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The cost of edge removal in graph domination

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ABSTRACT

A vertex set D of a graph G is a dominating set of G if each vertex of G is a member of D or is adjacent to a member of D. The domination number of G, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of G. In this paper two cost functions, $d_q(G)$ and $D_q(G)$, are considered which measure respectively the smallest possible and the largest possible increase in the cardinality of a dominating set, over and above $\gamma(G)$, if q edges were to be removed from G. Bounds are established on $d_q(G)$ and $D_q(G)$ for a general graph G, after which these bounds are sharpened or these parameters are determined exactly for a number of special graph classes, including paths, cycles, complete bipartite graphs and complete graphs.

KEYWORDS

Graph domination; edge removal; criticality

Let $G = (V, E)$ be a simple graph of order n. A set $D \subseteq V$ is a *dominating set* of G if each vertex of G is a member of D or is adjacent to G. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$.

Applications of the notion of domination abound: If the vertices of the graph G denote geographically dispersed facilities, and the edges model links between these facilities along which guards have line of sight, then a dominating set of G represents a collection of facility locations at which guards may be placed so that the entire complex of facilities modelled by G is protected (in the sense that if a security problem were to occur at facility u, there will either be a guard at that facility who can deal with the problem, or else a guard dealing with the problem from an adjacent facility v can signal an alarm due the visibility that exists between adjacent locations). In this application, the domination number represents the minimum number of guards required to protect the facility complex.

1. Edge removal

In applications conforming to the scenario described above one might seek the cost (in terms of the additional number of guards required over and above the minimum $\gamma(G)$ to protect an entire location complex G in the dominating sense) if a number of edges of G were to "fail" (i.e. a number of links were to be eliminated from the graph so that the guards no longer have vision along such disabled links).

In this paper, the notation $G - qe$ is used to denote the set of all non-isomorphic graphs obtained by removing $0 \leq q \leq m$ edges from a given graph G of size m. Furthermore, $\gamma(G - qe)$ denotes the set of values of $\gamma(H)$ as $H \in G - qe$ varies (for a fixed value of q). Walikar and Acharya [7, Proposition 2] were the first to note the following result.

Proposition 1. *Let G be any graph and e any edge of G. Then it follows that*

$$\gamma(G) \leq \gamma(G - e) \leq \gamma(G) + 1.$$

□

The following result follows immediately from Proposition 1.

Corollary 1 (Edge removal increases domination requirements). *For any graph G that is not edgeless* $\gamma(G) \leq \min \gamma(G - e) \leq \max \gamma(G - e) \leq \gamma(G) + 1$. □

The cost functions

$$d_q(G) = \min \gamma(G - qe) - \gamma(G)$$
$$D_q(G) = \max \gamma(G - qe) - \gamma(G)$$

are non-negative in view of Corollary 1 and measure respectively the *smallest possible* and the *largest possible* increase in the minimum number of guards required to dominate a member of $G - qe$, over and above the minimum number of guards required to dominate G, in the event that an arbitrary set of $0 \leq q \leq m$ edges are removed from G. Furthermore, cost sequences $\mathbf{d}(G) = d_0(G), d_1(G), d_2(G), \ldots, d_m(G)$ and $\mathbf{D}(G) = D_0(G), D_1(G), D_2(G), \ldots, D_m(G)$ can be constructed for any graph G.

The cost functions $d_q(G)$ and $D_q(G)$ were first introduced by Burger et al. [2] for the domination related parameter *secure domination*. For a graph G with secure domination number $\gamma_s(G)$ it follows that $\gamma(G) \leq \gamma_s(G)$ [3, Proposition 1].

Van Vuuren [6] studied the notion of *q-criticality* in a graph G. A graph G is q-critical if q is the smallest number of arbitrary edges of G whose removal from G necessarily

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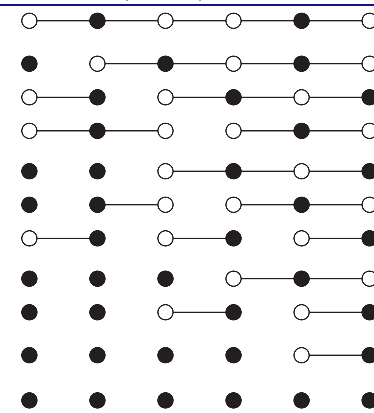
Table 1. The costs $d_q(P_6)$ and $D_q(P_6)$ for the path P_6.

q	$P_6 - qe$	γ	$d_q(P_6)$	$D_q(P_6)$	Graphical representation
0	P_6	2	0	0	
1	$P_1 \cup P_5$	3	0	1	
	$P_2 \cup P_4$	3			
	$2P_3$	2			
2	$2P_1 \cup P_4$	4	1	2	
	$P_1 \cup P_2 \cup P_3$	3			
	$3P_2$	3			
3	$3P_1 \cup P_3$	4	2	2	
	$2P_1 \cup 2P_2$	4			
4	$4P_1 \cup P_2$	5	3	3	
5	$6P_1$	6	4	4	



increases the domination number of the resulting graph. In this paper the cost sequence $\boldsymbol{d}(G)$ consequently produces the q-criticality of a graph G when $d_q(G) > 0$, but $d_{q-1}(G) = 0$. The notion of q-criticality have also been studied for other related graph parameters such as secure domination [4].

Proposition 2 (Cost function q-growth properties). *If G is a graph of size m and $0 \le q < m$, then*

(a) $d_q(G) \le d_{q+1}(G) \le d_q(G) + 1$, and
(b) $D_q(G) \le D_{q+1}(G) \le D_q(G) + 1$.

Proof: (a) By applying the result of Proposition 1 to each element of $G - qe$, it follows that

$$\begin{aligned} d_{q+1}(G) &= \min\{\gamma(G-(q+1)e)\} - \gamma(G) \\ &= \min\{\gamma((G-qe)-e)\} - \gamma(G) \\ &\ge \min\{\gamma(G-qe)\} - \gamma(G) \\ &= d_q(G), \end{aligned}$$

which establishes the first inequality. The second inequality holds because the domination number of a graph cannot increase by more than 1 if a single edge is removed from the graph by Proposition 1. The proof of part (b) is similar. □

The cost functions $d_q(P_6)$ and $D_q(P_6)$ are evaluated in Table 1 for the path P_6 of order 6 for all $0 \le q \le 5$. (These results may be verified by recalling from [3, Theorem 12] that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.)

2. General bounds on the cost sequences

The following general bounds hold with respect to the sequences $\boldsymbol{d}(G)$ and $\boldsymbol{D}(G)$ for any graph G.

Theorem 1. *For any graph G of order n and size m,*

$$n - m + q - \alpha(G) \le d_q(G) \le D_q(G) \le q.$$

Proof: It follows by Berge [1, Proposition 1, p. 304] that

$$\gamma(G) \ge n - m \tag{2.1}$$

for any graph G of order n and size m. Furthermore, from Haynes et al. [5] the independence number $\alpha(G)$ of a graph G is an upper bound on the domination number of G. Therefore

$$\gamma(G) \le \alpha(G) \tag{2.2}$$

for any graph G. It follows by (.1) and (.) that

$$d_q(G) = \min\{\gamma(G-qe)\} - \gamma(G) \ge n - (m-q) - \alpha(G).$$

Finally, by applying the result of Proposition 2(b) q times, it follows that $D_q(G) \le q$. □

The bounds in Theorem 1 are sharp; they are attained by taking G to be the vertex disjoint union of paths of order 1 and 2 (in which case $\alpha(G) = n - m$).

3. Special graph classes

In this section exact values of or bounds on the sequences $\boldsymbol{d}(G)$ and $\boldsymbol{D}(G)$ are established for a number of special classes of graphs, including paths, cycles, complete bipartite graphs and complete graphs.

3.1. Paths and cycles

In this section P_n and C_n denote a path and a cycle of order n, respectively. It follows by Theorem 1 that

$$1 + q - \left\lceil \frac{n}{2} \right\rceil \le d_q(P_n) \le D_q(P_n) \le q$$

for all $n \ge 2$ and $0 \le q \le n-1$, by noting that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. However, these bounds are weak, especially for small values of q. In this section the sequences $\boldsymbol{d}(P_n)$ and $\boldsymbol{D}(P_n)$ are determined exactly and these results are used to derive the sequences $\boldsymbol{d}(C_n)$ and $\boldsymbol{D}(C_n)$. For this purpose the following basic result is required.

Lemma 1.
(a) For $n \ge 4$ and any $1 \le k < n, \gamma(P_k \cup P_{n-k}) \ge \gamma(P_3 \cup P_{n-3})$.
(b) For $n \ge 5$ and any $1 \le k < n, \gamma(P_k \cup P_{n-k}) \le \gamma(P_4 \cup P_{n-4})$.

Proof: (a) Suppose $n \ge 4$ and let k be any positive integer not exceeding $n - 1$. Then

$$\begin{aligned}\gamma(P_k)+\gamma(P_{n-k}) &= \left\lceil\frac{k}{3}\right\rceil+\left\lceil\frac{n-k}{3}\right\rceil\\ &\geq \left\lceil\frac{n}{3}\right\rceil\\ &= 1+\left\lceil\frac{n}{3}-1\right\rceil\\ &= 1+\left\lceil\frac{n-3}{3}\right\rceil\\ &= \gamma(P_3)+\gamma(P_{n-3})\end{aligned}$$

by means of the identity $\lceil a\rceil+\lceil b-a\rceil\geq\lceil b\rceil$ for any $a,b\in\mathbb{R}$.
(b) Suppose $n\geq 5$ and let k be any positive integer not exceeding $n-1$. Then

$$\begin{aligned}\gamma(P_k)+\gamma(P_{n-k}) &= \left\lceil\frac{k}{3}\right\rceil+\left\lceil\frac{n-k}{3}\right\rceil\\ &= \left\lfloor\frac{k}{3}+\frac{2}{3}\right\rfloor+\left\lfloor\frac{n-k}{3}+\frac{2}{3}\right\rfloor\\ &\leq \left\lfloor\frac{n}{3}+\frac{2}{3}+\frac{2}{3}\right\rfloor\\ &= \left\lceil\frac{n}{3}+\frac{2}{3}\right\rceil\\ &= \left\lceil\frac{n+6-4}{3}\right\rceil\\ &= \left\lceil\frac{4}{3}\right\rceil+\left\lceil\frac{n-4}{3}\right\rceil\\ &= \gamma(P_4)+\gamma(P_{n-4})\end{aligned}$$

by (three times) using the identity $\left\lceil\frac{a}{b}\right\rceil=\lfloor\frac{a}{b}+\frac{b-1}{b}\rfloor$ for any $a,b\in\mathbb{R}$ with $b\neq 0$. □

The following intermediate results are also required.

Lemma 2. *Suppose $E,F\in P_n-qe$ respectively minimise and maximise $\gamma(P_n-qe)$.*

(a) If $2q\leq n\leq 3q$, then $E\cup P_2$ minimises $\gamma(P_{n+2}-(q+1)e)$.
(b) If $3q<n$, then $E\cup P_3$ minimises $\gamma(P_{n+3}-(q+1)e)$.
(c) If $n-3\leq q\leq n-1$, then $F\cup P_1$ maximises $\gamma(P_{n+1}-(q+1)e)$.
(d) If $q<n-3$, then $F\cup P_4$ maximises $\gamma(P_{n+4}-(q+1)e)$.

Proof: (a) By contradiction. Suppose $2q\leq n\leq 3q$ and that $G\in P_{n+2}-(q+1)e$ minimises $\gamma(P_{n+2}-(q+1)e)$, but that $\gamma(G)<\gamma(E\cup P_2)$. Then G contains no component isomorphic to P_2. It is next shown that it may be assumed that G is isolate-free. Since $\gamma(P_i)\leq\gamma(P_{i+1})$ for all $i\in\mathbb{N}$, it follows that $\gamma(P_2\cup P_\ell)\leq\gamma(P_1\cup P_{\ell+1})$. This means that if G were to contain a component of order 1, then G would have no component of order $i\geq 2$. But if G is the empty graph of order $n+2$, then $q=n+1$, which contradicts the supposition that $n\geq 2q$. Furthermore, G can have at most one component of order 3, since $\gamma(P_3\cup P_3)=2>3=\gamma(P_4\cup P_2)$. But then the order of G is $n+2>3(q+2)$, which contradicts the supposition that $n\leq 3q$.
(b) By contradiction. Suppose $3q<n$ and that $G\in P_{n+3}-(q+1)e$ minimises $\gamma(P_{n+3}-(q+1)e)$, but that $\gamma(G)<\gamma(E\cup P_3)$. Then G contains no component of order 3 and it follows by Lemma 1(a) that no two components of G together have more than three vertices. It is therefore assumed that $G\cong xP_2\cup yP_1$. By evaluating the number of components and the number of vertices of G, it follows that $x+y=q+2$ and $2x+y=n+3$, respectively. The unique solution to this simultaneous system of equations is $x=n-q+1$ and $y=2q-n+1$. Since $y\geq 0$ it follows that $2q\geq n-1$, contradicting the supposition.
(c) By contradiction. Suppose $n-3\leq q\leq n-1$ and that $H\in P_{n+1}-(q+1)e$ maximises $\gamma(P_{n+1}-(q+1)e)$, but that $\gamma(H)>\gamma(F\cup P_1)$. Then H is isolate-free and $\delta(H)\geq 2$. But then the order of H is $n+1>2(q+2)$, which contradicts the supposition that $n\leq q+3$.
(d) By contradiction. Suppose $q<n-3$ and that $H\in P_{n+4}-(q+1)e$ maximises $\gamma(P_{n+4}-(q+1)e)$, but that $\gamma(H)>\gamma(F\cup P_4)$. Then H contains no component of order 4 and it follows by Lemma 1(b) that no two components of H together have more than four vertices. Furthermore, the equality $\gamma(2P_2)=2=\gamma(P_3\cup P_1)$ show that there is at least one member of $P_{n+4}-(q+1)e$ which maximises $\gamma(P_{n+4}-(q+1)e)$ and which has at most one component which is not an isolate. It is therefore assumed that $G\cong P_i\cup xP_1$ for some $i\in\{2,3\}$. By evaluating the number of components and the number of vertices of H, it follows that $x+1=q+2$ and $x+i=n+4$, respectively, which together imply that $n=q+i-3$. However, this equality contradicts the supposition that $q<n-3$ for $i=2$, 3. □

It is now possible to establish the sequences $\boldsymbol{d}$ and $\boldsymbol{D}$ for paths.

Theorem 2 (The sequences $\boldsymbol{d}$ and $\boldsymbol{D}$ for paths)
Suppose $n\in\mathbb{N}$ and $q\in\mathbb{N}_0$ such that $q\leq n-1$. Then

$$d_q(P_n)=\begin{cases}0 & \text{if } q<\frac{n}{3}\\ q+1-\lceil\frac{n}{3}\rceil & \text{if } q\geq\frac{n}{3}\end{cases}$$

$$\text{and } D_q(P_n)=\left\lceil\frac{n+2q}{3}\right\rceil-\left\lceil\frac{n}{3}\right\rceil.$$

Proof: Both cases of the formula above for $d_q(P_n)$ are established by means of induction over q. Suppose $n>3q$, for which the base case is $d_0(P_n)=0$ and that $E_n\in P_n-\ell e$ minimises $\gamma(P_n-\ell e)$. Assume, as induction hypothesis that the desired formula holds for $q=\ell$, i.e. $\min\{\gamma(P_n-\ell e)\}=\lceil\frac{n}{3}\rceil$ for all $\ell<\frac{n}{3}$. To show that the formula also holds for $q=\ell+1$, a disjoint path P_3 is added to E_n for all $n>3\ell$. Then it follows by Lemma 2(b) that

$$\begin{aligned}\min\{\gamma(P_{n+3}-(\ell+1)e)\} &= \min\{\gamma(P_n-\ell e)\}+\gamma(P_3)\\ &= \left\lceil\frac{n}{3}\right\rceil+1=\left\lceil\frac{n+3}{3}\right\rceil,\end{aligned}$$

showing that $d_{\ell+1}(P_{n+3})=0$ for all $n>3(\ell+1)$ and thereby completing the induction process for this case.

Suppose next that $n\leq 3q$ and suppose that $E_n\in P_n-\ell e$ minimises $\gamma(P_n-\ell e)$ and assume, as induction hypothesis, that the formula holds for $q=\ell$, *i.e.* $\min\{\gamma(P_n-\ell e)\}=\ell+1$ for all $n\leq 3\ell$. To show that the formula also holds for $q=\ell+1$ a disjoint path P_2 is added to E_n for $2\ell\leq n\leq 3\ell$, thereby covering the required range of values of n for $q=\ell+1$, i.e. $2\ell+2\leq n\leq 3\ell+3$. Then it follows by Lemma 2(a) that

$$\min\{\gamma(P_{n+2}-(\ell+1)e)\} = \min\{\gamma(P_n-\ell e)\}+\gamma(P_2) = (\ell+1)+1,$$

thereby completing the induction process for $2\ell \le n \le 3\ell$.

Finally, suppose $n < 2q$ and consider $d_2(P_3)=2$ as base case. Assume, as induction hypothesis, that the formula holds for $q=\ell$, *i.e.* $\min\{\gamma(P_n-\ell e)\}=q+1$ for $n<3\ell$. Let $E_n \in P_n-\ell e$ and suppose the vertex set of E_n is $\{v_1,\ldots,v_n\}$. It is shown by contradiction that E_n has at least one isolated vertex. Assume, to the contrary, that E_n has no isolated vertex. Then it follows by the handshaking lemma that

$$n \le \sum_{i=1}^{n}\deg(v_i) = 2m = 2(n-1-\ell),$$

since each vertex has degree at least one. Therefore, $n \le 2(n-1-\ell)$, or equivalently $n \ge 2\ell+2$, which contradicts the fact that $n<2\ell+2$. Hence, E_n has at least one isolated vertex, and so

$$\min\{\gamma(P_{n+1}-(\ell+1)e)\} = \min\{\gamma(P_n-\ell e)\}+\gamma(P_1) = (\ell+1)+1,$$

thereby completing the induction process.

The formula above for $D_q(P_n)$ are established by induction over q and suppose that $q<n-3$ and suppose that $F_n \in P_n-\ell e$ maximises $\gamma(P_n-\ell e)$ and assume, as induction hypothesis, that the formula holds for $q=\ell$, i.e. $\max\{\gamma(P_n-\ell e)\}=\lceil\frac{n+2\ell}{3}\rceil$ for all $\ell<n-3$. To show that the formula also holds for $q=\ell+1$, a disjoint path P_4 is added to F_n for $q<n-3$, thereby covering the required range of values of n for $q=\ell+1$, i.e. $\ell+4<n-3$. Then it follows by Lemma 2(d) that

$$\begin{aligned}\max\{\gamma(P_{n+1}-(\ell+1)e)\} &= \max\{\gamma(P_n-qe)\}+\gamma(P_4)\\ &= \left\lceil\frac{n+2\ell}{3}\right\rceil+2\\ &= \left\lceil\frac{n+2\ell+6}{3}\right\rceil\\ &= \left\lceil\frac{(n+4)+2(\ell+1)}{3}\right\rceil,\end{aligned}$$

thereby completing the induction process for $\ell<n-3$.

Suppose next that $n-3\le\ell\le n-1$ and suppose that $F_n\in P_n-\ell e$ maximises $\gamma(P_n-\ell e)$. Assume, as induction hypothesis, that the formula holds for $q=\ell$, *i.e.* $\max\{\gamma(P_n-\ell e)\}=\lceil\frac{n+2\ell}{3}\rceil$ for all $n-3\le\ell\le n-1$. To show that the formula also holds for $q=\ell+1$, a disjoint path P_1 is added to F_n for $n-3\le\ell\le n-1$, thereby covering the required range of values of n for $q=\ell+1$, i.e. $n-3\le\ell+1\le n-1$. It follows by Lemma 2(c) that

$$\begin{aligned}\max\{\gamma(P_{n+1}-(\ell+1)e)\} &= \max\{\gamma(P_n-qe)\}+\gamma(P_1)\\ &= \left\lceil\frac{n+2\ell}{3}\right\rceil+1\\ &= \left\lceil\frac{n+2\ell+3}{3}\right\rceil\\ &= \left\lceil\frac{(n+1)+2(\ell+1)}{3}\right\rceil,\end{aligned}$$

thereby completing the induction process for $n-3\le\ell\le n-1$. □

The next result immediately follows from Theorem 2, because C_n-e contains a single element, which is isomorphic to P_n, for all $n\ge3$.

Corollary 2 (The sequences $\boldsymbol{d}$ and $\boldsymbol{D}$ for cycles) *Suppose $n\in\mathbb{N}$ and $q\in\mathbb{N}_0$ such that $q\le n$. Then*

$$d_q(C_n)=\begin{cases}0 & \text{if } q<\frac{n}{3}+1\\ q-\lceil\frac{n}{3}\rceil & \text{if } q\ge\frac{n}{3}+1.\end{cases}$$

$$\text{and } D_q(C_n)=\left\lceil\frac{n+\ 2q-2}{3}\right\rceil-\left\lceil\frac{n}{3}\right\rceil.$$

3.2. Complete bipartite graphs

It follows by Theorem 1 that $n-(j+1)(n-j)+q\le d_q(K_{j,n-j})\le D_q(K_{j,n-j})\le q$ for all $n-j\ge j$ and $0\le q\le j(n-j)$, by noting that $\alpha(K_{j,n-j})=n-j$. Again, these bounds seem to be weak for small values of q.

For the simplest class of complete bipartite graphs, namely stars, it is possible to determine the values of $\boldsymbol{d}$ and $\boldsymbol{D}$ exactly. For the simplest class of complete bipartite graphs, namely stars, it holds that

$$d_q(K_{1,n-1})=D_q(K_{1,n-1})=q.$$

Perhaps the most simple and most natural generalisation of a star, namely the graph $K_{2,n-2}$, is considered.

Theorem 3. *For the complete bipartite graph $K_{2,n-2}$ of order $n\ge4$,*

$$d_q(K_{2,n-2})=\begin{cases}0 & \text{if } q\le n-2\\ q-n+2 & \text{if } n-2<q\le 2n-4\end{cases}$$

and

$$D_q(K_{2,n-2}) = \begin{cases}\lfloor q/2\rfloor & \text{if } q\le 2(n-4)\\ n-4+\left\lceil\frac{2q-2-2(n-4)}{3}\right\rceil & \text{if } 2(n-4)<q\le 2(n-2)\end{cases}$$

Proof: Denote the partite sets of $K_{2,n-2}$ by $\{x,y\}$ and $V=\{v_1,\ldots,v_{n-2}\}$. Removing q edges from $K_{2,n-2}$ results in a subgraph $G\in K_{2,n-2}-qe=:K(n,q)$ and the partition $V=V_0^G\cup V_x^G\cup V_y^G\cup V_{xy}^G$, where V_0^G contains isolated vertices in G, V_x^G (V_y^G, respectively) contains the vertices adjacent to x only (y only, resp.) in G, and V_{xy}^G contains the common neighbours of x and y in G. Then, $2|V_0^G|+|V_x^G|+|V_y^G|=q$, so that

$$|V_0^G|+|V_x^G|+|V_y^G|=q-|V_0^G|. \tag{3.1}$$

In order to determine a minimum dominating set for G, two mutually exclusive cases are considered.

Case i: $|V_{xy}^G|\ne1$. In this case G is dominated by the vertices in $V_0^G\cup\{x,y\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

Case ii (a): $|V_{xy}^G| = 1$ *and* $|V_x^G| = |V_y^G| = 0$. In this case G is the vertex disjoint union of the isolated vertices in V_0^G and a star with universal vertex $\{z\} \in V_{xy}^G$. Therefore G is dominated by the vertices in $V_0^G \cup \{z\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

Case ii (b): $|V_{xy}^G| = 1$, $|V_x^G| > 0$ *and* $|V_y^G| > 0$. In this case G is again dominated by the vertices in $V_0^G \cup \{x, y\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

If $0 \le q \le n-2$, then the number of vertices in V_0^G is minimised by removing from $K_{2,n-2}$ the edges xv_1, xv_2, xv_3, and so on, in this order, until q edges have been removed. In this way, $|V_0^G| = |V_x^G| = 0$, $|V_y^G| = q$ and $|V_{xy}^G| = n-q-2$, resulting in the expression

$$d_q(K_{2,n-2}) = \min_{G \in K(n,q)} \{\gamma(G)\} - 2 = 0, \text{ if } 1 \le q \le n-2$$

as in Case i and Case ii (b). If $n-2 < n \le 2n-4$, then the number of vertices in V_0^G is minimised by removing the edges $xv_1, xv_2, \ldots, xv_{n-2}$ together with the edges yv_1, yv_2, yv_3, and so on, in this order, until q edges have been removed. In this way, $|V_0^G| = q-(n-2)$, $|V_x^G| = 0$, $|V_y^G| = (2n-4)-q$ and $|V_{xy}^G| = 0$, resulting in the expression

$$\begin{aligned} d_q(K_{2,n-2}) &= \min_{G \in K(n,q)} \{\gamma(G)\} - 2 \\ &= q - n + 2, \text{ if } n-2 < q \le 2n-4 \end{aligned}$$

as in Case i and Case ii (b).

The number of vertices in V_0^G is maximised by removing from $K_{2,n-2}$ the edges xv_1, yv_1, xv_2, yv_2, xv_3, yv_3, and so on, in this order, until q edges have been removed. In this way, $|V_0^G| = (q-1)/2$, $|V_x^G| = 0$, $|V_y^G| = 1$ and $|V_{xy}^G| = n-(q+5)/2$ if q is odd, while $|V_0^G| = q/2$, $|V_x^G| = |V_y^G| = 0$ and $|V_{xy}^G| = n-(q+4)/2$ if q is even. If $0 \le n \le 2(n-4)$, then

$$\begin{aligned} D_q(K_{2,n-2}) &= \max_{G \in K(n,q)} \{\gamma(G)\} - 2 \\ &= \begin{cases} q - \frac{q-1}{2} - 1, & \text{if } q \text{ is odd} \\ q - \frac{q}{2}, & \text{if } q \text{ is even} \end{cases} \\ &= \lfloor q/2 \rfloor \end{aligned}$$

as in Case i. If $2(n-4) < q \le 2(n-2)$, then the number of vertices in V_0^G is maximised by removing from $K_{2,n-2}$ the edges xv_1, yv_1, xv_2, yv_2, xv_3, yv_3, and so on, in this order, until $2n-8$ edges have been removed. It follows that $|V_0^G| = n-4$, $|V_x^G| = |V_y^G| = 0$ and $|V_{xy}^G| = 2$. Assume that $\{z_1, z_2\} \in V_{xy}^G$, then the vertices $\{x, y, z_1, z_2\}$ induce a cycle of order four, yielding the result

$$\begin{aligned} D_q(K_{2,n-2}) &= \max_{G \in K(n,q)} \{\gamma(G)\} - 2 \\ &= n - 4 + \left\lceil \frac{2q - 2 - 2(n-4)}{3} \right\rceil, \\ &\text{if } 2(n-4) < q \le 2(n-2) \end{aligned}$$

due to the result from Corollary 2 in conjunction with Case ii (a) and (b). □

From the results of Theorem 3 it is possible to generalise the result for the graph $K_{j,n-j}$, where $j > 2$ for the cost function $d_q(K_{j,n-j})$. This process is simplified by the realisation that $\gamma(K_{j,n-j}) = 2$ for all $n-j \ge j$ and $j \ge 3$. A simple sequence of edge removals can be shown to provide an exact formulation for $d_q(K_{j,n-j})$.

Theorem 4. *For the complete bipartite graph* $K_{j,n-j}$ *of order* $n-j \ge j \ge 3$, *then*

$$d_q(K_{j,n-j}) = \begin{cases} 0 & \text{if } 0 \le q \le (j-1)(n-j-1)+1 \\ q-(j-1)(n-j-1)-1 & \text{if } (j-1)(n-j-1)+2 \le q \le j(n-j) \end{cases}$$

Proof: Denote the partite sets of $K_{j,n-j}$ by $X = \{x_1, \ldots, x_j\}$ and $Y = \{y_1, \ldots, y_{n-j}\}$. The set $\{x_1, y_1\}$ is a minimum dominating set for $K_{j,n-j}$ by Cockayne et al. [3, Proposition 10(a)]. Removing q edges from $K_{j,n-j}$ results in a subgraph $G \in K_{j,n-j} =: K'(n,q)$ and the denote $E_{x_1'}^G$ as the set of edges incident to y_k for $k = 2, 3, \ldots, n-j$, and similarly, denote $E_{y_1'}^G$ as the set of edges incident with x_ℓ for $\ell = 2, 3, \ldots, j$. Finally, denote $E_{rem}^G = E(K_{j,n-j}) - E_{x_1'}^G - E_{y_1'}^G$. It follows that $|E_{x_1'}^G| = n-j-1$, $|E_{y_1'}^G| = j-1$ and $|E_{rem}^G| = j(n-j)-(n-j-1)-(j-1) = (j-1)(n-j-1)+1$. In order to determine a minimum dominating set for G, two mutually exclusive cases are considered.

Case i: $|E_{rem}^G| \ge 0$ *and* $|E_{x_1'}^G| = n-j-1$ *and* $|E_{x_1'}^G| = j-1$. In this case G is dominated by the vertex in $\{x_1, y_1\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

Case ii: $|E_{rem}^G| = 0$ *and* $|E_{x_1'}^G| \le n-j-1$ *and* $|E_{x_1'}^G| \le j-1$. In this case G is the vertex disjoint union of the isolated vertices, say V_0^G, and two disjoint stars with universal vertices x_1 and y_1, respectively. Therefore G is dominated by the vertices in $V_0^G \cup \{x_1, y_1\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

If $0 \le q \le (j-1)(n-j-1)+1$ the number of edges incident with the dominating set $\{x_1, y_1\}$ are not to be removed. The removal of any edge from the edge set $(x_k, y_\ell) \cup (x_1, y_1)$ where $k \ge \ell \ge 2$ yields a subgraph of $K_{j,n-j}$ for which $\{x_1, y_1\}$ is a dominating set of $K_{j,n-j} - qe$. By Case i, it follows that

$$\begin{aligned} d_q(K_{j,n-j}) = \min_{G \in K'(n,q)} \{\gamma(G)\} - 2 = 0, \text{ if } 0 < q \\ \le (j-1)(n-j-1)+1. \end{aligned}$$

For $(j-1)(n-j-1)+2 \le q \le j(n-j)$, the removal of the edges $x_\ell y_k$ for $k = 2, \ldots, n-j$ and $\ell = 2, \ldots, j$ and finally the edge x_1y_1 yields two disjoint stars, $K_{1,n-j}$ and $K_{1,j}$ with universal vertices x_1 and y_1, respectively. It follows that the removal of any subsequent edge from $K_{j,n-j}$ increases the domination number, and as a result it holds that

$$\begin{aligned} d_q(K_{j,n-j}) = \min_{G \in K'(n,q)} \{\gamma(G)\} - 2 = q - j', \text{ if } j' < q \\ \le j(n-j), \end{aligned}$$

by Case ii where $j' = (j-1)(n-j-1)+1$. □

It seems rather difficult to generalise the result of Theorem 3 for the cost function $D_q(K_{j,n-j})$ where $j > 2$, because of the large number of cases involved in a generalisation of the proof in Theorem 3. It is however possible to provide an algorithmic lower bound for $D_q(K_{j,n-j})$ where $j > 2$.

Algorithm 1: A lower bound on the sequence $\boldsymbol{D}$ for K_n or $K_{j,n-j}$

Input: The complete graph K_n or the complete bipartite graph $K_{j,n-j}$ of order n.
Output: A lower bound sequence `DBoundSequence` on $\boldsymbol{D}$.

`DValue` $\leftarrow 0$;
`DBoundSequence` $\leftarrow (0)$;
while $E(G) \neq \emptyset$ **do**
  if $G \cong K_{2,2}$ **then**
    Append(`DBoundSequence`, (`DValue`, `DValue`, `DValue` $+ 1$, `DValue` $+ 2$));
    $G \leftarrow \overline{K_n}$
  end
  else
    $x \leftarrow$ a vertex of minimum degree of G;
    Append(`DBoundSequence`, deg(x) $- 1$ *copies of* `DValue`);
    `DValue` $\leftarrow$ `DValue` $+ 1$;
    Append(`DBoundSequence`, `DValue`);
    $G \leftarrow G - \{x\}$;
  end
end
return `DBoundSequence`

A pseudo-code listing of this iterative procedure is given in the guise of a breadth-first search as Algorithm 1. The algorithm is based on the principle of iteratively isolating vertices of largest degree until the empty graph remains. The bounding sequence in Algorithm 1is expected to be good approximations of the sequences $\boldsymbol{D}(K_{j,n-j})$. The algorithm maintains a list `DBoundSequence`. This list is populated with appropriate lower bounds on $D_q(G)$ for a graph G during execution of the algorithm. For example, for the graph $K_{3,5}$ the list `DBoundSequence` is

$$0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 6.$$

3.3 Complete graphs

It follows by Theorem 1 that $n - \binom{n}{2} + q - 1 \leq d_q(K_n) \leq D_q(K_n) \leq q$, but these bounds are weak for small q.

Theorem 5. *For the complete graph K_n of order n, it follows that*

$$d_q(K_n) = \begin{cases} 0 & \text{if } 0 \leq q \leq \binom{n-1}{2} \\ q - \binom{n-1}{2} & \text{if } \binom{n-1}{2} < q \leq \binom{n}{2} \end{cases}$$

Proof: Let $x \in V(K_n)$, then $\{x\}$ is a minimum dominating set of G. Removing q edges from K_n results in a subgraph $G \in K_n - qe$ with partition $E^G = E^G_x \cup E^G_{\bar{x}}$, where E^G_x are the edges incident with the vertex x, and $E^G_{\bar{x}}$ are the set of edges incident with the vertex set $V(K_n)\ \{x\}$. In order to determine a minimum dominating set for G, two mutually exclusive cases are considered.

Case i: $|E^G_{\bar{x}}| \neq 0$ *and* $|E^G_x| = n - 1$. In this case G is dominated by the vertex in $\{x\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

Case ii: $|E^G_{\bar{x}}| = 0$ *and* $|E^G_x| \leq n - 1$. In this case G is the vertex disjoint union of the isolated vertices, say V^G_0, and a star with universal vertex $\{x\}$. Therefore G is dominated by the vertices in $V^G_0 \cup \{x\}$, and no smaller dominating set of G exists by Cockayne et al. [3, Proposition 10(a)].

Then it follows by Case i that

$$d_q(K_n) = \min_{G \in K_n - qe} \{\gamma(G)\} - 1 = 0, \text{ if } 0 < q \leq \binom{n-1}{2}.$$

For $\binom{n-1}{2} < q \leq \binom{n}{2}$, the removal of the edges $E^G_{\bar{x}}$, yields a star $K_{1,n-1}$ with x as universal vertex. Any subsequent edge removal increases the domination number of G and as a result follows holds that

$$d_q(K_n) = \min_{G \in K_n - qe} \{\gamma(G)\} - 1$$
$$= q - \binom{n-1}{2}, \text{ if } \binom{n-1}{2} < q \leq \binom{n}{2},$$

by Case ii. □

Again Algorithm 1 is considered to aid in providing a lower bound on $D_q(K_n)$. For the graph K_6, the list `DBoundSequence` is

$$0, 0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 5.$$

It is important to note that Algorithm 1 is not a suitable approximation of $D_q(G)$ for any graph G in general. Special graph classes such as the complete bipartite graph $K_{j,n-j}$ and complete graph K_n of orders n are suited candidates as input for Algorithm 1. However, it remains an open problem whether Algorithm 1 does provide the exact cost sequence $\boldsymbol{D}$ for complete graphs and complete bipartite graphs.

4. Conclusions

In this paper, two cost function sequences, $\boldsymbol{d}(G)$ and $\boldsymbol{D}(G)$ for a graph G were introduced and illustrated in §2. These sequences measure respectively the smallest and largest increase of $\gamma(G)$ as edges are removed from G. General bounds on $\boldsymbol{d}(G)$ and $\boldsymbol{D}(G)$ were established in §3, after which exact values for or bounds on these functions were determined in §4 for a number of special graph classes, including, paths, cycles, complete bipartite graphs and complete graphs.

Further, related work may include determining the value of γ for other graph classes, such as complete multipartite graphs, trees, circulant graphs and various Cartesian products. Furthermore, exact formulations on the cost sequence $\boldsymbol{D}$ for complete graphs and complete bipartite graphs remains open for further research.

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