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Algorithmic complexity of secure connected domination in graphs

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ABSTRACT
Let $G = (V, E)$ be a simple, undirected, and connected graph. A connected (total) dominating set $S \subseteq V$ is a secure connected (total) dominating set of G, if for each $u \in V \setminus S$, there exists $v \in S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a connected (total) dominating set of G. The minimum cardinality of a secure connected (total) dominating set of G denoted by $\gamma_{sc}(G)(\gamma_{st}(G))$, is called the secure connected (total) domination number of G. In this paper, we show that the decision problems corresponding to secure connected domination number and secure total domination number are NP-complete even when restricted to split graphs or bipartite graphs. The NP-complete reductions also show that these problems are w[2]-hard. We also prove that the secure connected domination problem is linear time solvable in block graphs and threshold graphs.

KEYWORDS
Domination; secure domination; secure connected domination; w[2]-hard

1. Introduction

Let $G(V, E)$ be a simple, undirected, and connected graph. For graph theoretic terminology we refer to [8]. For a vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \in V : uv \in E\}$, the *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. If $S \subseteq V$, then the open neighborhood of S is the set $N_G(S) = \cup_{v \in S} N_G(v)$. The closed neighborhood of S is $N_G[S] = S \cup N_G(S)$. Let $S \subseteq V$. Then a vertex $w \in V$ is called a private neighbor of v with respect to S if $N[w] \cap S = \{v\}$. If further $w \in V \setminus S$, then w is called an *external private neighbor (epn)* of v.

A subset S of V is a *dominating set* (DS) in G if for every $u \in V \setminus S$, there exists $v \in S$ such that $uv \in E$. The *domination number* of G is the minimum cardinality of a DS in G and is denoted by $\gamma(G)$. A set $S \subseteq V$ is said to be a *secure dominating set* (SDS) in G if for every $u \in V \setminus S$ there exists $v \in S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. We say that v *S-defends* u or u is defended by v [4]. The minimum cardinality of a SDS in G is called the *secure domination number* of G and is denoted by $\gamma_s(G)$. A dominating set S is said to be a *connected dominating set* (CDS), if the induced subgraph $G[S]$ is connected. A CDS S is said to be a *secure connected dominating set* (SCDS) in G if for each $u \in V \setminus S$, there exists $v \in S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a CDS in G. The minimum cardinality of a SCDS in G is called the *secure connected domination number* of G and is denoted by $\gamma_{sc}(G)$. A dominating set S is said to be a *total dominating set* (TDS), if the induced subgraph $G[S]$ has no isolated vertices. A TDS S is said to be a *secure total dominating set* (STDS) of G, if for each $u \in V \setminus S$, there exists $v \in S$ such that $uv \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a TDS in G. The minimum cardinality of a STDS in G is called the *secure total domination number* of G and is denoted by $\gamma_{st}(G)$. We need the following theorems.

Theorem 1. *([2]) Let G be a connected graph of order n. Then $\gamma_{sc}(G) = 1$ if and only if $G = K_n$.*

Theorem 2. *([2]) Let G be a connected graph of order $n \geq 3$. Let $L(G)$ and $S(G)$ be the set of pendant and support vertices of G respectively. Let X be a secure connected dominating set of G. Then (i) $L(G) \subseteq X$ and $S(G) \subseteq X$*

(ii) No vertex in $L(G) \cup S(G)$ is an X-defender.

Proposition 1. *([3]) Let S be a CDS in G. Then S is a SCDS in G if and only if the following conditions are satisfied.*

(i) *$epn(v, S) = \emptyset$ for all $v \in S$.*
(ii) *For every $u \in V \setminus S$, there exists $v \in S \cap N_G(u)$ such that $V(C) \cap N_G(u) \neq \emptyset$ for every component C of $G[S \setminus \{v\}]$.*

Proposition 2. *([2]) Let G be a non-complete connected graph and let S be a secure connected dominating set in G. Then the set $S \setminus \{v\}$ is a dominating set for every $v \in S$. In particular, $1 + \gamma(G) \leq \gamma_{sc}(G)$.*

2. Main results

We first determine the value of $\gamma_{sc}(G)$ for two families of graphs.

Theorem 3. *Let $W_n = v_1 + C_n$ be the wheel of order $n+1$ where $n \geq 3$. Let G be the graph obtained from W_{n+1} by subdividing all the edges of C_n. Then $\gamma_{sc}(G) = n + 1$.*

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Proof. Let $V(G) = \{v_1, v_2, ..., v_{2n+1}\}, d(v_{2n+1}) = n, d(v_i) = \begin{cases} 2 & \text{if } i \text{ is even} \\ 3 & \text{otherwise} \end{cases}$ and $N(v_i) = \{v_{i-1}, v_{i+1}\}$ if i is even. Then $S = \{v_i : i \text{ is odd}\}$ is a SCDS of G. Hence $\gamma_{sc}(G) \leq n+1$.

Now let D be any γ_{sc}-set of G. If $v_{2n+1} \notin D$ or if $v_{2n+1} \in D$ and defends a vertex v_i, then we get a connected dominating set $D_1 of G$ such that $|D_1| = |D|$ and $v_{2n+1} \notin D_1$. Hence $|D| = |D_1| \geq 2n-2$, which is a contradiction. Thus $v_{2n+1} \in D$ and v_{2n+1} does not defend any other vertex. Now let $v_i \in D$ for some $i where i$ is even. Since $G[D]$ is connected, one of v_{i-1} or v_{i+1} is in D. Also if $v_i \notin D$ for all even i, then $v_i \in D$ for all odd i. Hence $\gamma_{sc}(G) = |D| \geq n+1$. □

Theorem 4. *For the book graph $B_n = K_{1,n} \square K_2$, we have $\gamma_{sc}(B_n) = n+2$.*

Proof. Let S_1 and S_2 be the two copies of $K_{1,n}$ in B_n. Let $V(S_1) = \{v_1, v_2, ..., v_{n+1}\}$ and $V(S_2) = \{w_1, w_2, ..., w_{n+1}\}$. Let v_1 and w_1 be the central vertices of S_1, S_2, respectively. Let $v_i w_i \in E(B_n)$. Clearly $V(S_1) \cup \{w_1\}$ is an SCDS of B_n. Hence $\gamma_{sc}(B_n) \leq n+2$.

Now let D be any γ_{sc}-set of B_n. Since D is connected, either v_1 or w_1 is in D. If $w_1 \in D$ and $v_1 \notin D$, then $\{w_2, w_3, ..., w_{n+1}, v_2, v_3, ..., v_{n+1}\} \subseteq D$. Thus, $|D| \geq 2n+1$ which is a contradiction. Hence $v_1, w_1 \in D$. Now if both w_i and v_i are not in D for some $i \geq 2$, then $G[(D \setminus \{w_1\}) \cup \{w_i\}]$ and $G(D \setminus \{v_1\}) \cup \{v_i\}]$ are disconnected. Hence $|D \cap \{w_i, v_i\}| \geq 1$ for any $i \geq 2$ and $\gamma_{sc}(B_n) = |D| \geq n+2$. Thus, $\gamma_{sc}(B_n) = n+2$. □

Theorem 5. *Let $G = P_n \square P_2$ where $n \geq 3$. Then $\gamma_{sc}(G) = n + \lceil \frac{n}{3} \rceil$.*

Proof. Let $P = (v_1, v_2, ..., v_n)$ and $Q = (w_1, w_2, ..., w_n)$ be two copies of P_n in G such that $v_i w_i \in E(G)$. Let $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{w_1, w_2, ..., w_n\}$. Then $S = V_1 \cup \{w_i : i \equiv 2 (mod 3)\}$ is a SCDS of G. Hence $\gamma_{sc}(G) \leq n + \lceil \frac{n}{3} \rceil$.

Let D be any γ_{sc}-set of G. If $v_i, w_i \notin D$ for some i, where $2 \leq i \leq n-1$, then $G[D]$ is disconnected, which is a contradiction. Hence at least one of v_i, w_i is in D, where $2 \leq i \leq n-1$. If both v_1 and w_1 are not in D, then $G[(D \setminus \{v_2\}) \cup \{v_1\}]$ and $G[(D \setminus \{w_2\}) \cup \{w_1\}]$ are disconnected, which is a contradiction. Hence v_1 or w_1 is in D. Similarly, w_n or v_n is in D. We now claim that $D \cap V_1$ is a dominating set of P. Suppose there exists a vertex v_i such that v_i is not dominated by $D \cap V_1$. Then $w_i \in D$ and $G[(D \setminus \{w_i\}) \cup \{v_i\}]$ is disconnected, which is a contradiction. Hence $D \cap V_1$ is a dominating set of P. Similarly $D \cap V_2$ is a dominating set of Q. Now suppose $D \cap V_1 \subsetneq V_1$ and $D \cap V_2 \subsetneq V_2$. If three consecutive vertices of P say, v_i, v_{i+1}, v_{i+2} are not in D, then $w_i, w_{i+1}, w_{i+2} \in D$. However, $G[(D \setminus \{w_{i+1}\}) \cup \{v_{i+1}\}]$ is disconnected, which is a contradiction. Now suppose $v_i, v_{i+1} \notin D$. Then $v_{i-1}, v_{i+2}, w_i, w_{i+1} \in D$. Now since $G[D]$ is connected, it follows that $w_{i-1}, w_{i+2} \in D$. Hence $(D \setminus \{w_i, w_{i+1}\}) \cup \{v_i, v_{i+1}\}$ is also a SCDS of G. Thus by repeating the above process we get a SCDS of G, D_1 such that $|D_1| = |D|, D_1 \cap V_1 = V_1$ and $D_1 \cap V_2$ is a dominating set of Q. Thus, $|D| = |D_1| \geq n + \lceil \frac{n}{3} \rceil$. Therefore, $\gamma_{sc}(G) = n + \lceil \frac{n}{3} \rceil$. □

We now proceed to present results on algorithmic aspects such as NP-comppleteness and linear time algorithm for some classes of graphs.

Secure Connected Domination Problem (SCDM)

Instance: A connected graph G and a positive integer l.

Question: Does there exist a SCDS of size at most l in G?

The proof is by reduction from the Domination problem (DM), which is NP-complete [5].

Domination Problem (DM)

Instance: A graph G and a positive integer k.

Question: Does there exist a DS of size at most k in G?

Theorem 6. *SCDM is NP-complete.*

Proof. It can be easily verified that SCDM is in NP. Now let $G = (V, E)$ be a graph and let k be a positive integer. Let G^* be the graph with $V(G^*) = V \cup \{x\}$ and $E(G^*) = E \cup \{(u, x) : u \in V\}$ and let $l = k+1$. Clearly, G^* can be constructed from G in polynomial time.

Now if D is a dominating set of G with $|D| \leq k$, then $S = D \cup \{x\}$ is an SCDS of G^*. Conversely, let S^* be an SCDS of G^* with $|S^*| \leq k+1$. *If* $x \in S$, then it follows from Proposition 1 that $epn(x, S) = \emptyset$. Therefore, every vertex $u \in V(G^*) \setminus S$ is adjacent to a vertex in $S \setminus \{x\}$. Hence $S \setminus \{x\}$ is a DS of size at most k in G. If $x \notin S$, Proposition 2, the set $S \setminus \{v\}$, for any $v \in S$, is a DS of size at most k in G. □

Next, we define the decision version of total domination and secure total domination problems as follows.

Total Domination Problem (TDM)

Instance: A simple, undirected graph G without isolated vertices and a positive integer r.

Question: Does there exist a TDS of size at most r in G?

Secure Total Domination Problem (STDM)

Instance: A simple, undirected and connected graph G and a positive integer m.

Question: Does there exist a STDS of size at most $m$in$G$?

Theorem 7. *STDM is NP-complete.*

Proof. It is clear that STDM is in NP. The reduction given in the proof of Theorem 6 shows that STDM is NP-complete. □

We now give NP-completeness results even when restricted to bipartite graphs or split graphs. We formulate the SCDM for bipartite graphs as follows.

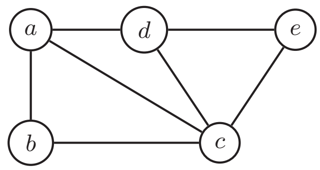
Secure Connected Domination Problem for Bipartite Graphs (SCDB)

Instance: A connected bipartite graph $G = (V_1, V_2, E)$ and a positive integer r.

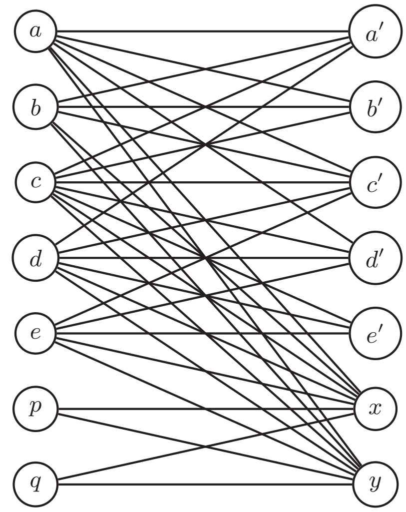
Question: Does there exist a SCDS of size at most r in G?

Theorem 8. *SCDB is NP-complete.*

Proof. It can be seen that SCDB is in NP. We transform an instance of SCDM problem to an instance of SCDB as follows. Given a graph G, we construct a graph $G^*(V_1, V_2, E)$ where $V_1(G^*) = V \cup \{p, q\}, V_2(G^*) = V'(G) \cup \{x, y\}$, here $V'(G)$ is another copy of V such that if u and v are two vertices in V then the corresponding vertices in



(a) Graph G



(b) Graph G^*

Figure 1. Construction of G^* from G.

$V'(G)$ are labeled as u' and v', and $E(G^*)$ consists of (i) edges uv' and $u'v$ for each edge $uv \in E$; (ii) edges of the form uu' for each vertex $u \in V$; and (iii) edges of the form ux and uy for every vertex $u \in V_1(G^*)$. Clearly G^* is a bipartite graph and can be constructed from G in polynomial time (Figure 1).

Next, we show that G has a SCDS of size at most r if and only if G^* has a SCDS of size at most $r+2$. If S is a SCDS of G with$|S| \leq r$, then it can be easily verified that $S^* = S \cup \{x, y\}$ is a SCDS of G^* with $|S^*| \leq r+2$.

Conversely, let S^* be an SCDS of G^* and $|S^*| \leq r+2$. Since $x$and$y$ are the only vertices in S^* which defend $p$and$q$, it follows that at least one of them must be in S^*. If $x \in S^*$ and $y \notin S^*$, then $G^*[(S^* \setminus \{x\}) \cup \{p\}]$ is disconnected, which is a contradiction. Hence $x, y \in S^*$. Let $S' = S^* \setminus \{x, y, p, q\}$ and $S'' = (S' \cup \{v : v' \in S' \cap V'(G)\}) \setminus \{v' : v' \in S' \cap V'(G)\}$. Clearly S'' forms a SCDS of size at most r in G. □

Theorem 9. *STDM is NP-complete for bipartite graphs.*

Proof. It is clear that STDM for bipartite graphs is in NP. The reduction given in the proof of Theorems 8 shows that STDM is NP-complete for bipartite graphs. □

Since the Domination problem is w[2]-complete for bipartite graphs [7] and the reductions in Theorems 8 and 9 are in the function of the parameter l, the following two corollaries are immediate.

Corollary 1. *SCDM is w[2]-hard in bipartite graphs.*

Corollary 2. *STDM is w[2]-hard in bipartite graphs.*

It has been shown that the DM and the TDM as NP-complete even when restricted to split graphs [1].

Theorem 10. *SCDM is NP-complete for split graphs.*

Proof. It is known that SCDM is a member of NP. We reduce DM for split graphs to SCDM for split graphs. Given a split graph G whose vertex set is partitioned into a clique Q and an independent set I, we construct a split graph G^* with a clique Q^* and an independent set I^* as follows:

$$V(G^*) = V \cup \{x, y\}, \text{ and}$$
$$E(G^*) = E \cup \{xu : u \in V\} \cup \{xy\}.$$

Note that G^* is a split graph, where $Q^* = Q \cup \{x\}$ and $I^* = I \cup \{y\}$ and G^* can be constructed from G in polynomial time.

Now let S be a DS of G with $|S| \leq k$. Then $S^* = S \cup \{x, y\}$ is a SCDS of G^* with $|S^*| \leq k+2$. Conversely, let S^* be a SCDS of G^* with $|S^*| \leq k+2$. It follows from Proposition 2 that $x, y \in S^*$. Clearly $S' = S^* \setminus \{x, y\}$ is a DS of G with $|S'| \leq k$. □

Theorem 11. *STDM is NP-complete for split graphs.*

Proof. It is clear that STDM for split graphs is in NP. The reduction given in the proof of Theorem 10 shows that STDM is NP-complete for split graphs. □

Since the Domination problem is w[2]-complete for split graphs [7] and the reductions in Theorems 10 and 11 are in the function of the parameter c, the following two corollaries are immediate.

Corollary 3. *SCDM is w[2]-hard in split graphs.*

Corollary 4. *STDM is w[2]-hard in split graphs.*

In the next two theorems we prove that $\gamma_{sc}(G)$ can be computed in linear time for block graphs and threshold graphs.

Let $G = (V, E)$ be a connected graph. A vertex v is called a cut-vertex of G if $G - v$ is a disconnected graph. A graph G with no cut-vertex is called a block. A block B of a graph is a maximal connected induced subgraph of G such that B has no cut-vertex. In block B, vertices which are not cut vertices of G are called block vertices. A graph G is called a *block graph* if all its blocks are cliques.

Definition 1. A graph G is called a block graph if all the blocks of G are cliques.

Theorem 12. *Let G be a block graph having r blocks and k cut vertices. Then* $\gamma_{sc}(G) = k + r - r'$, *where* r' *is the number of blocks such that all vertices of the block are cut vertices.*

Proof. Let A denote the set of all cut vertices of G. Let $B_1, B_2, ..., B_{r'}, B_{r'+1}, ..., B_r$ be the blocks of G where every vertex of B_i is a cut vertex of G if $1 \leq i \leq r'$. Let $T = \{v_i : 1 \leq i \leq r - r'$ and v_i is a non-cut vertex of $B_{r'+i}\}$. Let $S = A \cup T$.Since$A$ contains all cut-vertices of G, it follows that $G[S]$ is connected. Also if $v \in V \setminus S$, then v is not a cut-vertex. Now there exists a vertex $u \in T$ such that $uv \in E$and$(S \setminus \{v\}) \cup \{u\}$ is a CDS of G. Hence, S is a SCDS of G. Therefore, $\gamma_{sc}(G) \leq k + r - r'$.

Now let D be any γ_{sc}-set of G. Since $G[D]$ is connected, $D \supseteq A$. Further, a cut-vertex cannot defend any other vertex and hence D contains at least one non-cut vertex from each block B_i where $r' + 1 \leq i \leq r$. Hence $\gamma_{sc}(G) = |D| \geq |S| = k + r - r'$. Thus $\gamma_{sc}(G) = k + r - r'$. □

Corollary 5. *Let G be a block graph with r blocks and exactly one cut-vertex. Then* $\gamma_{sc}(G) = r + 1$.

Corollary 6. *For any tree T with n vertices,* $\gamma_{sc}(T) = n$.

Proof. Here $r = n - 1, r' = n - 1 - l$ and $k = n - l$ where l is the number of leaves in T. Therefore, $\gamma_{sc}(T) = n$. □

Corollary 7. *SCDM is linear time solvable for block graphs.*

Proof. Since number of blocks and number of cut-vertices of block graph can be determined in linear time, the result follows. □

Definition 2. A graph $G = (V, E)$ is called a threshold graph if there is a real number t and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(S) \leq t$.

Threshold graphs considered are assumed to be non-complete and connected. We use the following characterization of threshold graphs given in [6] to prove that secure connected domination number can be computed in linear time for threshold graphs.

A graph G is a *threshold graph* if and only if it is a split graph and for split partition (C, I) of V, there is an ordering $(x_1, x_2, ..., x_p)$ of vertices of C such that $N[x_1] \subseteq N[x_2] \subseteq ... \subseteq N[x_p]$, and there is an ordering $(y_1, y_2, ..., y_q)$ of the vertices of I such that $N(y_1) \supseteq N(y_2) \supseteq ...N(y_q)$.

Theorem 13. *Let G be a connected threshold graph. Then* $\gamma_{sc}(G) = 2 + l$, *where l is the number of pendant vertices.*

Proof. Let $S = \{x_p, x_{p-1}\} \cup \{v \in I : v \in N(x_p) \setminus N(x_{p-1})\}$. Clearly $G[S]$ is a star with center x_p. Also every vertex $v \in V \setminus S$ is defended by x_{p-1} and $G[(S \setminus \{x_{p-1}\}) \cup \{v\}]$ is connected. Thus, S is a SCDS of G.Hence$\gamma_{sc}(G) \leq 2 + l$.

Let D be any γ_{sc}-set of G. It follows from Theorem 2 that $|D| \geq l + 1$.*If* $|D| = l + 1$, then exactly one vertex of C say, u is a support vertex. Hence no vertex of $C \setminus \{u\}$ is D-defended, which is a contradiction. Hence $\gamma_{sc}(G) = |D| \geq 2 + l$. Thus $\gamma_{sc}(G) = 2 + l$. □

Theorem 14. *SCDM is linear time solvable for threshold graphs.*

Proof. Since the ordering of the vertices of the clique in a threshold graph can be determined in linear time [6], the result follows. □

3. Conclusion

In this paper, it is shown that secure connected (total) domination problem is NP-complete even when restricted to bipartite graphs, or split graphs. Since split graphs form a proper subclass of chordal graphs, these problems are also NP-complete for chordal graphs. We have proved that secure connected domination problem is linear time solvable for block graphs and threshold graphs. It will be interesting to investigate the algorithmic complexity of secure connected (total) domination problem for subclasses of chordal and bipartite graphs.

Disclosure statement

No potential conflict of interest was reported by the authors.

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