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Decomposition of complete bipartite graphs into cycles and stars with four edges

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ABSTRACT

Let C_k, S_k denote a cycle, star with k edges and let $K_{m,n}$ denotes a complete bipartite graph with m and n vertices in the parts. In this paper, we obtain necessary and sufficient conditions for the existence of a decomposition of complete bipartite graphs into cycles and stars with four edges.

KEYWORDS
Cycles; star; graph decomposition

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1. Introduction

All graphs considered here are finite. For the standard graph-theoretic terminology the reader is referred to [4]. Let C_k, S_k denote a cycle, star with k edges and let $K_{m,n}$ denotes a complete bipartite graph with m and n vertices in the parts. Also we denote the cycle C_k with vertices $x_0, x_1, \cdots, x_{k-1}$ and edges $x_0x_1, x_1x_2, \cdots, x_{k-2}x_{k-1}, x_{k-1}x_0$ as $(x_0, x_1, \cdots, x_{k-1}, x_0)$ and a star S_k consists of a centre vertex x_0 of degree k (i.e., $d(x_0) = k$) and k end vertices $x_1, x_2, \cdots, x_k$ as $(x_0; x_1, \cdots, x_k)$. If there are t stars with same end vertices $x_1, x_2, \cdots, x_k$ and different centres $y_1, y_2, \cdots, y_t$, we denote it by $(y_1, y_2, \cdots, y_t;\ x_1, x_2, \cdots, x_k)$. Note that S_k is isomorphic to $K_{1,k}$. By a *decomposition* of G, we mean a list of edge-disjoint subgraphs of G whose union is G (ignoring isolated vertices). For the graph G, if $E(G)$ can be partitioned into $E_1, \cdots, E_k$ such that the subgraph induced by E_i is H_i, for all i, $1 \leq i \leq k$, then we say that $H_1, \cdots, H_k$ decompose G and we write $G = H_1 \oplus \cdots \oplus H_k$. For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has a *H-decomposition* and it is denoted by $H|G$. If G can be decomposed into p copies of H_1 and q copies of H_2, then we say that G has a *$\{pH_1, qH_2\}$-decomposition* or *(H_1, H_2)-multidecomposition*. If such a decomposition exits for all p and q satisfying trivial necessary conditions, then we say that G has a *$\{H_1, H_2\}_{\{p,q\}}$-decomposition* or *complete $\{H_1, H_2\}$-decomposition*. We denote the number of edges of G by $e(G)$.

Cycle decomposition of graphs and star decomposition of graphs are popular topic of research in graph theory; see [3, 12–14]. The study of (K, H) -multidecomposition has been introduced by Atif Abueida and M. Daven [1]. Moreover, Atif Abueida and Theresa O'Neil [2] have settled the existence of (K, H)-multidecomposition of $K_m(\lambda)$ when $(K, H) = (K_{1,n-1}, C_n)$ for $n = 3, 4, 5$. Priyadharsini and Muthusamy [9] established necessary and sufficient condition for the existence of (G_n, H_n)-multidecomposition of λK_n where $G_n, H_n \in \{C_n, P_{n-1}, S_{n-1}\}$. Lee [7], gave necessary and sufficient condition for the multidecomposition of $K_{m,n}$ into at least one copy of C_k and S_k. Lee and J.J. Lin [8], have obtained necessary and sufficient condition for the decomposition of complete bipartite graph minus a one factor into cycles and stars. Shyu [10] considered the existence of a decomposition of $K_{m,n}$ into paths and stars with k edges, giving a necessary and sufficient condition for $k = 3$. Jeevadoss and Muthusamy [5] have obtained some necessary and sufficient condition for the existence of a decomposition of complete bipartite graphs into paths and cycles. Recently, Lee [6] established necessary and sufficient conditions for the existence of a decomposition of complete bipartite multigraph into cycles and stars with at least one copy of each. In this paper, we study about the existence of a decomposition of complete bipartite graphs into p copies of C_4 and q copies of S_4 for all possible values of p and q. We abbreviate the notation for such a decomposition as $\{pC_4, qS_4\}$-decomposition. In fact, we establish necessary and sufficient conditions for the existence of $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$.

To prove our main results, we state the following:

Theorem 1.1. [11] *Let* m, n and $l \in \mathbb{Z}_+$. *There exists an* C_{2l}*-decomposition of* $K_{m,n}$ *if and only if* m and n *are even,* $m, n \geq l \geq 2$ and $mn \equiv 0 \pmod{2l}$.

Theorem 1.2. [14] *Let* k, m and $n \in \mathbb{Z}_+$ with $m \leq n$. *There exists an* S_k*-decomposition of* $K_{m,n}$ *if and only if one of the following holds:*

(i) $k \leq m$ and $mn \equiv 0 \pmod{k}$;
(ii) $m < k \leq n$ and $n \equiv 0 \pmod{k}$.

Remarks.

1. If G_1 and G_2 have a $\{pC_4, qS_4\}$-decomposition, then $G_1 \oplus G_2$ has a such decomposition.

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2. Let $X+Y=\{(x_1+y_1,x_2+y_2)\mid (x_1,x_2)\in X,(y_1,y_2)\in Y\}$ and rX is the sum of r copies of X.

2. Necessary conditions

The following Lemmas gives necessary conditions for the existence of a $\{pC_4,qS_4\}$-decomposition of $K_{m,n}$.

Lemma 2.1. *Let $p,q\geq 0$ and even $n\geq 2$. If $K_{2,n}$ has a $\{pC_4,qS_4\}$-decomposition, then q must be even.*

Proof. Let D be an arbitrary $\{pC_4,qS_4\}$-decomposition of $K_{2,n}$. Then $4(p+q)=e(K_{2,n})$. Let $V(K_{2,n})=(X_1,X_2)$, where $X_1=\{x_{11},x_{12}\}$ and $X_2=\{x_{21},\cdots,x_{2n}\}$. Since x_{11} and x_{12} must be a centre vertex of each S_4's in D and each C_4's in D must contains both x_{11} and x_{12}. Therefore the number of copies of S_4 centered in both x_{11} and x_{12} are the same. □

Lemma 2.2. *Let $p,q\geq 0$ and even $n\geq 4$. If $K_{4,n}$ has a $\{pC_4,qS_4\}$-decomposition, then $p,q\neq 1$.*

Proof. Let D be an arbitrary $\{pC_4,qS_4\}$-decomposition of $K_{m,n}$. Then $4(p+q)=e(K_{m,n})$. On the contrary, suppose that $q=1$. Let S_4^1 denote the only star in D. It follows that each end vertex of S_4^1 has odd degree in $K_{4,n}\backslash E(S_4^1)$, which cannot have a cycle decomposition and hence a contradiction. On the other hand, let $V(K_{4,n})=(X_1,X_2)$, where $X_1=\{x_{11},\cdots,x_{14}\}$ and $X_2=\{x_{21},\cdots,x_{2n}\}$. Assume that there exist a $(4;1,n-1)$-decomposition of $K_{4,n}$. Without loss of generality, let $C_4^1=(x_{11},x_{21},x_{12},x_{22},x_{11})$ be the only C_4 in D. Then by assumption $K_{4,n}\backslash E(C_4^1)=G$ has an S_4-decomposition. In G, the vertices x_{21} and x_{22} have exactly degree 2. So it could not be a centre vertex of any stars in D. Therefore we have two stars S_4^1 and S_4^2 whose centre vertex is x_{13} and x_{14} respectively, which consist of x_{21} and x_{22} as end vertices. That is, $S_4^1=(x_{13};x_{21},x_{22},x_{2i},x_{2j})$ and $S_4^2=(x_{14};x_{21},x_{22},x_{2i'},x_{2j'})$ for some $i\neq j$ and $i'\neq j'\in\mathbb{Z}_+$. We define $D'=\{S_4\in D\mid$ Centre vertex of S_4 is $x_{14}\}$; $|D'|=r$, where $r\in\mathbb{Z}_+$ and let $X_2'=\{y\in X_2\mid y$ is an end vertex of $S_4\in D'\}$. Then clearly $S_4^2\in D'$ and $|X_2'|=4r$. Every vertex in $X_2\backslash X_2'$ must be centre vertex of some stars in $D\backslash D'$. Otherwise we cannot use the edges between $x_{14}\in X_1$ and $X_2\backslash X_2'$ in any star in D, whose centre vertex is not x_{14}.

Now we collect all the stars whose centre in $X_2\backslash X_2'$ and denote it D''. That is, $D''=\{S_4\in D\mid$ Centre vertex of S_4 in $X_2\backslash X_2'\}$. Then the new graph $G'=K_{4,n}\backslash\{E(C_4^1)\cup E(D')\cup E(D'')\cup E(S_4^1)\}\cong K_{3,4r-2}\backslash\{x_{13}x_{2i}\}\cup\{x_{13}x_{2j}\}$. Let $V(K_{3,4r-2})=(Y_1,Y_2)$, where $Y_1=X_1\backslash\{x_{14}\}$ and $Y_2=X_2\backslash(X_2\backslash X_2'\cup\{x_{21},x_{22}\})$. By our assumption G' has an S_4-decomposition. In G', $d(x_{11})=d(x_{12})=4r-2$ and $d(x_{13})=4r-4$ also $d(x_{2i})=d(x_{2j})=2$ and $d(x_{2k})=3$, where $x_{2k}\neq x_{2i},x_{2j}$ and $x_{2k}\in X_2$. It follows that no vertex of Y_2 can be a centre vertex of any stars, so the centre vertices of stars must be in Y_1. Since $d(x_{11})=d(x_{12})=4r-2\not\equiv 0 \pmod 4$, by Theorem 1.2, the graph G' cannot have S_4-decomposition, which is contradiction to our assumption. □

Lemma 2.3. *Let $p,q\geq 0$ and even $m,n\geq 6$ with $m\leq n$. If $K_{m,n}$ has a $\{pC_4,qS_4\}$-decomposition, then $q\neq 1$.*

Proof. Let D be an arbitrary $\{pC_4,qS_4\}$-decomposition of $K_{m,n}$. On the contrary, suppose $q=1$, we obtain a contradiction as in Lemma 2.2. □

Lemma 2.4. *Let p,q be nonnegative integers, m is odd (resp., n is odd) and $n\equiv 0 \pmod 4$(resp., $m\equiv 0 \pmod 4$) such that $m<n$. If $K_{m,n}$ has a $\{pC_4,qS_4\}$-decomposition, then $q\geq\frac{n}{4}$ (resp., $q\geq\frac{m}{4}$).*

Proof. Let D be an arbitrary $\{pC_4,qS_4\}$-decomposition of $K_{m,n}$. Let the q copies of stars be $S_4^i\in D$, $1\leq i\leq q$ and $G=K_{m,n}\backslash E(\sum_{i=1}^{q}S_4^i)$. By hypothesis, G has a C_4-decomposition. It follows that every vertex of G must be of even degree. Note that when n is even (resp., m), each vertex of X_2 (resp., X_1) must be either an end vertex or the centre of some S_4^i, $1\leq i\leq q$. It implies that $4q\geq n$ (resp., $4q\geq m$). □

3. Sufficient conditions

The following sequence of lemmas we show that the above necessary conditions are also sufficient.

Lemma 3.1. *If $m,n\in 2\mathbb{Z}_+$ with $2\leq m\leq n\leq 8$, then there exists a $\{pC_4,qS_4\}$-decomposition of $K_{m,n}$.*

Proof. **Case 1.** For $m=2$ and $n=2$, trivially one C_4. For $n=4$, let $V(K_{2,4})=(X_1,X_2)$, where $X_1=\{x_{11},x_{12}\}$ and $X_2=\{x_{21},\cdots,x_{24}\}$. Then the required $\{pC_4,qS_4\}$-decompositions are as given below:

1. $p=2$ and $q=0$.
   By Theorem 1.1, we get the required $2C_4$'s.
2. $p=0$ and $q=2$.
   By Theorem 1.2, we get the required $2S_4$'s.

For $n=6$, let $V(K_{2,6})=(X_1,X_2)$, where $X_1=\{x_{11},x_{12}\}$ and $X_2=\{x_{21},\cdots,x_{26}\}$. Then the required $\{pC_4,qS_4\}$-decompositions are as given below:

1. $p=3$ and $q=0$.
   By Theorem 1.1, we get the required $3C_4$'s.
2. $p=1$ and $q=2$.
   $(x_{11},x_{21},x_{12},x_{22},x_{11})$ and $(x_{11},x_{12};x_{23},x_{24},x_{25},x_{26})$.

For $n=8$, we can write, $K_{2,8}=K_{2,4}\oplus K_{2,4}$. Then the graph $K_{2,4}$ has a $\{pC_4,qS_4\}$-decomposition, by the starting of the proof. Hence, by the remark, the graph $K_{2,8}$ has a desired decomposition.

Case 2. For $m=4$ and $n=4$, let $V(K_{4,4})=(X_1,X_2)$, where $X_1=\{x_{11},\cdots,x_{14}\}$ and $X_2=\{x_{21},\cdots,x_{24}\}$. Then the required $\{pC_4,qS_4\}$-decompositions are as given below:

1. $p = 4$ and $q = 0$.
$(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11}), (x_{13}, x_{21}, x_{14}, x_{22}, x_{13})$ and $(x_{13}, x_{23}, x_{14}, x_{24}, x_{13})$.
2. $p = 2$ and $q = 2$.
The first two C_4's in (1) and the $2S_4$'s $(x_{13}, x_{14}; x_{21}, x_{22}, x_{23}, x_{24})$ gives the required decomposition.
3. $p = 0$ and $q = 4$.
By Theorem 1.2, we get the required $4S_4$'s.

For $m = 4$ and $n = 6$, let $V(K_{4,6}) = (X_1, X_2)$, where $X_1 = \{x_{11}, \cdots, x_{14}\}$ and $X_2 = \{x_{21}, \cdots, x_{26}\}$. Then the required $\{pC_4, qS_4\}$-decompositions are as given below:

1. $p = 6$ and $q = 0$.
$(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11}), (x_{11}, x_{25}, x_{12}, x_{26}, x_{11}), (x_{13}, x_{21}, x_{14}, x_{22}, x_{13})$,
$(x_{13}, x_{23}, x_{14}, x_{24}, x_{13})$ and $(x_{13}, x_{25}, x_{14}, x_{26}, x_{13})$.
2. $p = 4$ and $q = 2$.
The first four C_4's in (1) and the $2S_4$'s $(x_{13}, x_{14}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
3. $p = 3$ and $q = 3$.
The first three C_4's in (1) and the $3S_4$'s $(x_{11}; x_{21}, x_{22}, x_{23}, x_{24})$, $(x_{12}; x_{21}, x_{22}, x_{25}, x_{26})$, $(x_{13}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
4. $p = 2$ and $q = 4$.
The $2C_4$'s $(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{13}, x_{21}, x_{14}, x_{22}, x_{13})$ and the $4S_4$'s $(x_{11}, x_{12}, x_{13}, x_{14}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
5. $p = 0$ and $q = 6$.
By Theorem 1.2, we get the required $6S_4$'s.

For $n = 8$, we can write, $K_{4,8} = K_{4,6} \oplus K_{4,2}$. Both the graphs $K_{4,6}$ and $K_{4,2}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{4,8}$ has a desired decomposition.
Case 3. For $m = 6$ and $n = 6$, let $V(K_{6,6}) = (X_1, X_2)$, where $X_1 = \{x_{11}, \cdots, x_{16}\}$ and $X_2 = \{x_{21}, \cdots, x_{26}\}$. Then the required $\{pC_4, qS_4\}$ -decompositions are as given below:

1. $p = 9$ and $q = 0$.
$(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11}), (x_{11}, x_{25}, x_{12}, x_{26}, x_{11}), (x_{13}, x_{21}, x_{14}, x_{22}, x_{13}), (x_{13}, x_{23}, x_{14}, x_{24}, x_{13}), (x_{13}, x_{25}, x_{14}, x_{26}, x_{13}), (x_{15}, x_{21}, x_{16}, x_{22}, x_{15}), (x_{15}, x_{23}, x_{16}, x_{24}, x_{15}), (x_{15}, x_{25}, x_{16}, x_{26}, x_{15})$.
2. $p = 7$ and $q = 2$.
The first four and last three C_4's in (1) and the $2S_4$'s $(x_{13}, x_{14}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
3. $p = 6$ and $q = 3$.
The first three and last three C_4's in (1) and the $3S_4$'s $(x_{11}; x_{21}, x_{22}, x_{23}, x_{24})$, $(x_{12}; x_{21}, x_{22}, x_{25}, x_{26})$, $(x_{13}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
4. $p = 5$ and $q = 4$.
The last three C_4's in (1) along with $(x_{11}, x_{21}, x_{12}, x_{22}, x_{11})$, $(x_{13}, x_{21}, x_{14}, x_{22}, x_{13})$ and the $4S_4$'s $(x_{11}, x_{12}, x_{13}, x_{14}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
5. $p = 4$ and $q = 5$.
The $4C_4$'s $(x_{11}, x_{25}, x_{13}, x_{26}, x_{11}), (x_{12}, x_{23}, x_{14}, x_{24}, x_{12})$, $(x_{13}, x_{21}, x_{14}, x_{22}, x_{13})$, $(x_{15}, x_{21}, x_{16}, x_{22}, x_{15})$ and the $5S_4$'s $(x_{11}, x_{12}; x_{21}, x_{22}, x_{25}, x_{26})$, $(x_{13}, x_{15}, x_{16}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
6. $p = 3$ and $q = 6$.
The $3C_4$'s $(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{13}, x_{21}, x_{14}, x_{22}, x_{13})$, $(x_{15}, x_{21}, x_{16}, x_{22}, x_{15})$ and the $6S_4$'s $(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}; x_{23}, x_{24}, x_{25}, x_{26})$ gives the required decomposition.
7. $p = 2$ and $q = 7$.
The $2C_4$'s $(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11})$ and the $7S_4$'s $(x_{13}; x_{21}, x_{22}, x_{23}, x_{26})$, $(x_{14}; x_{21}, x_{22}, x_{23}, x_{26})$, $(x_{15}; x_{21}, x_{22}, x_{23}, x_{24})$, $(x_{16}; x_{21}, x_{22}, x_{23}, x_{24})$, $(x_{24}; x_{11}, x_{12}, x_{13}, x_{14})$, $(x_{25}; x_{13}, x_{14}, x_{15}, x_{16})$, $(x_{26}; x_{11}, x_{12}, x_{15}, x_{16})$ gives the required decomposition.
8. $p = 1$ and $q = 8$.
$(x_{11}, x_{21}, x_{12}, x_{22}, x_{11})$ and the $8S_4$'s $(x_{15}, x_{16}; x_{23}, x_{24}, x_{25}, x_{26})$, $(x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}; x_{11}, x_{12}, x_{13}, x_{14})$ gives the required decomposition.
9. $p = 0$ and $q = 9$.
By Theorem 1.2, we get the required $9S_4$'s.

For $n = 8$, we can write, $K_{6,8} = K_{6,4} \oplus K_{6,4}$. Then we obtained $(p, q) \in \{(12, 0), (10, 2), \cdots, (2, 10), (0, 12)\}$. The case $(1, 11)$ can be obtain by taking, $K_{6,8} = K_{6,2} \oplus K_{6,6}$, we have $(1, 11) = (1, 2) + (0, 9)$, by the Cases 1 and 3 above procedure. Hence, the graph $K_{6,8}$ has a desired decomposition.
Case 4. For $m = 8$ and $n = 8$, we can write, $K_{8,8} = 2K_{4,6} \oplus 2K_{2,4}$. By Case 1 above, we obtained $(p, q) \in \{(16, 0), (14, 2), \cdots, (2, 14), (0, 16)\}$. The case $(1, 15)$ can be obtain by taking, $K_{8,8} = K_{6,8} \oplus 2K_{2,4}$, we have $(1, 15) = (1, 11) + (0, 4)$, by the Case 1 and 3 above procedure. Hence, the graph $K_{8,8}$ has a desired decomposition. □

Lemma 3.2. *If $m, n \in 2\mathbb{Z}_+$ with $2 \le m \le 8$ and $n \ge 10$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$.*

Proof. **Case 1.** For $m = 2$, we distinguish two subcases.
Subcase 1. $n \equiv 2 \pmod 4$, we have $n = 4x + 2$, where $x \ge 2$.
For $x = 2$ we have, $K_{2,10} = K_{2,8} \oplus K_{2,2}$. Then the graph $K_{2,10}$ has a $(4; p, q)$-decomposition, by Lemma 3.1. For $x \ge 3$, we can write, $K_{2,4x+2} = K_{2,10} \oplus (x - 2)K_{2,4}$. By the above procedure and Lemma 3.1, both the graphs $K_{2,10}$ and $K_{2,4}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{2,4x+2}$ has a desired decomposition.
Subcase 2. $n \equiv 0 \pmod 4$, we have $n = 4x$, where $x \ge 3$. We can write, $K_{2,4x} = xK_{2,4}$. Hence, the graph $K_{2,4x}$ has a $\{pC_4, qS_4\}$-decomposition, by Lemma 3.1.
For $m = 4, 6$ and $n = 10$, we can write, $K_{m,10} = K_{m,6} \oplus K_{m,4}$. Hence, the graph $K_{m,10}$ has a $\{pC_4, qS_4\}$-decomposition, by Lemma 3.1.
Let $n > 10$, we have $n = 4x + y$, where $3 \le x \in \mathbb{Z}_+$ and $y = 0, 2$. We can write, $K_{m,n} = K_{m,4x+y}$.

Case 2. For $m = 4$, we can write, $K_{4,4x+y} = K_{4,10} \oplus \left(\frac{4x+y}{2} - 5\right)K_{4,2}$. By the above procedure and Lemma 3.1, both the graphs $K_{4,10}$ and $K_{4,2}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{4,4x+y}$ has the desired decomposition.

Case 3. For $m = 6$. We distinguish two subcases.

Subcase 1. $n \equiv 2 \pmod 4$, we have $n = 4x + 2$, where $3 \le x \in \mathbb{Z}_+$, we can write, $K_{6,4x+2} = (x-1)K_{6,4} \oplus K_{6,6}$. Hence, the graph $K_{6,4x+2}$ has a $\{pC_4, qS_4\}$-decomposition, by Lemma 3.1.

Subcase 2. $n \equiv 0 \pmod 4$, we have $n = 4x$, where $3 \le x \in \mathbb{Z}_+$, we can write, $K_{6,4x} = (x-1)K_{6,4} \oplus K_{6,4}$. Hence, the graph $K_{6,4x}$ has a $\{pC_4, qS_4\}$-decomposition, by Lemma 3.1.

Case 4. For $m = 8$ and $n = 2x$, where $x \ge 5$, we can write, $K_{8,2x} = K_{8,8} \oplus (x-4)K_{8,2}$. Note that $K_{8,2} \cong K_{2,8}$. Hence, the graph $K_{8,2x}$ has a $\{pC_4, qS_4\}$-decomposition, by Lemma 3.1. □

Lemma 3.3. *If $m \in 2\mathbb{Z}_+$ with $m \equiv 0 \pmod 4 \ge 12$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{\frac{m}{2},m}$, where $q \ne 1$.*

Proof. We distinguish two cases.

Case 1. $m = 12$ and $q \ne 1$.

We can write, $K_{6,12} = K_{6,6} \oplus K_{6,6}$. By Lemma 3.1, the graph $K_{6,6}$ has a $\{pC_4, qS_4\}$-decomposition. Hence, the graph $K_{6,12}$ has the desired decomposition

Case 2. $m > 12$ and $q \ne 1$. We can write,

$$K_{\frac{m}{2},m} = K_{6,12} \oplus K_{6,m-12} \oplus \left(\frac{m-12}{4}\right)(K_{2,12} \oplus K_{2,m-12}).$$

By the Case 1 above and Lemma 3.2, the graphs $K_{6,12}, K_{2,12}$ and $K_{2,m-12}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{\frac{m}{2},m}$ has the desired decomposition. □

Lemma 3.4. *If $m \in 2\mathbb{Z}_+$ with $m \equiv 0 \pmod 4 \ge 12$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,m}$, where $q \ne 1$.*

Proof. For $m \ge 12$ and $q \ne 1$. We can write,

$$K_{m,m} = K_{8,8} \oplus \left(\frac{m-12}{4}\right)K_{4,4} \oplus \left\{\bigoplus_{i}^{m-4} K_{i,4}\right\} \oplus \left\{\bigoplus_{j}^{m-4} K_{4,j}\right\},$$

where $i \equiv 0 \pmod 4) \ge 4$ and $j \equiv 0 \pmod 4) \ge 8$. Note that $K_{i,4} \cong K_{4,i}$. By Lemmas 3.1 and 3.2, the graphs $K_{4,4}, K_{8,8}, K_{i,4}$ and $K_{4,j}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{m,m}$ has the desired decomposition. □

Lemma 3.5. *If $m \in 2\mathbb{Z}_+$ with $m \equiv 2 \pmod 4 \ge 10$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,m}$, where $q \ne 1$.*

Proof. We can write,

$$K_{m,m} = K_{6,6} \oplus \left(\frac{m-6}{4}\right)K_{4,4} \oplus \{\oplus_i^{m-4} K_{4,i}\} \oplus \{\oplus_j^{m-4} K_{j,4}\},$$

where $i, j \equiv 2 \pmod 4) \ge 6$. Note that $K_{j,4} \cong K_{4,j}$. By Lemmas 3.1 and 3.2, the graphs $K_{4,4}, K_{6,6}, K_{4,i}$ and $K_{j,4}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{m,m}$ has the desired decomposition. □

Lemma 3.6. *If $m, n \in 2\mathbb{Z}_+$ with $n \ge m \ge 12$ and $m, n \equiv 0 \pmod 4$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \ne 1$.*

Proof. We can write,

$$K_{m,n} = K_{m,m} \oplus \left(\frac{n-m}{4}\right)K_{m,4}.$$

Note that $K_{m,4} \cong K_{4,m}$. By Lemmas 3.2 and 3.4, the graphs $K_{m,4}$ and $K_{m,m}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{m,n}$ has the desired decomposition. □

Lemma 3.7. *If $m, n \in 2\mathbb{Z}_+$ with $n \ge m \ge 12$; $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \ne 1$.*

Proof. Let $m = 4x$ and $n = 4y + 2$, where $x, y \in \mathbb{Z}_+$ and $y \ge x \ge 3$. Hence $K_{m,n} = K_{4x,4y+2}$. We can write, $K_{4x,4y+2} = K_{4x,4(y-1)} \oplus K_{4x,4+2}$. Since the graph $K_{4x,4(y-1)}$ can be viewed as $(y-1)$ copies of $K_{4x,4}$. Note that $K_{4x,6} \cong K_{6,4x}$. By Lemma 3.2, both the graphs $K_{4x,4}$ and $K_{4x,6}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{m,n}$ has the desired decomposition. □

Lemma 3.8. *If $m, n \in 2\mathbb{Z}_+$ with $n \ge m \ge 10$; $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \ne 1$.*

Proof. By the similar argument as in Lemma 3.7, we get a required decomposition. □

Lemma 3.9. *If $m, n \in 2\mathbb{Z}_+$ with $n \ge m \ge 10$; $m, n \equiv 2 \pmod 4$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \ne 1$.*

Proof. We can write,

$$K_{m,n} = K_{m,m} \oplus \left(\frac{n-m}{4}\right)K_{m,4}.$$

Note that $K_{m,4} \cong K_{4,m}$. By Lemma 3.2 and 3.5, the graphs $K_{m,4}$ and $K_{m,m}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{m,n}$ has the desired decomposition. □

Lemma 3.10. *If $m \in \{3, 5, 7\}$ and $n = 4$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$.*

Proof. We distinguish three cases.

Case 1. For $m = 3$ and $n = 4$. Let $V(K_{3,4}) = (X_1, X_2)$, where $X_1 = \{x_{11}, x_{12}, x_{13}\}$ and $X_2 = \{x_{21}, \cdots, x_{24}\}$. Then the required $\{pC_4, qS_4\}$-decompositions are as given below:

1. $p = 0$ and $q = 3$.
   $(x_{11}, x_{12}, x_{13}; x_{21}, x_{22}, x_{23}, x_{24})$.

2. $p = 2$ and $q = 1$.
$(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11})$ and $(x_{13}; x_{21}, x_{22}, x_{23}, x_{24})$.

Case 2. For $m = 5$ and $n = 4$. We can write, $K_{5,4} = K_{4,4} \oplus K_{1,4}$. Note that $K_{5,4} \cong K_{4,5}$. By Lemma 3.1, the graph $K_{4,4}$ has a $\{pC_4, qS_4\}$-decomposition and trivially the graph $K_{1,4}$ is S_4. Hence, the graph $K_{5,4}$ has the desired decomposition.
Case 3. For $m = 7$ and $n = 4$. We can write, $K_{7,4} = K_{6,4} \oplus K_{1,4}$. By Lemma 3.1 the graph $K_{6,4}$ has a $\{pC_4, qS_4\}$-decomposition and trivially the graph $K_{1,4}$ is S_4. Hence, the graph $K_{7,4}$ has the desired decomposition. □

Lemma 3.11. *If $m = 3$ and $n \in 2\mathbb{Z}_+$ with $n \equiv 0 \pmod 4 \geq 8$, then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \geq \frac{n}{4}$.*

Proof. We distinguish two cases.
Case 1. For $m = 3$ and $n = 8$, let $V(K_{3,8}) = V(X_1, X_2)$, where $X_1 = \{x_{11}, x_{12}, x_{13}\}$, $X_2 = \{x_{21}, \cdots, x_{28}\}$ and $E(K_{3,8}) = \{x_{1i}x_{2j} | \ i = 1, 2, 3$ and $j = 1, \cdots, 8\}$. Then the required $\{pC_4, qS_4\}$-decompositions are as given below:

1. $p = 4$ and $q = 2$.
The $4C_4$'s $(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11}), (x_{11}, x_{25}, x_{12}, x_{26}, x_{11}), (x_{11}, x_{27}, x_{12}, x_{28}, x_{11})$ and the $2S_4$'s $(x_{13}; x_{21}, x_{22}, x_{23}, x_{24})$, $(x_{13}; x_{25}, x_{26}, x_{27}, x_{28})$ gives the required decomposition.
2. $p = 3$ and $q = 3$.
The $3C_4$'s $(x_{11}, x_{21}, x_{23}, x_{22}, x_{11}), (x_{11}, x_{24}, x_{13}, x_{25}, x_{11}), (x_{12}, x_{26}, x_{13}, x_{28}, x_{12})$ and the $3S_4$'s $(x_{11}; x_{22}, x_{26}, x_{27}, x_{28})$, $(x_{12}; x_{22}, x_{24}, x_{25}, x_{27})$, $(x_{13}; x_{21}, x_{22}, x_{23}, x_{27})$ gives the required decomposition.
3. $p = 2$ and $q = 4$.
The $2C_4$'s $(x_{11}, x_{21}, x_{12}, x_{22}, x_{11}), (x_{11}, x_{23}, x_{12}, x_{24}, x_{11})$ and the $4S_4$'s $(x_{13}; x_{21}, x_{22}, x_{23}, x_{24}), (x_{11}, x_{12}, x_{13}; x_{25}, x_{26}, x_{27}, x_{28})$ gives the required decomposition.
4. $p = 0$ and $q = 6$.
The $6S_4$'s $(x_{11}, x_{12}, x_{13}; x_{21}, x_{22}, x_{23}, x_{24}), (x_{11}, x_{12}, x_{13}; x_{25}, x_{26}, x_{27}, x_{28})$.

Case 2. For $m = 3, n > 8$ and $q \geq \frac{n}{4}$. We can write,

$$K_{3,n} = K_{3,8} \oplus \left(\frac{n-8}{4}\right) K_{3,4}.$$

By Lemma 3.10 and the Case 1 above, the graphs $K_{3,4}$ and $K_{3,8}$ have a $\{pC_4, qS_4\}$-decomposition. Hence, by the remark, the graph $K_{3,n}$ has the desired decomposition. □

Lemma 3.12. *Let m be an odd integer and $n \in 2\mathbb{Z}_+$ with $2 < m < n$ and $n \equiv 0 \pmod 4 \geq 4$. Then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \geq \frac{n}{4}$.*

Proof. Let $m = 4x + s$ and $n = 4y$, where $x, y \in \mathbb{Z}_+$ with $y \geq x \geq 1$ and $s = 1, 3$. We can write, $K_{m,n} = K_{m-1,n} \oplus K_{1,n}$, we have $K_{4x+s,4y} = K_{4x+s-1,4y} \oplus K_{1,4y}$. Since $K_{4x+s-1,4y}$ can be viewed as x copies of $K_{4+s-1,4y}$. Then the graph $K_{4+s-1,4y}$ has a $\{pC_4, qS_4\}$-decomposition, by Lemma 3.2 and trivially the graph $K_{1,4y}$ has an S_4-decomposition. Hence, the graph $K_{4x+s,4y}$ has the desired decomposition. □

Lemma 3.13. *Let n be an odd integer and $m \equiv 0 \pmod 4), n > m \geq 4$. Then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$, where $q \geq \frac{m}{4}$.*

Proof. By the similar argument as in Lemma 3.12, we get a required decomposition. □

4. Conclusion

As a consequence of Lemmas 2.1–2.4 and 3.1–3.13, our main result immediately follows.

Theorem 4.1. *Let p and q be nonnegative integers, and let m and n be positive integers such that $m \leq n$. Then there exists a $\{pC_4, qS_4\}$-decomposition of $K_{m,n}$ if and only if one of the following holds:*

1. *q is even, when $m = 2$ and even $n \geq 2$;*
2. *$p, q \neq 1$, when $m = 4$ and even $n \geq 4$;*
3. *$q \neq 1$ when even $m, n \geq 6$;*
4. *$q \geq \frac{n}{4}$ (resp., $q \geq \frac{m}{4}$), when m (resp., n) is an odd integer and $n \equiv 0 \pmod 4$(resp., $m \equiv 0 \pmod 4$).*

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