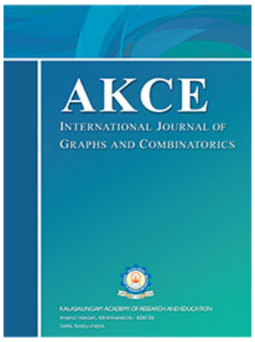

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On the structure of spikes

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ABSTRACT

Spikes are an important class of 3-connected matroids. For an integer $r \geq 3$, there is a unique binary r-spike denoted by Z_r. When a circuit-hyperplane of Z_r is relaxed, we obtain another spike and repeating this procedure will produce other non-binary spikes. The *es*-splitting operation on a binary spike of rank *r*, may not yield a spike. In this paper, we give a necessary and sufficient condition for the *es*-splitting operation to construct Z_{r+1} directly from Z_r. Indeed, all binary spikes and many of non-binary spikes of each rank can be derived from the spike Z_3 by a sequence of the *es*-splitting operations and circuit-hyperplane relaxations.

KEYWORDS

Binary matroid; *es*-splitting operation; relaxation; spike

1. Introduction

Azanchiler [1, 2] extended the notion of n-line splitting operation from graphs to binary matroids. He characterized the n-line splitting operation of graphs in terms of cycles of the respective graph and then extended this operation to binary matroids as follows. Let M be a binary matroid on a set E and let X be a subset of E with $e \in X$. Suppose A is a matrix that represents M over $GF(2)$. Let A_X^e be a matrix obtained from A by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to the elements of X where it takes the value 1, and then adjoining two columns labeled α and γ to the resulting matrix such that the column labeled α is zero everywhere except in the last row where it takes the value 1, and γ is the sum of the two column vectors corresponding to the elements α and e. The vector matroid of the matrix A_X^e is denoted by M_X^e. The transition from M to M_X^e is called an *es-splitting operation*. We call the matroid M_X^e as *es-splitting matroid*.

Let M be a matroid and $X \subseteq E(M)$, a circuit C of M is called an *OX-circuit* if C contains an odd number of elements of X, and C is an *EX-circuit* if C contains an even number of elements of X. The following proposition characterizes the circuits of the matroid M_X^e in terms of the circuits of the matroid M.

Proposition 1. [1] *Let* $M = (E, \mathcal{C})$ *be a binary matroid together with the collection of circuits* $\mathcal{C}$. *Suppose* $X \subseteq E, e \in X$ *and* $\alpha, \gamma \notin E$. *Then* $M_X^e = (E \cup \{\alpha, \gamma\}, \mathcal{C}')$ *where* $\mathcal{C}' = (\cup_{i=0}^{5} \mathcal{C}_i) \cup \Lambda$ *with* $\Lambda = \{e, \alpha, \gamma\}$ *and*

$\mathcal{C}_0 = \{C \in \mathcal{C} : C \text{ is an EX-circuit}\}$;

$\mathcal{C}_1 = \{C \cup \{\alpha\} : C \in \mathcal{C} \text{ and } C \text{ is an OX-circuit}\}$;

$\mathcal{C}_2 = \{C \cup \{e, \gamma\} : C \in \mathcal{C}, e \notin C \text{ and } C \text{ is an OX-circuit}\}$;

$\mathcal{C}_3 = \{(C \setminus e) \cup \{\gamma\} : C \in \mathcal{C}, e \in C \text{ and } C \text{ is an OX-circuit}\}$;

$\mathcal{C}_4 = \{(C \setminus e) \cup \{\alpha, \gamma\} : C \in \mathcal{C}, e \in C \text{ and } C \text{ is an EX-circuit}\}$;

$\mathcal{C}_5 =$ *The set of minimal members of* $\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset$ *and each of* C_1 *and* C_2 *is an OX-circuit*$\}$.

It is observed that the *es*-splitting operation on a 3-connected binary matroid may not yield a 3-connected binary matroid. The following result, provide a sufficient condition under which the *es*-splitting operation on a 3-connected binary matroid yields a 3-connected binary matroid.

Proposition 2. [3] *Let* M *be a 3-connected binary matroid,* $X \subseteq E(M)$ *and* $e \in X$. *Suppose that* M *has an OX-circuit not containing e. Then* M_X^e *is a 3-connected binary matroid.*

To define rank-r spikes, let $E = \{x_1, x_2, ..., x_r, y_1, y_2 ..., y_r, t\}$ for some $r \geq 3$. Let $\mathcal{C}_1 = \{\{t, x_i, y_i\} : 1 \leq i \leq r\}$ and $\mathcal{C}_2 = \{\{x_i, y_i, x_j, y_j : 1 \leq i < j \leq r\}$. The set of circuits of every spike on E includes $\mathcal{C}_1 \cup \mathcal{C}_2$. Let $\mathcal{C}_3$ be a, possibly empty, subsets of $\{\{z_1, z_2, ... z_r\} : z_i$ is in $\{x_i, y_i\}$ for all $i\}$ such that no two members of $\mathcal{C}_3$ have more than $r - 2$ common elements. Finally, let $\mathcal{C}_4$ be the collection of all $(r+1)$-element subsets of E that contain no member of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$.

Proposition 3. [4] *There is a rank-r matroid* M *on* E *whose collection* $\mathcal{C}$ *of circuits is* $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

The matroid M on E with collection $\mathcal{C}$ of circuits in the last proposition is called a *rank-r spike with tip t* and legs $L_1, L_2, ... L_r$ where $L_i = \{t, x_i, y_i\}$ for all i. In the construction of a spike, if $\mathcal{C}_3$ is empty, the corresponding spike is called the *rank-r free spike with tip t*. In an arbitrary spike M, each circuit in $\mathcal{C}_3$ is also a hyperplane of M. Evidently, when such a circuit-hyperplane is relaxed, we obtain another spike. Repeating this procedure until all of the circuit-hyperplanes in $\mathcal{C}_3$ have been relaxed will produce the free spike. Now let J_r and $\mathbf{1}$ be the $r \times r$ and $r \times 1$ matrices of all ones. For $r \geq$

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3, let A_r be the $r \times (2r+1)$ matrix $[I_r|J_r - I_r|\mathbf{1}]$ over $GF(2)$ whose columns are labeled, in order, $x_1, x_2, ..., x_r, y_1, y_2 ..., y_r, t$. The vector matroid $M[A_r]$ of this matrix is called the *rank-r binary spike with tip t* and denoted by Z_r. Oxley [4] showed that all rank-r, 3-connected binary matroids without a 4-wheel minor can be obtained from a binary r-spike by deleting at most two elements.

2. Circuits of Z_r

In this section, we characterize the collection of circuits of Z_r. To do this, we use the next well-known theorem.

Theorem 4. [4] *A matroid M is binary if and only if for every two distinct circuits C_1 and C_2 of M, their symmetric difference, $C_1 \Delta C_2$, contains a circuit of M.*

Now let $M = (E, \mathcal{C})$ be a binary matroid on the set E together with the set $\mathcal{C}$ of circuits where $E = \{x_1, x_2, ..., x_r, y_1, y_2 ..., y_r, t\}$ for some $r \geq 3$. Suppose $Y = \{y_1, y_2 ..., y_r\}$. For k in $\{1, 2, 3, 4\}$, we define φ_k as follows.

$\varphi_1 = \{L_i = \{t, x_i, y_i\} : 1 \leq i \leq r\}$;

$\varphi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \leq i < j \leq r\}$;

$\varphi_3 = \{Z \subseteq E : |Z| = r,\ |Z \cap Y|$ is odd and $|Z \cap \{y_i, x_i\}| = 1$ where $1 \leq i \leq r\}$; and

$$\varphi_4 = \begin{cases} \{E - C : C \in \varphi_3\}, & \text{if } r \text{ is odd;} \\ \{(E - C)\Delta\{x_{r-1}, y_{r-1}\} : C \in \varphi_3\}, & \text{if } r \text{ is even.} \end{cases}$$

Theorem 5. *A matroid whose collection $\mathcal{C}$ of circuits is $\varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4$, is the rank-r binary spike.*

Proof. Let M be a matroid on the set $E = \{x_1, x_2, ..., x_r, y_1, y_2 ..., y_r, t\}$ such that $\mathcal{C}(M) = \varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4$. Suppose $Y = \{y_1, y_2, ... y_r\}$. Then, for every two distinct circuits C_1 and C_2 of φ_3, we have $C_1 \cap Y \neq C_2 \cap Y$ and $|C_j \cap \{x_i, y_i\}| = 1$ for all i and j with $1 \leq i \leq r$ and $j \in \{1, 2\}$. We conclude that there is at least one y_i in C_1 such that $y_i \notin C_2$ and so x_i is in C_2 but it is not in C_1. Thus, no two members of φ_3 have more than r – 2 common elements. It is clear that every member of φ_4 has $(r+1)$-elements and contains no member of $\varphi_1 \cup \varphi_2 \cup \varphi_3$. By Proposition 3, we conclude that M is a rank-r spike. It is straightforward to show that for every two distinct members of $\mathcal{C}$, their symmetric difference contains a circuit of M. Thus, by Theorem 4, M is a binary spike. □

It is not difficult to check that if r is odd, then the intersection of every two members of φ_3 has odd cardinality and the intersection of every two members of φ_4 has even cardinality and if r is even, then the intersection of every two members of φ_3 has even cardinality and the intersection of every two members of φ_4 has odd cardinality. Clearly, $|\varphi_1| = r$, $|\varphi_2| = \frac{r(r-1)}{2}$ and $|\varphi_3| = |\varphi_4| = 2^{r-1}$. Therefore, every rank-r binary spike has $2^r + \frac{r(r+1)}{2}$ circuits. Moreover, $\cap_{i=1}^r L_i \neq \emptyset$ and $|C \cap \{x_i, y_i\}| = 1$ where $1 \leq i \leq r$ and C is a member of $\varphi_3 \cup \varphi_4$.

3. The *es*-splitting operation on Z_r

By applying the *es*-splitting operation on a given binary matroid with k elements, we obtain a matroid with $k+2$ elements. In this section, our main goal is to give a necessary and sufficient condition for $X \subseteq E(Z_r)$ with $e \in X$, to obtain Z_{r+1} by applying the *es*-splitting operation on X. Now suppose that $M = Z_r$ be a binary rank-r spike with the matrix representation $[I_r|J_r - I_r|\mathbf{1}]$ over $GF(2)$ whose columns are labeled, in order $x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t$. Suppose $\varphi = \varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4$ be the collection of circuits of Z_r defined in section 2. Let $X_1 = \{x_1, x_2, ..., x_r\}$ and $Y_1 = \{y_1, y_2, ..., y_r\}$ and let X be a subset of $E(Z_r)$. By the following lemmas, we give six conditions for membership of X such that, for every element e of this set, $(Z_r)^e_X$ is not the spike Z_{r+1}.

Lemma 6. *If $r \geq 4$ and $t \notin X$, then, for every element e of X, the matroid $(Z_r)^e_X$ is not the spike Z_{r+1}.*

Proof. Suppose that $t \notin X$. Without loss of generality, we may assume that there exist i in $\{1, 2, ..., r\}$ such that $x_i \in X$ and $e = x_i$. By Proposition 1, the set $\Lambda = \{x_i, \alpha, \gamma\}$ is a circuit of $(Z_r)^e_X$. Now consider the leg $L_i = \{t, x_i, y_i\}$, we have the following two cases.

(i) If $y_i \in X$, then $|L_i \cap X|$ is even. By Proposition 1, the leg L_i is a circuit of $(Z_r)^e_X$. Now if all other legs of Z_r have an odd number of elements of X, by Proposition 1, we observe that these legs transform to circuits of cardinality 4 and 5. So there are exactly two 3-circuits in $(Z_r)^e_X$. If not, there is a $j \neq i$ such that L_j is a 3-circuit of $(Z_r)^e_X$ and $(\Lambda \cap L_i \cap L_j) = \emptyset$. Since Z_{r+1} has $r+1$ legs and the intersection of the legs of Z_{r+1} is nonempty, we conclude that in each case, for every element e of X, the matroid $(Z_r)^e_X$ is not the spike Z_{r+1}.

(ii) If $y_i \notin X$, then $|L_i \cap X|$ is odd. By Proposition 1, $(L_i \setminus x_i) \cup \gamma$ is a circuit of $(Z_r)^e_X$. Now if there is the other leg L_j such that $|L_j \cap X|$ is even, then L_j is a circuit of $(Z_r)^e_X$. But $(L_j \cap \Lambda \cap ((L_i \setminus x_i) \cup \gamma)) = \emptyset$, so $(Z_r)^e_X$ is not the spike Z_{r+1}. We conclude that every leg L_j with $j \neq i$ has an odd number of elements of X. Since $x_i \notin L_j$, by Proposition 1 again, L_j is not a 3-circuit in $(Z_r)^e_X$. Therefore, $(Z_r)^e_X$ has only two 3-circuits and so, for every element e of X, the matroid $(Z_r)^e_X$ is not the spike Z_{r+1}. □

Lemma 7. *If $r \geq 4$ and $e \neq t$, then, for every element e of X – t, the matroid $(Z_r)^e_X$ is not the spike Z_{r+1}.*

Proof. Suppose that $e \neq t$. Without loss of generality, we may assume that there exist i in $\{1, 2, ..., r\}$ such that $x_i \in X$ and $e = x_i$. By Proposition 1, the set $\Lambda = \{x_i, \alpha, \gamma\}$ is a circuit of $(Z_r)^e_X$ and by Lemma 6, to obtain Z_{r+1}, the element t is contained in X. Now consider the leg $L_i = \{t, x_i, y_i\}$. We have the following two cases.

(i) If $y_i \in X$, then $|L_i \cap X|$ is odd. By Proposition 1, $L_i \cup \alpha$ and $(L_i \setminus x_i) \cup \gamma$ are circuits of $(Z_r)^e_X$. Now if there is the other leg L_j such that $|L_j \cap X|$ is even, then L_j is a circuit of $(Z_r)^e_X$. But $(L_j \cap \Lambda \cap ((L_i \setminus x_i) \cup \gamma)) = \emptyset$, so $(Z_r)^e_X$ is not the spike Z_{r+1}. We conclude that every leg L_j with $j \neq i$ has an odd number of elements of X. Since $x_i \notin L_j$, by Proposition 1 again, L_j is not a 3-circuit in $(Z_r)^e_X$. Therefore $(Z_r)^e_X$ has only two 3-circuits and so $(Z_r)^e_X$ is not the spike Z_{r+1}.

(ii) If $y_i \notin X$, then $|L_i \cap X|$ is even. So L_i is a circuit of $(Z_r)^e_X$. By similar arguments as in Lemma 6 (i), one can show that for every element e of $X - t$, the matroid $(Z_r)^e_X$ is not the spike Z_{r+1}. □

Next by Lemmas 6 and 7, to obtain the spike Z_{r+1}, we take t in X and $e = t$.

Lemma 8. *If $r \geq 4$ and there is a circuit C of φ_3 such that $|C \cap X|$ is even, then the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}.*

Proof. Suppose that C is a circuit of Z_r such that C is a member of φ_3 and $|C \cap X|$ is even. Then, by Proposition 1, the circuit C is preserved under the *es*-splitting operation. So C is a circuit of $(Z_r)^t_X$. But $|C| = r$. Now if $r > 4$, then C cannot be a circuit of Z_{r+1}, since it has no r-circuit, and if $r = 4$, then, to preserve the members of φ_2 in Z_4 under the *es*-splitting operation and to have at least one member of φ_3 which has even number of elements of X, the set X must be $E(Z_r) - t$ or t. But in each case $(Z_4)^t_X$ has exactly fourteen 4-circuits, so it is not the spike Z_5, since this spike has exactly ten 4-circuit. We conclude that the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}. □

Lemma 9. *If $r \geq 4$ and $|X \cap \{x_i, y_i\}| = 2$, for i in $\{1, 2, \ldots, r\}$, then the matroid $(Z_r)^t_X$ is not the spike Z_{r+1} unless r is odd and for all i, $\{x_i, y_i\} \subset X$, in which case Z_{r+1} has γ as a tip.*

Proof. Suppose that $\{x_i, y_i\} \subset X$ for $i \in \{1, 2, \ldots, r\}$. Since $t \in X$ and $e = t$, after applying the *es*-splitting operation, the leg $\{t, x_i, y_i\}$ turns into two circuits $\{t, x_i, y_i, \alpha\}$ and $\{x_i, y_i, \gamma\}$. Now consider the leg $L_j = \{t, x_j, y_j\}$ where $j \neq i$. If $|L_j \cap X|$ is even (this means $\{x_j, y_j\} \not\subseteq X$), then L_j is a circuit of $(Z_r)^e_X$. But $\{x_i, y_i, \gamma\} \cap \{t, x_j, y_j\} = \emptyset$ and this contradicts the fact that the intersection of the legs of a spike is not the empty set. So $\{x_j, y_j\}$ must be a subset of X. We conclude that $\{x_k, y_k\} \subset X$ for all $k \neq i$. Thus $X = E(Z_r)$. But in this case, r cannot be even since every circuit in φ_3 has even cardinality and by Lemma 8, the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}.

Now we show that if $X = E(Z_r)$, and r is odd, then $(Z_r)^t_X$ is the spike Z_{r+1} with tip γ. Clearly, every leg of Z_r has an odd number of elements of X. Using Proposition 1, after applying the *es*-splitting operation, we have the following changes.

For i in $\{1, 2, \ldots, r\}$, L_i transforms to two circuits $(L_i \setminus t) \cup \gamma$ and $L_i \cup \alpha$, every member of φ_2 is preserved, and if $C \in \varphi_3$, then $C \cup \alpha$ and $C \cup \{t, \gamma\}$ are circuits of $(Z_r)^t_X$. Finally, if $C \in \varphi_4$, then C and $(C \setminus t) \cup \{\alpha, \gamma\}$ are circuits of $(Z_r)^t_X$. Note that, since $X = E(Z_r)$ with $e = t$, there are no two disjoint OX-circuits in Z_r such that their union be minimal. Therefore the collection $\mathcal{C}_5$ in Proposition 1 is empty. Now suppose that α and t play the roles of x_{r+1} and y_{r+1}, respectively, and γ plays the role of tip. Then we have the spike Z_{r+1} with tip γ whose collection ψ of circuits is $\psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4$ where

$$\psi_1 = \{(L_i \setminus t) \cup \gamma : 1 \leq i \leq r\} \cup \Lambda;$$

$$\psi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \leq i < j \leq r\} \cup \{(L_i \cup \alpha : 1 \leq i \leq r\};$$

$$\psi_3 = \{C \cup \alpha : C \in \varphi_3\} \cup \{C : C \in \varphi_4\};$$

$$\psi_4 = \{C \cup \{t, \gamma\} : C \in \varphi_3\} \cup \{(C \setminus t) \cup \{\alpha, \gamma\} : C \in \varphi_4\}.$$

□

In the following lemma, we shall use the well-known facts that if a matroid M is n-connected with $E(M) \geq 2(n-1)$, then all circuits and all cocircuits of M have at least n elements, and if A is a matrix that represents M over $GF(2)$, then the cocircuit space of M equals the row space of A.

Lemma 10. *If $|X| \leq r$, then the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}.*

Proof. Suppose $X \subset E(Z_r)$ such that $|X| \leq r$. Then, by Lemmas 6, 7 and 8, $t \in X$ with $e = t$ and $|X \cap \{x_i, y_i\}| = 1$ for all i in $\{1, 2, \ldots, r\}$. Therefore, there are at least two elements x_j and y_j with $i \neq j$ not contained in X and so the leg $L_j = \{t, x_j, y_j\}$ has an odd number of elements of X. Thus, after applying the *es*-splitting operation L_j transforms to $\{x_j, y_j, \gamma\}$. Now let $L_k = \{t, x_k, y_k\}$ be another leg of Z_r. If $|L_k \cap X|$ is even, then L_k is a circuit of $(Z_r)^t_X$. But $(L_k \cap \Lambda \cap \{x_j, y_j, \gamma\}) = \emptyset$. Hence, in this case, the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}. We may now assume that every other leg of Z_r has an odd number of elements of X. Then, for all $j \neq i$, the elements x_j and y_j are not contained in X. We conclude that $|X| = 1$ and in the last row of the matrix that represents the matroid $(Z_r)^t_X$ there are two entries 1 in the corresponding columns of t and α. Hence, $(Z_r)^t_X$ has a 2-cocircuit and it is not the matroid Z_{r+1} since spikes are 3-connected matroids. □

By Lemmas 9 and 10, we must check that if $|X| = r + 1$, then, by using the *es*-splitting operation, can we build the spike Z_{r+1}?

Lemma 11. *If $r \geq 4$ and $|X \cap X_1|$ be odd, then the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}.*

Proof. Suppose that r is even and $|X \cap X_1|$ is odd. Since $t \in X$ and $|X| = r + 1$, so $|X \cap Y_1|$ must be odd. Therefore the set X must be $C \cup t$ where $C \in \varphi_3$. But $|C \cap X|$ is even and by Lemma 8, the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}. Now Suppose that r is odd, $r \geq 4$ and $|X \cap X_1|$ is odd. Then $|X \cap Y_1|$ must be even and so $X = C$ where $C \in \varphi_4$. By definition of binary spikes, there is a circuit C' in φ_4 such that $C' = C \Delta \{x_i, y_i, x_j, y_j\}$ for all i and j with $1 \leq i < j \leq r$. Clearly, $|(E - C') \cap X| = 2$. Since $(E - C')$ is a circuit of Z_r and is a member of φ_3, by Lemma 8, the matroid $(Z_r)^t_X$ is not the spike Z_{r+1}. □

Now suppose that M is a binary rank-r spike with tip t and $r \geq 4$. Let $X \subseteq E(M)$ and $e \in X$ and let $E(M) - E(M^e_X) =$

$\{\alpha, \gamma\}$ such that $\{e, \alpha, \gamma\}$ is a circuit of M_X^e. Suppose $\varphi = \cup_{i=0}^4 \varphi_i$ be the collection of circuits of M where φ_i is defined in section 2. With these preliminaries, the next two theorems are the main results of this paper.

Theorem 12. *Suppose that r is an even integer greater than three. Let M be a rank-r binary spike with tip t. Then M_X^e is a rank-$(r+1)$ binary spike if and only if $X = C$ where $C \in \varphi_4$ and $e = t$.*

Proof. Suppose that $M = Z_r$ and $X \subseteq E(M)$ and r is even. Then, by combining the last six lemmas, $|X| = r+1$; and X contains an even number of elements of X_1 with $t \in X$. The only subsets of $E(Z_r)$ with these properties are members of φ_4. Therefore $X = C$ where $C \in \varphi_4$ and by Lemma 7, $e = t$. Conversely, let $X = C$ where $C \in \varphi_4$. Then, by using Proposition 1, every leg of Z_r is preserved under the *es*-splitting operation since they have an even number of elements of X. Moreover, for $i \in \{1, 2, \ldots, r\}$, every leg L_i contains e where $e = t$. So $L_i \setminus t$ contains an odd number of elements of X and by Proposition 1, the set $(L_i \setminus t) \cup \{\alpha, \gamma\}$ is a circuit of M_X^t. Clearly, every member of φ_2 is preserved. Now let $C' \in \varphi_3$. Then $t \notin C'$. We have the following two cases.

(i) Let $C' = (E - X)\Delta\{x_{r-1}, y_{r-1}\}$. Then $|C' \cap X| = 1$ and by Proposition 1, $C' \cup \alpha$ and $C' \cup \{t, \gamma\}$ are circuits of M_X^t.

(ii) Let $C' = (E - C'')\Delta\{x_{r-1}, y_{r-1}\}$ where $C'' \neq X$ and $C'' \in \varphi_4$. Since $|X| = r+1$ and $|C'' \cap X|$ is odd, the cardinality of the set $X \cap (E - C'')$ is even and so $|C' \cap X|$ is odd. Therefore, by Proposition 1 again, $C' \cup \alpha$ and $C' \cup \{t, \gamma\}$ are circuits of M_X^t.

Evidently, if $C \in \varphi_4$, then $|C \cap X|$ is odd and by Proposition 1, $C \cup \alpha$ and $(C \setminus t) \cup \gamma$ are circuits of M_X^t. Moreover, there are no two disjoint OX-circuits in φ. So the collection $\mathcal{C}_5$ in Proposition 1 is empty. To complete the proof, suppose that α and γ play the roles of x_{r+1} and y_{r+1}, respectively, then we have the spike Z_{r+1} with collection of circuits $\psi = \psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4$ where

$$\psi_1 = \{L_i = \{t, x_i, y_i\} : 1 \le i \le r\} \cup \Lambda;$$

$$\psi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \le i < j \le r\} \cup \{(L_i \setminus t) \cup \{\alpha, \gamma\} : 1 \le i \le r\};$$

$$\psi_3 = \{C \cup \alpha : C \in \varphi_3\} \cup \{(C \setminus t) \cup \gamma : C \in \varphi_4\};$$

$$\psi_4 = \{C \cup \{t, \gamma\} : C \in \varphi_3\} \cup \{C \cup \alpha : C \in \varphi_4\}.$$

Theorem 13. *Suppose that r is an odd integer greater than three. Let M be a rank-r binary spike with tip t. Then M_X^e is a rank-$(r+1)$ binary spike if and only if $X = C \cup t$ where $C \in \varphi_3$ or $X = E(M)$, and $e = t$.*

Proof. Suppose that $M = Z_r$ and $X \subseteq E(M)$. Let $X = E(M)$. Then, by Lemma 9, the matroid M_X^t is the spike Z_{r+1} with tip γ. Now, by combining the last six lemmas., $|X| = r+1$ and X contains an even number of elements of X_1 with $t \in X$. The only subsets of $E(Z_r)$ with these properties are in $\{C \cup t : C \in \varphi_3\}$. Conversely, let $X = C \cup t$ where $C \in \varphi_3$. Clearly, every member of φ_3 contains an odd number of elements of X. Now let C' be a member of φ_4. If $C' = E(Z_r) - C$, then C' contains an odd number of elements of X. If $C' \neq E(Z_r) - C$, then there is a $C'' \in \varphi_3$ such that $C' = E(Z_r) - C''$. Therefore $|C \cap C'| = |C \cap (E(Z_r) - C'')| = |C - (C \cap C'')|$ and so $|C \cap C'|$ is even. So C' contains an odd number of elements of X and, by Proposition 1 again $C' \cup \alpha$ and $(C \setminus t) \cup \gamma$ are circuits of M_X^t. Evidently, if C_1 and C_2 are disjoint OX-circuits of Z_r, then one of C_1 and C_2 is in φ_3 and the other is in φ_4 where $C_2 = E(Z_r) - C_1$. Moreover, as $C_1 \cup C_2$ is not minimal, it follows by Proposition 1 that $\mathcal{C}_5$ is empty. Now if α and γ play the roles of x_{r+1} and y_{r+1}, respectively, then M_X^t is the spike Z_{r+1} with collection of circuits $\psi = \psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4$ where

$$\psi_1 = \{L_i = \{t, x_i, y_i\} : 1 \le i \le r\} \cup \Lambda;$$

$$\psi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \le i < j \le r\} \cup \{(L_i \setminus t) \cup \{\alpha, \gamma\} : 1 \le i \le r\};$$

$$\psi_3 = \{C \cup \alpha : C \in \varphi_3\} \cup \{(C \setminus t) \cup \gamma : C \in \varphi_4\};$$

$$\psi_4 = \{C \cup \{t, \gamma\} : C \in \varphi_3\} \cup \{C \cup \alpha : C \in \varphi_4\}.$$

□

Remark 14. Note that the binary rank-3 spike is the Fano matroid denoted by F_7. It is straightforward to check that any one of the seven elements of F_7 can be taken as the tip, and F_7 satisfies the conditions of Theorem 13 for any tip. So, there are exactly 35 subset X of $E(F_7)$ such that $(F_7)_X^e$ is the binary 4-spike where e is a tip of it. Therefore, by Theorem 13, these subsets are $X = E(F_7)$ for every element e of X and $C \cup z$ for every element z in $E(F_7)$ not contained in C with $e = z$ where C is a 3-circuit of F_7.

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