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To cite this article: Elizabeth Donovan \& Timothy Schroeder (2020): An algorithm for an -homological test for the planarity of a graph, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2019.08.013

To link to this article: https://doi.org/10.1016/j.akcej.2019.08.013

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Published online: 24 Apr 2020.

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# An algorithm for an $\ell^{2}$-homological test for the planarity of a graph 

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#### Abstract

Given a finite simple graph $\Gamma$, one is able to define the presentation of an associate Coxeter group $W(\Gamma)$ and construct a CW-complex on which the associated Coxeter group acts. The space is the so-called Davis Complex, denoted $\Sigma(\Gamma)$, and the given graph carries much of the local topological information of the space. This paper summarizes these connections including those between the $\ell^{2}$-homology of $\Sigma(\Gamma)$ and the planarity (or genus) of $\Gamma$. The main purpose of this paper is to further investigate this interesting connection between a main topic in geometric group theory (discrete group actions on cellular complexes) and the detection of planar graphs by creating an algorithm we call the $\ell^{2}$-test.


## KEYWORDS

Graph theory; planarity;
Coxeter group; Davis
Complex; $\ell^{2}$-Betti numbers

## 1. Introduction

Given any finite simple graph $\Gamma$, one can, with proper integer edge weights, use the graph to define the presentation of an associated Coxeter system. Indeed, let $S$ denote the vertex set of $\Gamma$, and let $m_{s t} \geq 2$ denote the label (or weight) on the edge $\{s, t\}$. We define a group $W_{\Gamma}$ by the following presentation:

$$
\begin{aligned}
& W_{\Gamma}=\langle S| s^{2}=1, \forall s \in S \text {, and }(s t)^{m_{s t}}=1 \\
& \text { whenever }\{s, t\} \in \operatorname{Edge}(\Gamma)\rangle .
\end{aligned}
$$

The pair $\left(W_{\Gamma}, S\right)$ is called a Coxeter system.
In several papers ([2-4]), M. Davis describes a construction which associates to any Coxeter system $(W, S)$ a CWcomplex $\Sigma(W, S)$, or simply $\Sigma$ when the Coxeter system is clear, on which $W$ acts properly and cocompactly. Furthermore, if the graph $\Gamma$ defines the Coxeter system $\left(W_{\Gamma}, S\right)$ as above, then the 1 -skeleton of the link of each 0 -cell of $\Sigma$ is $\Gamma$. This implies two things: (1) The graph $\Gamma$ carries the local topological data of $\Sigma$, and (2) If $\Gamma$ is the 1 -skeleton of a triangulation of an $(n-1)$-sphere, then $\Sigma$ is an $n$-manifold. There is a variation of Singer's Conjecture regarding the $\ell^{2}$-homology of such $\Sigma$ :

Singer's Conjecture for Coxeter groups 1.1. Let (W, S) be a Coxeter system for which the corresponding Davis complex $\Sigma$ is an n-manifold. Then $\mathcal{H}_{i}(\Sigma)=0$ for all $i$ not equal to $\frac{n}{2}$, where $\mathcal{H}_{i}(\Sigma)$ denotes the $i^{\text {th }} \ell^{2}$-(co)homology group of $\Sigma$.

For details on $\ell^{2}$-homology theory, see [4], and [7]. For further details on Coxeter systems and their corresponding Davis manifolds, see [5, 10-13], and [14].

Now, Singer's Conjecture for Coxeter groups holds for elementary reasons in dimensions 1 and 2. The key result
for our purposes is found in [14], where the author proves that Conjecture 1.1 holds in dimension 3; that is, it holds for Coxeter systems whose corresponding graphs are the 1 skeleton of a triangulation of $\mathbb{S}^{2}$. It is this result that connects the study of these Davis manifolds with the planarity of graphs. Indeed, the goal of this paper is to implement a test for the planarity of a given, finite, simple graph by utilizing the connection between this graph and the corresponding the Coxeter systems and Davis complexes. So, initially, we outline the proof of the following result:

Main Theorem. Let $\Gamma$ be a simple, connected graph, with $V>2$ vertices. If $\Gamma$ admits a metric flag labeling where $n_{e}$ (an integer $>2$ ) is the label on the edge $e$ with

$$
1-\frac{V}{2}+\frac{1}{2}\left(\sum_{\text {edges } e} \frac{1}{n_{e}}\right)>0
$$

then $\Gamma$ is not planar.
Here we note two things: (1) The result in Theorem 1 implies what we'll call the 'classical test' for planarity, specifically that the number of edges $E$ and vertices $V$ of a planar graph satisfy $E \leq 3 V-6$ in general, and $E \leq 2 V-4$ for triangle free graphs; and (2) $\ell^{2}$-homology theory gives insight into the genus of a graph. In particular, assuming a conjecture about the $\ell^{2}$-homology of Davis complexes associated to nerves that are triangulations of a genus $g$ surface, if

$$
\begin{equation*}
1-\frac{V}{2}+\frac{1}{2}\left(\sum_{\text {edges }} \frac{1}{n_{e}}\right)>g \tag{1.1}
\end{equation*}
$$

then $\Gamma$ does not embed in a surface of genus $g$. The assumed conjecture is not too far a stretch, since there is

[^0]evidence for it that coincides nicely with classical theory on the genus of a graph. See [16] for a full discussion of the inequality in 1.1, in this paper we will focus on the test for planarity.

Of course, planar graphs are well studied and algorithms to detect the planarity of a given graph abound in the literature, see $[1,17]$ for example, and especially Kuratowski's 1930 paper [9]. The purpose of this paper isn't to challenge those algorithms, per se. Rather, by exploiting the inherent connection between Coxeter group actions on the Davis complex and finite simple graphs, we demonstrate an enhanced version of the classical test, and present a test for planarity that is, in our opinion, very accessible.

## 2. Background: graphs, Coxeter groups, and $\ell^{2}$-homology

Let $S$ be a finite set of generators. A Coxeter matrix on $S$ is a symmetric $S \times S$ matrix $M=\left(m_{s t}\right)$ with entries in $\mathbb{N} \cup \infty$ such that each diagonal entry is 1 and each off diagonal entry is $\geq 2$. The matrix $M$ gives a presentation of an associated Coxeter group W:

$$
\begin{equation*}
\left.W=\langle S|(s t)^{m_{s t}}=1, \text { for each pair }(s, t) \text { with } m_{s t} \neq \infty\right\rangle \tag{2.1}
\end{equation*}
$$

The pair $(W, S)$ is called a Coxeter system. Given a subset $U$ of $S$, define $W_{U}$ to be the subgroup of $W$ generated by the elements of $U$. A subset $T$ of $S$ is spherical if $W_{T}$ is a finite subgroup of $W$. In this case, we will also say that the subgroup $W_{T}$ is spherical. The nerve of $(W, S)$ is a simplicial complex $L$ with vertex set $S$ and with the property that a non-empty subset of vertices $T$ spans a simplex of $L$ if and only if $T$ is spherical.

Define a labeling on the edges of $L$ by the map $m$ : $\operatorname{Edge}(L) \rightarrow\{2,3, \ldots\}$, where $\{s, t\} \mapsto m_{s t}$. This labeling accomplishes two things: (1) the Coxeter system can be recovered (up to isomorphism) from the labeled $L$ and (2) the underlying graph of $L$ inherits a natural piecewise spherical structure in which the edge $\{s, t\}$ has length $\pi-\frac{\pi}{m_{s}}$. $L$ is then a metric flag simplicial complex (see [3, Definition I.7.1,2]). This means that any finite set of vertices, which are pairwise connected by edges, spans a simplex of $L$ if an only if it is possible to find some spherical simplex with the given edge lengths. In other words, $L$ is metrically determined by its underlying graph. In summary, if one is given a Coxeter system ( $W, S$ ), then there corresponds a labeled, metric flag simplicial complex $L$.

The idea then, is to turn this correspondence around. That is, we wish to begin with a simplicial complex $L$, and from it define a corresponding Coxeter system for which $L$ is itself the nerve. In particular, we're taking the initial complex to be a finite, connected, simple graph, and the desire is to define a Coxeter system for which the graph itself is the nerve, not just the underlying graph, or 1 -skeleton, of a higher dimensional simplicial complex. To that end, suppose $\Gamma$ is a finite, simple, connected graph with vertex set $S$. From the presentation in Equation (2.1), the procedure may seem quite obvious: simply label the edges of the given graph with integers $\geq 2$ and define the presentation so that
each vertex corresponds to a generator with order 2 , and include relations $(s t)^{m_{s t}}=1$ when $m_{s t}$ is the label on edge $\{s$, $t\}$. The problem here is that if vertices $r, s, t$ are pairwise connected, and if the labels $m_{r s}, m_{s t}$, and $m_{r t}$ are such that

$$
\frac{1}{m_{r s}}+\frac{1}{m_{s t}}+\frac{1}{m_{r t}}>1
$$

(all 2's, for instance), then $\{r, s, t\}$ is a spherical subset and therefore the nerve of the corresponding Coxeter system must have a 2 -simplex with vertex set $\{r, s, t\}$. This would result in the defining graph $\Gamma$ not being the nerve of the corresponding Coxeter system, only the 1 -skeleton of the nerve. We must avoid this type of issue.

To develop the presentation of $\left(W_{\Gamma}, S\right)$, take $S$ as the set of generators, each of order two. (This means that we will be treating the vertices of the graph as elements of the group $W_{\Gamma .}$ ) Next, for each edge $\{s, t\}$ choose an integer $m_{s t}=m_{t s} \geq 2$ and in the presentation, include a relator of the form $(s t)^{m_{s t}}=1$. (This implies also that $(t s)^{m_{s t}}=1$.) But, as mentioned above, we must choose such labels carefully. The graph $\Gamma$ is (viewed as) a 1-dimensional simplicial complex, it has no 2 -simplices. So if we desire $\Gamma$ to be the nerve of $\left(W_{\Gamma}, S\right)$, we must have no spherical subsets with three (or more) vertices. So, a proper labeling scheme requires us to understand the types of subsets of generators/vertices that are spherical, specifically those subsets of size three. But note that finite Coxeter systems are fully classified by their "Dynkin diagrams," see [8], for instance, and so it is possible to label $\Gamma$ is such a way as to avoid unwanted spherical subsets.

First note that since we only included relators for adjacent vertices, any pair of non-adjacent vertices generates an infinite dihedral group, so spherical subsets must be comprised of pairwise connected vertices. So our concern is with 3-cycles contained in $\Gamma$. By conferring with the so-called "Dynkin diagrams" mentioned above, we see that a set $\{r, s$, $t\}$ of pairwise connected vertices generates a finite Coxeter group if and only if their edge labels satisfy

$$
\frac{1}{m_{r s}}+\frac{1}{m_{s t}}+\frac{1}{m_{r t}}>1 .
$$

So we have the following definition and resulting edge labeling strategy.
Definition 2.1. Let $\Gamma$ be a finite simple graph with vertex set $S$. We say the edge labeling $m: \operatorname{Edge}(\Gamma) \rightarrow\{2,3, \ldots\}$ is metric flag if

$$
\frac{1}{m_{r s}}+\frac{1}{m_{s t}}+\frac{1}{m_{r t}} \leq 1
$$

whenever $\{r, s, t\}$ are pairwise connected (form a 3-cycle). Here, we also say that the graph $\Gamma$ is metric flag to mean that $\Gamma$ has a metric flag labeling.

Note that Definition 2.1 can be extended to an arbitrary simplicial complex $L$ by requiring a subset of vertices $T$ to define a simplex if and only if the Coxeter system generated by the vertices $T$ with relators induced from the labeling is finite. We also call such simplicial complexes metric flag. Ultimately, we are looking at ways to take the labeled graph
or simplicial complex as the initial data, instead of a specific Coxeter system. See [14].

Now, for a given graph, there are many labelings that result in the graph being a metric flag complex. We observe that a labeling of all 3's will always work in defining a metric flag labeling of a graph $\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1\right)$, and that if a graph $\Gamma$ contains a 3 -cycle, then $\Gamma$ cannot correspond to the labeled nerve of a so-called "right-angled" Coxeter system, where all edge labels are 2 . So, our emphasis is not only be on a given graph, but also on a specific labeling of edges.

In summary, given a metric flag graph $\Gamma$, we define an associated Coxeter system $\left(W_{\Gamma}, S\right)$ where

- The set of generators $S$ is the set of vertices of $\Gamma$ (or $L$ ).
- Each generator has order 2.
- For every edge weight $m_{s t}$ on the edge $\{s, t\}$, we have a relator $(s t)^{m_{s t}}$.
- $\quad \Gamma$ is the nerve of $\left(W_{\Gamma}, S\right)$.

That is, $W_{\Gamma}$ is defined by the group presentation

$$
\begin{aligned}
& W_{\Gamma}=\langle S| s^{2}=1, \forall s \in S, \text { and }(s t)^{m_{s t}}=1 \\
& \text { whenever }\{s, t\} \in \operatorname{Edge}(\Gamma)\rangle .
\end{aligned}
$$

Example 2.2. Let $\Gamma$ be a triangle with vertices $r, s$, and $t$, with each edge labeled 3. Then $W$ is infinite, and the (empty) triangle $\Gamma$ is the nerve of the corresponding Coxeter system. Indeed, $W \cong \tilde{A}_{3}$.

### 2.1. The Davis complex

Now, to a given a metric flag graph $\Gamma$ or, more generally, such a simplicial complex $L$ with vertex set $S$, there is an associated CW-complex $\Sigma_{L}$, or simply $\Sigma$ when the nerve is clear on which $W_{L}$ acts geometrically. In lieu of a complete description of the construction of $\Sigma$, we refer the reader to [3,5], or [14], and reference its key properties through the following from [3, Proposition 7.3.4].
Proposition 2.3. There is a natural cell structure on $\Sigma_{L}$ so that

- Its vertex set is $W_{L}$ and its underlying graph is the Cayley graph of $\left(W_{L}, S\right)$.
- The link of every 0 -cell is isomorphic to $L$, the nerve of $\left(W_{L}, S\right)$.
The second bullet point of Proposition 2.3 means that if the nerve $L$ is a triangulation of an $(n-1)$-sphere, that $\Sigma$ is an $n$-dimensional manifold. For our purposes, note that if we begin with a metric flag graph $\Gamma, \Sigma$ is a 2 -dimensional complex with the property that at a given vertex $v$, the edges containing $v$ are in 1-1 correspondence with the vertices of $\Gamma$, and 2-cells containing $v$ are in 1-1 correspondence with the edges of $\Gamma$.
Example 2.4. Let $\Gamma$ be a triangle with vertices $r, s$, and $t$, and each edge labeled 3. (As in Example 2.2.) Each of $\{r, s\},\{s, t\}$, and $\{r, t\}$ generate a dihedral group of order six acting on a


Figure 1. $\Sigma_{\Gamma} \cong \mathbb{R}^{2}$, tiled by hexagons.
hexagon. There are 3 such spherical subsets, so in $\Sigma$, three hexagons meet at every vertex, and $\Sigma$ can be geometrically realized as the Euclidean plane tiled by hexagons. See Figure 1.

Example 2.5. If $\Gamma$ is the complete graph on 5 vertices, $K_{5}$, with each edge labeled 3, then $\Sigma$ is more complicated. It is not Euclidean space, and not even a manifold. However, it can be understood in terms of Example 2.4. Indeed, in this case, we can understand $\Sigma$ as having 10 types of planes, each tiled by hexagons as in Figure 1, corresponding to the $\binom{5}{3}$ different versions of the triangle in Example 2.4. The space has significant branching: each hexagon is contained in 4 different planes. Note that in $\Sigma$, the link of every vertex is isomorphic to $K_{5}$.
$\ell^{2}$-Homology and planar graphs. Let $\Sigma_{L}$ be the Davis complex constructed from the Coxeter system with nerve $L$ (in general a simplicial complex). To each cellular dimension $i$ of $\Sigma_{L}$, we can assign a non-negative rational number $\beta_{i}(L)$, called the $i^{\text {th }} \ell^{2}$-Betti number of $\Sigma_{L}$. This is the so-called von Neumann dimension of the $i^{\text {th }} \ell^{2}$-homology group of $\Sigma_{L}$. The key property of the $\ell^{2}$-Betti numbers is that $\beta_{i}(L) \geq$ 0 and $\beta_{i}(L)=0$ if and only if the $i^{\text {th }} \ell^{2}$-homology group of $\Sigma_{L}$ is 0 . We restate Singer's conjecture.

Singer's Conjecture for Coxeter groups 2.5.1. Let ( $W, S$ ) be a Coxeter system such that its nerve $L$ is a triangulation of an $(n-1)$-sphere. Then $\beta_{i}(L)=0$ for all $i$ not equal to $\frac{n}{2}$.

It is shown in [14] that Conjecture 2.5.1 is true in dimension 3. Specifically, if $L$ is a triangulation of a 2 -sphere, then $\beta_{i}(L)=0$ for all $i$.

For a graph $\Gamma$ viewed as a full subcomplex of $L$, the subgroup $W_{\Gamma}$ generated by the vertex set of $\Gamma$ is a subgroup of $W_{L}$, and $\Sigma_{\Gamma}$ is naturally a subcomplex of $\Sigma_{L}$. We use $\beta_{i}(\Gamma)$ to represent the $i^{\text {th }} \ell^{2}$-Betti number of this space. We then have the following from [15].

Theorem 2.6. Let $L$ be a metric flag triangulation of a 2sphere, $\Gamma \subseteq L$ a full subcomplex with every edge in $L-\Gamma$ labeled with 2. Then


Figure 2. $\chi^{\text {orb }}(\Gamma)=\frac{1}{4}$.

$$
\beta_{i}(\Gamma)=0 \text { for } i>1
$$

### 2.2. Orbihedral Euler characteristic

Let $\Gamma$ be metric flag. The orbihedral Euler characteristic of the quotient space of $\Sigma_{\Gamma}$ under the action of $W_{\Gamma}$, denoted $\chi^{\text {orb }}(\Gamma)$, is the rational number defined by

$$
\begin{equation*}
\chi^{\mathrm{orb}}(\Gamma)=\sum_{\sigma} \frac{(-1)^{\operatorname{dim}}(\sigma)}{\left|\operatorname{Stab}_{W}(\sigma)\right|} \tag{2.2}
\end{equation*}
$$

where the summation is over the simplices of the quotient space $\Sigma_{\Gamma} / W_{\Gamma}$, and $\left|\operatorname{Stab}_{W_{\Gamma}}(\sigma)\right|$ denotes the order of the stabilizer of the cell $\sigma$ in $W_{\Gamma}$. Then, a standard argument (see [7]) Atiyah's formula. That Equation 2.2 can be re-written in terms of the $\ell^{2}$-Betti numbers:

$$
\begin{equation*}
\chi^{\mathrm{orb}}(\Gamma)=\sum_{i=0}^{n}(-1)^{n} \beta_{i}(\Gamma) \tag{2.3}
\end{equation*}
$$

The orbihedral Euler characteristic is the last piece of the puzzle that allows us to state the $\ell^{2}$-test for planarity.

### 2.3. The $\ell^{2}$-test for planarity

Now suppose $\Gamma$ is a simple, connected graph, with metric flag labeling. The key observation is that if $\Gamma$ is a planar graph, then it can be embedded as a full subcomplex of a triangulation $L$ of a 2 -sphere. Furthermore, if $\Gamma$ also has a metric flag labeling, then this labeling can be extended to all of $L$ so that $L$ is metric flag, the labeling of $\Gamma$ is preserved, and each edge in $L-\Gamma$ is labeled with 2 . So Theorem 2.6 holds, and combining this with Atiyah's formula in Equation (2.3), we get that

$$
\chi^{\mathrm{orb}}(\Gamma)=\beta_{0}(\Gamma)-\beta_{1}(\Gamma)
$$

But, as long as the vertices of $\Gamma$ generate an infinite group (which occurs whenever $\Gamma$ has metric flag labeling and isn't
a single edge nor a single vertex), we have from [5] that $\beta_{0}(\Gamma)=0$ and so for planar graphs $\Gamma$, we have that

$$
\begin{equation*}
\chi^{\text {orb }}(\Gamma)=-\beta_{1}(\Gamma) \leq 0 \tag{2.4}
\end{equation*}
$$

So we see that the test for planarity comes down to a calculation of a modified Euler characteristic. Well, we have the following specific calculations of $\chi^{\text {orb }}(\Gamma)$ for a metric flag finite simple graph $\Gamma$, with $V=$ number of vertices, and with $n_{e}$ the edge label on edge $e$ (see [15]):

$$
\begin{equation*}
\chi^{\text {orb }}(\Gamma)=1+\frac{V}{2}+\sum_{e \in \operatorname{Edge}(\Gamma)} \frac{1}{2 n_{e}} \tag{2.5}
\end{equation*}
$$

Thus, using this formula along with that for planar graphs in Equation 2.4, we have the following test for detecting non-planar graphs, which is the main result of [15].

Theorem 2.7. ( $\ell^{2}$-test for planarity) Let $\Gamma$ be a finite, simple, connected graph, with $V>2$ vertices. If $\Gamma$ admits a metric flag labeling where $n_{e}$ (an integer $\geq 2$ ) is the label on edge $e$ with

$$
1-\frac{V}{2}+\frac{1}{2}\left(\sum_{e \in \operatorname{Edge}(\Gamma)} \frac{1}{n_{e}}\right)>0
$$

then $\Gamma$ is not planar.

### 2.4. Kuratowski's graphs: $K_{3,3}$ and $K_{5}$

Theorem 2.7 does detect that both of Kuratowski's graphs are non planar. Indeed, if $\Gamma$ is the complete bipartite graph on $3+3$ vertices, then we can label each edge with a 2 (there are no 3-cycles) and we get that

$$
\chi^{\mathrm{orb}}\left(K_{\Gamma}\right)=1-6 / 2+9 / 4=1 / 4>0
$$

If $\Gamma$ is the complete graph on 5 vertices, then we can use a uniform labeling with 3's and we have

$$
\chi^{\mathrm{orb}}\left(K_{\Gamma}\right)=1-5 / 2+10 / 6=1 / 6>0
$$

We acknowledge that there is a large similarity between the $\ell^{2}$-methods put forth above and what we'll call the "classical" argument using the usual Euler characteristic to detect nonplanar graphs. But we note that the classical inequalities actually follow from our scheme.

Corollary 2.8. Let $\Gamma$ be a finite, simple, connected, planar graph, with $V>2$ vertices and $E$ edges. Then $E \leq 3 V-6$. If, moreover, $\Gamma$ contains no 3 -cycles, then $E \leq 2 V-4$.

Proof. Take a uniform labeling of 3 on the edges of $\Gamma$. Then

$$
\chi^{\mathrm{orb}}\left(K_{\Gamma}\right)=1-\frac{V}{2}+\frac{E}{6} \leq 0
$$

which implies that $E \leq 3 V-6$. If $\Gamma$ contains no 3 -cycles, then a uniform labeling of 2 on each edge is metric flag, and we have that

$$
\chi^{\mathrm{orb}}\left(K_{\Gamma}\right)=1-\frac{V}{2}+\frac{E}{4} \leq 0
$$

which implies that $E \leq 2 V-4$.

While this may seem like a quite complicated procedure to obtain the classically known results, it is valuable since the $\ell^{2}$-test for planarity is actually stronger than these classical inequalities, meaning that the $\ell^{2}$-methods allow enable us to detect non-planar graphs that the first inequality in Corollary 2.8 cannot. For example, consider the graph $\Gamma$ in Figure 2, a member of the Petersen family of graphs. $\Gamma$ does contain 3 -cycles and we have $V=8$ and $E=15$, so $E<$ $3 V-6$; but with the indicated labeling, we have $\chi^{\text {orb }}(\Gamma)=$ $\frac{1}{4}$, and thus we know $\Gamma$ is not planar.

### 2.5. Summary

We close this by section pointing out two things about Theorem 2.7. (1) In order to use the $\ell^{2}$-test to detect a nonplanar graph, one wants to choose a metric flag labeling that produces as large an orbihedral Euler characteristic as possible. For example, labeling $K_{5}$ with 4's results in a metric flag labeling, but not a positive orbihedral Euler characteristic, and therefore that labeling does not detect $K_{5}$ as a nonplanar graph. So, our goal is to produce an algorithm that will attempt to maximize $\chi^{\text {orb }}$. (2) It is clear from Equation (2.5) that increasing any one edge label of $\Gamma$ decreases the corresponding orbihedral Euler characteristic. So, a labeling of 2's on each edge will produce the largest possible orbihedral Euler characteristic though this labeling is only metric flag for triangle-free graphs. In other words, for triangle-free graphs, the $\ell^{2}$-test is not stronger than the classical inequality $E \leq 2 V-4$, meaning it will not detect non-planar graphs the classical inequality misses. However, as shown in Figure 2, the presence of triangles means that the slight flexibility we have in labeling can be beneficial. We therefore summarize the useful "jurisdiction" of our algorithm in Figure 3.

- The region for which $E>3 V-6$ are classically detected non planar graphs. These are not missed by the $\ell^{2}$-test, since this inequality follows from a labeling of 3's. See Corollary 2.8.
- The region for which $E \leq 2 V-4$ may have non-planar graphs, but these are not detectable by the classical nor the $\ell^{2}$-methods.
- The $\ell^{2}$-test finds its advantage in the shaded region dubbed the " $\ell^{2}$-zone," on graphs $\Gamma$ containing a triangle.


## 3. An implementation of the $\ell^{2}$-test for planarity

As shown in Figure 2, and in other examples in [15], the $\ell^{2}$-homological test for planarity is more flexible than the classical test. Moreover, it is computational, which makes it more practical, in some cases, than the non-constructive criteria Kuratowski lays out for planar vs. non-planar graphs.

To implement the $\ell^{2}$-homological test, it is clear that the idea is to maximize the orbihedral Euler characteristic formula in Equation (2.5). Hence, it is also clear that a general labeling strategy should be to use as many 2's as possible. But, in order for the labeling to be metric flag, one must ensure that the sum of the reciprocals of the labels around


Figure 3. The useful jurisdiction of the $\ell^{2}$-test.
any triangle is bounded above by 1 . In particular, one cannot have two labels of 2 in any triangle. As a result, if an edge of any triangle is labeled 2, then the the naive choice for the labels on the other edges should be either a 3 and a 6 , or two 4's.

A few notes before spelling out the algorithm in Algorithm 3.1. First, note that the algorithm returns the largest orbihedral Euler characteristic (simply denoted $\chi^{\text {orb }}$, the graph $\Gamma$ is understood) resulting from three metric flag labelings: $X=\chi^{\text {orb }}$ resulting from a labeling with 2-3-6 labeled triangles, $Y=\chi^{\text {orb }}$ resulting from a labeling with 2 -4-4 triangles, and $Z=\chi^{\text {orb }}$ resulting from a constant labeling of all 3's. Second, while not advantageous in detecting non planar graphs, the algorithm will label triangle-free graphs with all 2's, and will return $X=Y$ (these values will be greater than $Z$ ). Third, one can see that the algorithm actually produces labelings, and can be adjusted to return the actual labeling scheme (instead of the largest $\chi^{\text {orb }}$ ). Finally, a full analysis of the speed of our algorithm is the focus of ongoing study, here we look at the relative strength of the algorithm, but it seems apparent to us that, in the worst case, it runs in polynomial time.

### 3.1. Algorithm

Labeling Algorithm: Given a finite, simple, connected graph $\Gamma$ with $V$ vertices and $E$ edges, proceed in the following way.

1. Recursively remove any pendant vertices until none remain. If no vertices remain, return " $-\infty$." Note that in this case, $\Gamma$ is a tree and, thus, planar.
2. For each remaining edge $e$, determine the number of triangles in which it is contained. Then give e label $n_{e}$ as follows:
(a) If $e$ is in no triangles, set $n_{e}:=2$.
(b) If $e$ is in at least three triangles, set $n_{e}:=3$.


Figure 4. Examples of graphs and their associated $(X, Y, Z)$, with an optimal labeling illustrated.
(c) If $e$ is in one triangle, if no other edge of this triangle is labeled with a 2 , set $n_{e}:=2$, else set $n_{e}:=3$.
(d) Edges not fulfilling these requirements remain unlabeled.
3. The remaining unlabeled edges are contained in exactly two triangles. Loop through the vertices, from greatest degree to least degree, from the neighbor with the largest degree to the smallest degree, choosing as a label for these edges the smallest from $\{2,3,6\}$ which still satisfies the upper bound on both triangle sums. Break any tie on degrees arbitrarily.
4. Compute $X:=\chi^{\text {orb }}$.
5. Swap all 3's and 6's for 4's. Compute $Y:=\chi^{\text {orb }}$.
6. Compute $Z:=\chi^{\mathrm{orb}}$ with 3 's on every edge.
7. Return $\max \{X, Y, Z\}$.

### 3.2. Examples

We've applied the above algorithm to the graphs shown in Figure 4, with a maximum $\chi^{\text {orb }}$ edge label indicated. In Figure 3.3, we see the 2-3-6 labeling scheme yields the largest $\chi^{\text {orb }}$ from among our three options, while in Figure 3.3 using only 2's and 4's give the optimal algorithm output. The graph given in Figure 4(c) is of particular interest since a labeling by 2's, 3's and 6's in our algorithm yields the same $\chi^{\text {orb }}$ result as labeling with 2's and 4's. In Figure 4(d), we illustrate what we call an almost $\ell^{2}$-nonplanar graph, meaning the algorithm almost detects it as nonplanar (see Section 3.3). Though not detected by our algorithm, the graph is nonplanar as it contains $K_{3,3}$ as a subgraph.

Table 1. Results of the $\ell^{2}$-tests on graphs $\Gamma$ with $3 \leq V \leq 8$.

| $V=$ | $\triangle \subset \Gamma$ | nonplanar $\Gamma$ | $E>3 V-6$ | $\chi^{\text {orb }}>0$ | $\chi^{\text {orb }}=0$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 3 | 1 | 0 | 0 | 0 | 0 |
| 4 | 2 | 0 | 0 | 0 | 0 |
| 5 | 9 | 1 | 1 | 1 | 0 |
| 6 | 55 | 11 | 4 | 6 | 4 |
| 7 | 490 | 172 | 39 | 69 | 43 |
| 8 | 7360 | 4240 | 812 | 1503 | 772 |

### 3.3. Results

To gauge the relative strength of our algorithm, we ran it on connected graphs with $3 \leq V \leq 8$, which contain a triangle and which have minimum degree $>1$. The first requirement is discussed above and is the natural limitation of this method in improving detection of nonplanar, triangle-free graphs. In the table, we've denoted this count as $\triangle \subset \Gamma$. The latter requirement eliminates trees and graphs with pendant vertices from consideration thus eliminating redundancy in our counts, since the removal of any vertex of degree one yields a graph of smaller order whose analysis would be already be included under the given conditions. For each $V$, we record four items:

1. The number of nonplanar graphs. This is known for graphs of small order.
2. The number of nonplanar graphs detected by the classical inequality, i.e. graphs for which $Z>0$. Note that these nonplanar graphs are detected by the $\ell^{2}$-test, since the classical inequality corresponds to a labeling by 3 's (Corollary 2.8).
3. The number $\ell^{2}$-nonplanar graphs, i.e. those for which the output is $>0$. The difference of these two lines gives the number of graphs detected by $\ell^{2}$-methods that are not detected by the classical inequality. That is, it demonstrates the relative strength of the $\ell^{2}$-test.
4. The number of almost $\ell^{2}$-nonplanar graphs, i.e. nonplanar graphs for which the output is 0 , so they are 'almost' detected by our algorithm. We borrow this language from [6], in which a characterization of almost planar graphs are studied. Here, we've applied this term only to known nonplanar graphs, but in general one could have planar graphs for which $\chi^{\text {orb }}=0$, and the concept would be the same: Any additional edge would result in $\chi^{\text {orb }}>0$.

One can see that the $\ell^{2}$-methods nearly double the rate of detection of nonplanar graphs over the classical inequality, and give a good indication that several more graphs are nonplanar. We acknowledge that the methods shown herein do not supplant Kuratowski's or Wagner's full classification of planar graphs, that they contain $K_{5}$ or $K_{3,3}$ as minors, but we note that Kuratowski's Theorem does not yield a fast recognition algorithm, whereas the method included here is as readily applied as Euler's formula, and, as shown in Table 1, more robust.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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