, any vertex in any C_3 is different from all other vertices in any other C_3. Now we must show that each vertex in each C_3 is different from the vertices within the same C_3.

- $V(\mathrm{I})$:
  - $v_{\frac{n}{2}}$ is contained in two C_3s.
  - $v_{\frac{n}{2}-1}$ is contained in three C_3s.
  - $v_{\frac{n}{2}-2}$ is contained in one C_3.
- $V(\mathrm{II})$:
  - v_n is distance two away from v_2
  - v_1 is distance one away from v_2
  - $v_{\frac{n}{2}-1}$ is contained in 3 C_3s
- $V(\mathrm{III})$:
  - v_n is distance two away from v_2
  - $v_{\frac{n}{2}}$ is distance three away from v_2
  - $v_{\frac{n}{2}-1}$ is contained in 3 C_3s.
- $V(\mathrm{IV})$: Each vertex in this vertex set is differentiated by a unique ordered pair (x, y), where x is the distance to $v_{\frac{n}{2}-1}$ and y is the distance to v_2
  - v_{n-1}: (2, 3)
  - v_{n-2}: (2, 2)
  - v_{n-3}: (3, 2)
- $V(\mathrm{V})$:
  - $v_{\frac{n}{4}}$ is adjacent to the special edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$
  - $v_{\frac{n}{4}-1}$ is always has a greater distance to v_2 than $v_{\frac{n}{4}+1}$
  - $v_{\frac{n}{4}+1}$ is always closer to v_2 than $v_{\frac{n}{4}-1}$.
- $V(\mathrm{VI})$:
  - $v_{\frac{3n}{4}}$ is adjacent to the special edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$
  - $v_{\frac{3n}{4}-1}$ is always has a greater distance to v_2 than $v_{\frac{3n}{4}+1}$
  - $v_{\frac{3n}{4}+1}$ is always closer to v_2 than $v_{\frac{3n}{4}-1}$

Vertices that are not contained in a 3-cycle

We can divide the vertices that are not contained in a 3-cycle into two sets.

- $V(\mathrm{VII}) = \{v_2, v_3, v_4, v_{\frac{n}{2}-4}, v_{\frac{n}{2}-3}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}\}$
  - v_2 is distance one away from $V(\mathrm{I})$ and one away from $V(\mathrm{II})$.
  - v_3 is distance two away from $V(\mathrm{II})$, distance three away from $V(\mathrm{IV})$ and distance one away from v_2.
  - v_4 is distance four away from $V(\mathrm{III})$ and distance three away from $V(\mathrm{IV})$.
  - $v_{\frac{n}{2}-4}$ is distance two away from $V(\mathrm{I})$, and distance two away from $V(\mathrm{IV})$.
  - $v_{\frac{n}{2}-3}$ is distance one away from $V(\mathrm{I})$.
  - $v_{\frac{n}{2}+1}$ is distance one away from $V(\mathrm{I})$ and distance one away from $V(\mathrm{IV})$.
  - $v_{\frac{n}{2}+2}$ is distance one away from $V(\mathrm{IV})$, and distance two away from $V(\mathrm{II})$.
  - $v_{\frac{n}{2}+3}$ is distance one away from $V(\mathrm{IV})$, and distance three away from $V(\mathrm{II})$.
  In this way the vertices in Set VII are all unique.
- $V(\mathrm{VIII})$: Vertices that are added as the graph gets larger. We can divide the remaining vertices, or the vertices that are added as the graph becomes larger, into the following vertex sets:
  - $A = \{v_x \in G : 4 < x < \frac{n}{4}\}$
  - $B = \{v_x \in G : \frac{n}{4} < x < \frac{n}{2} - 4\}$
  - $C = \{v_x \in G : \frac{n}{2} + 3 < x < \frac{3n}{4}\}$
  - $D = \{v_x \in G : \frac{3n}{4} < x < n - 3\}$

In order to show that vertices in $V(A)$, $V(B)$, $V(C)$, and $V(D)$ are unique, we must compare vertices in each vertex set to vertices in every other vertex set. We must also compare every vertex in each set to all other vertices within that same set.

- Comparing Sets A and B to Sets C and D:
  When comparing a vertex in set $A \cup B$ to a vertex in set $C \cup D$, a vertex in $A \cup B$ has either the same distance to the vertex $v_{\frac{n}{4}}$ as a vertex in $C \cup D$ has to $v_{\frac{3n}{4}}$, or a vertex in $A \cup B$ has a different distance to the vertex $v_{\frac{n}{4}}$ as a vertex in $C \cup D$ has to $v_{\frac{3n}{4}}$. As a result, we have two cases.
  - Case I: A vertex in $A \cup B$ is the same distance to the vertex $v_{\frac{n}{4}}$, as a vertex in $C \cup D$ is to $v_{\frac{3n}{4}}$.
    Recall that the vertex $v_{\frac{n}{4}}$ is different from the vertex $v_{\frac{3n}{4}}$ because the vertex $v_{\frac{n}{4}}$ has a greater number of shortest paths to $V(\mathrm{I})$ that begin with the edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$, compared to the vertex $v_{\frac{3n}{4}}$. (Note that the edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$ is unique because it connects two C_3s whose vertices are all in only one C_3, and all have a distance greater than one to another C_3). When comparing two vertices that are equidistant from $v_{\frac{n}{4}}$ and $v_{\frac{3n}{4}}$ respectively, we say that each vertex is i away from $v_{\frac{n}{4}}$ and $v_{\frac{3n}{4}}$, respectively. Let $i = |\frac{n}{4} - x|$. Any v_x distance i away from $v_{\frac{n}{4}}$ or $v_{\frac{3n}{4}}$ will have a path to $V(\mathrm{I})$ that passes through the edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$ in the $i+1$ position of the path. If we ensure that the edge $v_{\frac{n}{4}}, v_{\frac{3n}{4}}$ is used at the $i+1$ position of the path to $V(\mathrm{I})$, we know that a vertex in $A \cup B$ will have a greater number of shortest paths that use the edge $v_{\frac{n}{4}}v_{\frac{3n}{4}}$ at the $i+1$ position of the path to $V(\mathrm{I})$, than the vertices in $C \cup D$. In this way, vertices in $A \cup B$ that are i away from $v_{\frac{n}{4}}$ are unique from vertices in $C \cup D$ that are i away from $v_{\frac{3n}{4}}$.
  - Case II: A vertex in $A \cup B$ is a different distance from the vertex $v_{\frac{n}{4}}$, than a vertex in $C \cup D$ is to $v_{\frac{3n}{4}}$.

Let a vertex in $C \cup B$ be distance l from $v_{\frac{3n}{4}}$, where $l = |\frac{3n}{4} - x|$, and let a vertex in $A \cup B$ be distance k from $v_{\lceil \frac{n}{4} \rceil}$, where $k = |\frac{3n}{4} - x|$, and $k \neq l$. A vertex k from $v_{\lceil \frac{n}{4} \rceil}$ has a shortest path to $V(\mathrm{I})$ that uses the edge $v_{\frac{n}{4}} v_{\frac{3n}{4}}$ at the $k+1$ position of the path to $V(\mathrm{I})$. A vertex l from $v_{\frac{3n}{4}}$ has a shortest path to $V(\mathrm{I})$ that uses the edge $v_{\frac{n}{4}} v_{\frac{3n}{4}}$ at the $l+1$ position of the path to $V(\mathrm{I})$. Since $k \neq l$, the vertices being compared will use the edge $v_{\frac{n}{4}} v_{\frac{3n}{4}}$ in their path to $V(\mathrm{I})$ at different times. Since the edge $v_{\frac{n}{4}} v_{\frac{3n}{4}}$ is unique, and since the vertices in $A \cup B$ use the edge $v_{\frac{n}{4}} v_{\frac{3n}{4}}$ to get to $V(\mathrm{I})$ at a different time than the vertices in $C \cup B$, the vertices in $A \cup B$ that are k away from $v_{\frac{n}{4}}$ are unique from vertices in $C \cup D$ that are l away from $v_{\frac{3n}{4}}$.

In this way, vertices in $A \cup B$ are unique from vertices in set $C \cup D$.

- Comparing vertices in Set A to Set B:
  The edge $v_4 v_{\frac{n}{2}-4}$ is unique because v_4 and $v_{\frac{n}{2}-4}$ are unique vertices: v_4 is always distance four from $V(\mathrm{III})$ and is always distance three from $V(\mathrm{IV})$, and $v_{\frac{n}{2}-4}$ is always distance two from $V(\mathrm{I})$ and is always distance two from $V(\mathrm{IV})$. We compare the vertices in Set A to the vertices in Set B by comparing the path from v_x to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_4 v_{\frac{n}{2}-4}$. We find that the path from any vertex v_x in Set A to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_4 v_{\frac{n}{2}-4}$, is always i edges short, where $i = x - 3$, from a $2i$ cycle. Whereas, the path from any vertex v_x in Set B to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_4 v_{\frac{n}{2}-4}$, is always j edges short, where $j = \frac{n}{2} - 4 - x$, from a $2j + 2$ cycle. When comparing vertices in Set A to vertices in Set B we assign an ordered pair (r, s) to each v_x in each set, where s is the number of edges that v_x is short of an r-cycle. In Set A each v_x is assigned the ordered pair $(2i, i)$ and in Set B each v_x is assigned the ordered pair $(2j + 2, j)$. We will show that a vertex in Set A will never have the same ordered pair as a vertex in set B. There are two cases:
  - Case 1: $i = j$
    When $i = j$, then $(2i, i)$ and $(2j + 2, j)$ will never be the same, as if $i = j$, we know $2i \neq 2j + 2$. Thus, the $(2i, i) \neq (2j + 2, j)$, and the vertices in Set A are unique from the vertices in Set B.
  - Case 2: $i \neq j$
    If $i \neq j$, then $(2i, i)$ and $(2j + 2, j)$ are different. In this way the vertices in Set A are always unique from the vertices in Set B.

  In this way the vertices in Set A are always unique from the vertices in Set B.
- Comparing vertices within Set A:
  We compare a vertex v_x in Set A to all other vertices within Set A by studying the path from v_x to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_4 v_{\frac{n}{2}-4}$. This path will always be i edges short, where $i = x - 3$, from a $2i$ cycle. Every v_x in Set A has a unique i value because $i = x - 3$ and every vertex has a unique x value assigned. Thus, every vertex in Set A is unique, as every vertex is short of a cycle by a unique number of edges. In this way, every vertex in Set A is unique.
- Comparing vertices within Set B:
  We compare a vertex v_x in Set B to all other vertices within Set B by studying the path from v_x to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_4 v_{\frac{n}{2}-4}$. This path will always be j edges short of a cycle, where $j = \frac{n}{2} - 4 - x$, from a $2j + 2$ cycle. Every v_x in Set B has a unique j value because $j = \frac{n}{2} - 4 - x$ and every vertex has a uniquet x value assigned. Thus, every vertex in Set B is unique, as every vertex is short of a cycle by a unique number of edges. In this way, every vertex in Set B is unique.
- Comparing vertices in Set D to Set C:
  The edge $v_{n-3} v_{\frac{n}{2}+3}$ is unique because v_{n-3} and $v_{\frac{n}{2}+3}$ are unique vertices: v_{n-3} is in $V(\mathrm{IV})$ and distance three from n, and $v_{\frac{n}{2}+3}$ is always distance one from $V(\mathrm{IV})$ and is always distance three from $V(\mathrm{II})$. We compare the vertices in Set D to the vertices in Set C by comparing the path from v_x to the vertex $v_{\frac{n}{2}+3}$ that uses the edge $v_{n-3} v_{\frac{n}{2}+3}$. We find that the path from any vertex v_x in Set D to the vertex $v_{\frac{n}{2}+3}$, that uses the edge $v_{n-3} v_{\frac{n}{2}+3}$, is always m edges short, where $m = n - x - 2$, from a $2m$ cycle. Whereas, the path from any vertex v_x in Set C to the vertex $v_{\frac{n}{2}+3}$, that uses the edge $v_{n-3} v_{\frac{n}{2}+3}$, is always q edges short, where $q = x - \frac{n}{2} + 3$, from a $2q + 2$ cycle. When comparing vertices in Set D to vertices in Set C we assign an ordered pair (r, s) to each v_x in each set, where s is the number of edges that v_x is short of an r-cycle. In Set C each v_x is assigned the ordered pair $(2q + 2, q)$ and in Set D each v_x is assigned the ordered pair $(2m, m)$. We will show that a vertex in Set D will never have the same ordered pair as a vertex in set C. There are two cases:
  - Case 1: $m = q$
    When $m = q$, then $(2m, m)$ and $(2q + 2, q)$ will never be the same, as if $m = q$, we know $2m \neq 2q + 2$. Thus, the $(2m, m) \neq (2q + 2, q)$, and the vertices in Set D are unique from the vertices in Set C.
  - Case 2: $m \neq q$
    If $m \neq q$, then $(2m, m)$ and $(2q + 2, q)$ are different. In this way the vertices in Set D are always unique from the vertices in Set C.

  In this way the vertices in Set D are always unique from the vertices in Set C.
- Comparing vertices within Set C:
  We compare a vertex v_x in Set C to all other vertices within Set C by studying the path from v_x to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_{n-3} v_{\frac{n}{2}+3}$. This path will always be m edges short, where $m = n - x - 2$, from a $2m$ cycle. Every v_x in Set C has a unique m value because $m = n - x - 2$ and every vertex has a unique x value assigned. Thus, every vertex in Set C is unique, as every vertex is short of a cycle by a unique number of edges. In this way, every vertex in Set C is unique.
- Comparing vertices within Set D:
  We compare a vertex v_x in Set D to all other vertices within Set D by studying the path from v_x to the vertex $v_{\frac{n}{2}-4}$, that uses the edge $v_{n-3} v_{\frac{n}{2}+3}$. This path will always be q edges short, where $q = x - \frac{n}{2} + 3$, from a $2q + 2$ cycle. Every v_x in Set D has a unique q value because

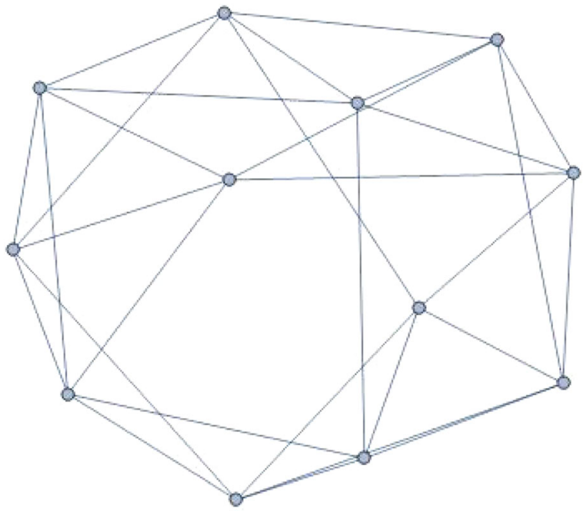

Figure 6. An asymmetric 5-regular Hamiltonian graph with 12 vertices.

$q = x - \frac{n}{2} + 3$ and every vertex has a unique x value assigned. Thus, every vertex in Set D is unique, as every vertex is short of a cycle by a unique number of edges. In this way, every vertex in Set D is unique.
In this way, any vertex in any set A, B, C, and D, is unique from all other vertices.

In this way, all vertices are unique in this construction of a quartic Hamiltonian graph on any $n \geq 20, n \equiv 0 \mod 4$. Therefore, the graph is asymmetric.

4.5.2. When $n \equiv 2 \mod 4$

Vertices that are always contained in a 3-cycle

Vertices $v_1, v_2, v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{n-3}, v_{n-2}, v_{n-1}, v_n$ are the only vertices contained in any C_3 subgraph.

- The vertex $v_{\frac{n}{2}}$ is unique because it is the only vertex contained in two distinct C_3s.
- The vertices v_2 and v_{n-3} are the only vertices contained in a C_3 that is not distance one away from another C_3.
  - v_2 is distance three away from the vertex $v_{\frac{n}{2}}$, while the vertex v_{n-3} is not.
- The vertices $v_1, v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}, v_{n-2}, v_{n-1}, v_{\frac{n}{2}-2}, v_{\frac{n}{2}+2}$ and v_n are all contained in a C_3 and are one away from a C_3.
  - v_1 and v_n are adjacent to v_2.
    - v_1 is only contained in one C_4 whereas v_n is contained in two C_4s.
  - $v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}-2}$, and $v_{\frac{n}{2}+2}$ are adjacent to the vertex $v_{\frac{n}{2}}$.
    - $v_{\frac{n}{2}-1}$ is adjacent to v_n, while $v_{\frac{n}{2}+1}, v_{\frac{n}{2}-2}$, and $v_{\frac{n}{2}+2}$ are not.
    - $v_{\frac{n}{2}+1}$ is adjacent to v_1, while $v_{\frac{n}{2}-1}, v_{\frac{n}{2}-2}$, and $v_{\frac{n}{2}+2}$ are not.
    - $v_{\frac{n}{2}-2}$ is distance two away from v_n, while $v_{\frac{n}{2}+2}$ is not.
  - v_{n-1} and v_{n-2} are not adjacent to $v_{\frac{n}{2}}$ nor v_2.
  - v_{n-1} is adjacent to v_n, while the vertex v_{n-2} is not.

Vertices that are not contained in a 3-cycle

Let any vertex v_k such that $n-6 \leq k \leq 6, \frac{n}{2} - 5 \leq k \leq \frac{n}{2} + 4$ be an element of $\mathcal{U}$.

In order to show that the vertices, $6 < x < \frac{n}{2} - 5$ and $\frac{n}{2} + 4 < x < n - 6$, are unique, we begin with a vertex v_x and check to see if its adjacent vertices are elements of $\mathcal{U}$. If all of the vertices adjacent to v_x are unique, we can say that the vertex is unique and conclude it belongs to $\mathcal{U}$.

We begin with the vertex v_{6+t}, where $t = 0$ and visit the vertices adjacent to v_{6+t}, that is: $v_{5+t}, v_{7+t}, v_{n-8-t}$, and v_{n-4-t}. We check if the adjacent vertices are elements of $\mathcal{U}$. If at least two of these vertices belong to $\mathcal{U}$, then compare the distance from each of the vertices to v_{6+t-3}. We find that the shortest path from v_{7+t} to v_{6+t-3} is always two, and the shortest path from v_{n-8-t} to v_{6+t-3} is always four. Thus, the vertices are different, as they have different distances to v_{6+t-3}. We can now conclude the vertices $v_{5+t}, v_{7+t}, v_{n-8-t}$, and v_{n-4-t} belong to $\mathcal{U}$.

If there are fewer than two vertices belonging to $\mathcal{U}$, then check the distance from each vertex to v_{6+t-3}. We find that the shortest paths from the vertices v_{7+t}, v_{n-8-t}, and v_{n-4-t} to the vertex v_{6+t-3} are, two, four, and two, respectively. Since two vertices have the same distance to v_{6+t-3}, we compare their distances to v_{6+t-4}. Since the shortest path distance from v_{7+t} to v_{6+t-4} is always three, and since the shortest path from v_{n-4-t} to v_{6+t-4} is always one, the vertices are unique, as each vertex has a unique distance to both v_{6+t-3} and v_{6+t-4}. We can now say the vertices $v_{5+t}, v_{7+t}, v_{n-8-t}$, and v_{n-4-t} belong to $\mathcal{U}$.

We then add one to t and check the adjacent vertices as described, until we reach the vertex $v_{\frac{n}{2}-6}$. We know the vertex $v_{\frac{n}{2}-6}$ is unique because all of its adjacent vertices are elements of $\mathcal{U}$. At this point, the cardinality of $\mathcal{U}$ is equal to the number of vertices in the graph, which implies the graph is asymmetric.

5. Conclusion

We constructed and presented an infinite family of cubic and quartic Hamiltonian graphs of even orders, starting at $n = 12$. We note that the complement of an asymmetric 3-regular Hamiltonian graph on n vertices is an asymmetric $(n-4)$-regular graph. An application of Dirac's Theorem shows that this graph is Hamiltonian. This covers all even $n \geq 8$. The complement of an asymmetric 4-regular Hamiltonian graph on n vertices is an asymmetric $(n-5)$-regular graph. An application of Dirac's Theorem shows that this graph is Hamiltonian. This covers all odd $n \geq 7$.

In Figure 6, we present an asymmetric 5-regular Hamiltonian graph with 12 vertices. The complement of this graph will be an asymmetric 6-regular graph on 12 vertices. It follows by Dirac's Theorem that this graph is Hamiltonian.

We established the existence of infinite families of k-regular asymmetric Hamiltonian graphs for $k = 3$ and $k = 4$. It would be interesting to determine for which $k > 4$ there exists an infinite family of asymmetric k-regular Hamiltonian graphs.

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