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On inclusive *d*-distance irregularity strength on triangular ladder graph and path

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ABSTRACT

The length of a shortest path between two vertices u and v in a simple and connected graph G, denoted by $d(u, v)$, is called the distance of u and v. An inclusive vertex irregular d-distance labeling is a labeling defined as $\lambda : V(G) \to \{1, ..., k\}$ such that the vertex weight, that is $\text{wt}(v) = \lambda(v) + \sum_{\{u:1 \le d(u,v) \le d\}} \lambda(u)$, are all distinct. The minimal value of the largest label used over all such labeling of graph G, denoted by $\text{dis}_d^0(G)$, is defined as inclusive d-distance irregularity strength of G. Others studies have concluded the lower bound value of $\text{dis}_d^0(G)$ and the value of $\text{dis}_1^0(\mathbb{L}_n)$. In this paper, we generalize the lower bound value of $\text{dis}_d^0(G)$ for $d \ge 2$. We used the lower bound value of $\text{dis}_d^0(G)$ and the previous result of $\text{dis}_1^0(\mathbb{L}_n)$ to investigate the value of $\text{dis}_2^0(\mathbb{L}_n)$. As a result, we found the exact values of $\text{dis}_2^0(\mathbb{L}_n)$ for the cases $n \equiv 0, 3, 4, 5, 8 \pmod 9$, $n = 7$, and the value of the upper bound of $\text{dis}_2^0(\mathbb{L}_n)$ for other n. We also found the relation of the value of $\text{dis}_d^0(\mathbb{L}_n)$ and the value of $\text{dis}_{2d}^0(P_{2n})$. Further investigation on path brought us to conclude the exact value of $\text{dis}_2^0(P_n), \text{dis}_3^0(P_n)$ and $\text{dis}_4^0(P_n)$ for some n.

1. Introduction

Consider a simple and connected graph G with $V(G)$ as a vertex set of G and $E(G)$ as an edge set of G. A distance between vertex u and vertex v, with $u, v \in V(G)$, is defined as the length of the shortest path between them and denoted by $d(u, v)$. The set of all vertices u with $d(u, v) = 1$ is called the neighbor of vertex v and denoted by $N(v)$ [4]. We use the notation $N_d(v)$ to represent the set of all vertices u with $d(u, v) \le d$ and $N_d[v]$ to represent $N_d(v) \cup \{v\}$. A degree of $v \in G$ is the number of adjacent vertices of v, or simply denoted by $|N(v)|$. The smallest degree of G is denoted by $\delta(G)$, while the largest degree is denoted by $\Delta(G)$.

Motivated by distance labeling that is introduced by Miller et al. [5] and irregular assignment by Chartrand et al. [3], Slamin [6] introduced a new variation of vertex labeling, that is called distance irregular labeling. It is a vertex labeling $f : V(G) \to \{1, ..., k\}$, such that the sum of adjacent vertex labels at each vertex are distinct. In this labeling, f does not need to be injective. The weight of vertex v, $\text{wt}(v)$, in this labeling is generalized to the sum of all labels of vertices u which $d(u, v) \le d$. Moreover, the labeling is generalized to non-inclusive and inclusive vertex irregular d-distance labeling by Bong et al. [2]. In the inclusive vertex labeling, the label of vertex v is included when computing $\text{wt}(v)$, while in the non-inclusive, it is excluded. The definition of the inclusive vertex irregular d-distance labeling is given below.

Definition 1.1 [2]. An inclusive vertex irregular d-distance labeling λ is an irregular labeling of vertices in a graph G where the weight of a vertex $v \in V(G)$ is the sum of the label of v and all labels up to the distance d from v, that is $\text{wt}(v) = \lambda(v) + \sum_{\{u:1 \le d(u,v) \le d\}} \lambda(u)$. The largest label used in this labeling is called the inclusive d-distance irregularity strength of G and denoted by $\text{dis}_d^0(G)$.

If such a labeling exists for any graph G, then the value of $\text{dis}_d^0(G)$ can be determined. Meanwhile, if it does not exist, the value of $\text{dis}_d^0(G)$ is defined as ∞. Theorem 1.1 gives a sufficient and necessary condition for $\text{dis}_d^0(G) < \infty$, while Lemma 1.1 gives a lower bound value of $\text{dis}_d^0(G)$ for any graph G.

Theorem 1.1 [1]. *For any graph G, $\text{dis}_1^0(G) = \infty$ if and only if there exist two distinct vertices $u, v \in V(G)$ such that $N[u] = N[v]$.*

Lemma 1.1 [2]. *Let G be a graph with the number of vertices $|V(G)|$, the smallest degree $\delta(G)$, and the largest degree $\Delta(G)$. Then $\text{dis}_1^0(G) \ge \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil$.*

We derived our results in this paper based on Theorem 1.1, Lemma 1.1 and the previous study on triangular ladder graph $\mathbb{L}_n$ by Utami et al. [7]. Illustration of the graph $\mathbb{L}_n$ is shown in Figure 1. Triangular ladder graph is obtained from ladder graph with some edges addition, those are $v_i v_{i+1}$ for i even
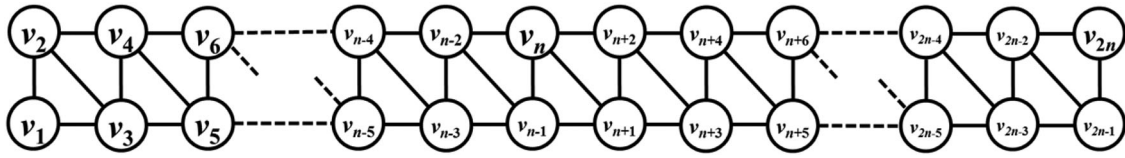
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Figure 1. Illustration of triangular ladder graph $\mathbb{L}_n$.

and $2 \leq i \leq 2n - 2$. The exact value of $\mathrm{dis}_1^0(\mathbb{L}_n)$ is already found for some cases, which is stated in Theorem 1.2.

Theorem 1.2 [7]. *Let $\mathbb{L}_n$ be triangular ladder graph with order 2n, for $n \geq 3$, then*

$$\mathrm{dis}_1^0(\mathbb{L}_n) = \begin{cases} \left\lceil \dfrac{2n+2}{5} \right\rceil, & n \not\equiv 4 \pmod 5; \\ 3, & n = 4. \end{cases}$$

If $n \equiv 4 \pmod 5, n > 4$, then

$$\frac{2n+2}{5} \leq \mathrm{dis}_1^0(\mathbb{L}_n) \leq \frac{2n+2}{5} + 1.$$

We proved Theorem 1.2 by using the following idea of labeling with $d = 1$ for the graph $\mathbb{L}_n$.

- Let the largest label used is k. Label vertex v_n and all the vertices in $N(v_n)$, those are $v_{n-2}, v_{n-1}, v_{n+1}$ and v_{n+2}, with k. Thus, vertex v_n has the largest weight, which is $5k$.
- Label five vertices before v_{n-2}, those are v_i for $n - 7 \leq i \leq n - 3$, with $k - 2$, and the next five vertices before them, those are v_i for $n - 12 \leq i \leq n - 8$, with $k - 4$, and so on, so that the weights of vertices before v_n form a decreasing sequence of even numbers, those are $5k - 2, 5k - 4, 5k - 6, ..., 6$.
- Label one vertex after v_{n+2} with $k - 1$ and four vertices after it, those are v_i for $n + 4 \leq i \leq n + 7$, with $k - 2$, and the next one and four vertices after it, those are v_{n+8} and v_i for $n + 9 \leq i \leq n + 12$, with $k - 3$ and $k - 4$, and so on, so that the weights of vertices after v_n form a decreasing sequence of odd numbers, those are $5k - 1, 5k - 3, 5k - 5, ..., 5$.

We modified this labeling idea to form the labeling with $d = 2$ for graph $\mathbb{L}_n$ with $n \equiv 5 \pmod 9$. The illustration of the modified labeling is shown in Figure 2. As a result, we determine the exact value of $\mathrm{dis}_2^0(\mathbb{L}_n)$ with $n \equiv 5 \pmod 9$. And then, we modified the obtained labeling to determine $\mathrm{dis}_2^0(\mathbb{L}_n)$ for others values of $n \pmod 9$. Moreover, we investigate the value of $\mathrm{dis}_d^0(P_n)$ with $d \geq 1$ using the result on graph $\mathbb{L}_n$.

2. Results

First, we generalized the lower bound value of $\mathrm{dis}_1^0(G)$ which is stated in Lemma 1.1 for $d \geq 1$. Let $N_d(v)$ denotes a set of vertices where their distance from vertex v are less than or equal to d. Based on Definition 1.1, it is obvious that the minimum possible value of $\mathrm{wt}(v)$ is $1 + \min|N_d(v)|$. Thus, the largest possible value of $\mathrm{wt}(v)$ is at least $|V(G)| + \min|N_d(v)|$. Therefore, the largest label is at least $\left\lceil \frac{|V(G)|+\min|N_d(v)|}{\max|N_d(v)|+1} \right\rceil$. If the

label used is less than $\left\lceil \frac{|V(G)|+\min|N_d(v)|}{\max|N_d(v)|+1} \right\rceil$, then the weight of vertex v with $\max|N_d(v)|$ will be less than $|V(G)| + \min|N_d(v)|$. In other words, $\min|N_d(v)| + 1 \leq \mathrm{wt}(v) < |V(G)| + \min|N_d(v)|$. However, there are $|V(G)|$ vertices in graph G, so it is impossible to have all the vertex weights distinct. Thus, the label cannot be less than $\left\lceil \frac{|V(G)|+\min|N_d(v)|}{\max|N_d(v)|+1} \right\rceil$. This result is stated in Lemma 2.1.

Lemma 2.1. *Let G be a graph with $|V(G)|$ vertices, $\delta_d(G) = \min|N_d(v)|$ and $\Delta_d(G) = \max|N_d(v)|$, then $\mathrm{dis}_d^0(G) \geq \left\lceil \frac{|V(G)|+\delta_d(G)}{\Delta_d(G)+1} \right\rceil$.*

Lemma 2.2 states the sufficient condition for $\mathrm{dis}_d^0(G) < \infty$. We conclude this result based on Theorem 1.1 and Definition 1.1.

Lemma 2.2. *For a graph G, if there exist two distinct vertices $u, v \in V(G)$ such that $N_d[u] = N_d[v]$, then $\mathrm{dis}_d^0(G) = \infty$.*

We used these results to find the value of $\mathrm{dis}_d^0(G)$ for triangular ladder graph and path as follows.

2.1. Triangular ladder graph

Let $\mathbb{L}_n$ with $n \geq 2$ be a triangular ladder graph with $2n$ vertices. We denote the vertices and edges of this graph as follows.

$$V(\mathbb{L}_n) = \{v_i : i = 1, 2, ..., 2n\},$$
$$E(\mathbb{L}_n) = \{v_i v_{i+1}, v_i v_{i+2} : i = 1, 2, ..., 2n - 2\} \cup \{v_{2n-1}v_{2n}\}.$$

Based on Lemma 2.2, it is easy to see that $\mathrm{dis}_2^0(\mathbb{L}_n) = \infty$ for $n < 5$. Meanwhile, for another value of n, we proved the following. Labeling illustration for the following results is shown in Figure 2. Note that w_i in the Figure is the notation for $\mathrm{wt}(v_i)$.

Lemma 2.1.1. *Let $\mathbb{L}_n$ be a triangular ladder graph with $n \geq 5$ and $n \equiv 5 \pmod 9$. Then*

$$\mathrm{dis}_2^0(\mathbb{L}_n) = \left\lceil \frac{2n+4}{9} \right\rceil.$$

Proof. Using Lemma 2.1, we can obtain $\mathrm{dis}_2^0(\mathbb{L}) \geq \left\lceil \frac{|V(G)|+\delta_d(G)}{\Delta_d(G)+1} \right\rceil = \left\lceil \frac{2n+4}{9} \right\rceil$. Next, we show that the largest label used in the labeling is equal to this lower bound.

First, let λ be the labeling that is defined as follows.

$$\lambda(v_i) = \begin{cases} 2/9(i+8), & i \equiv 1 \pmod 9, 1 \leq i \leq n - 4 \\ 2/9(2n - i) + 1, & i \equiv 1 \pmod 9, n + 5 \leq i \leq 2n \\ \lambda(v_{i-1}) - 1, & i \equiv 2 \pmod 9, n + 6 \leq i \leq 2n - 8 \\ \lambda(v_{i-1}), & \text{otherwise.} \end{cases}$$

Based on this definition of λ, the largest label used is $2/9(i+8)$ with $i = n - 4$, since $n \equiv 5 \pmod 9$, then the
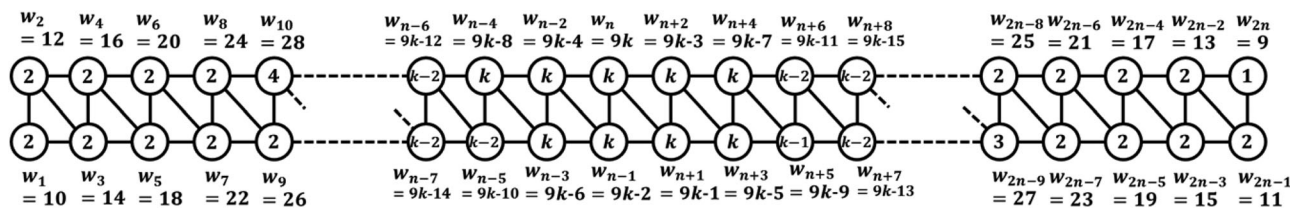
Figure 2. Illustration of inclusive vertex irregular 2-distance labeling for the case $n \equiv 5 \pmod 9$.

largest label used is equal to $\lceil \frac{2n+4}{9} \rceil$. The weights of vertices obtained for this graph can be formulated as follows.

$$\mathrm{wt}(v_i) = \begin{cases} 2i + 8, & 1 \le i \le n \\ 4n - 2i + 9, & n+1 \le i \le 2n. \end{cases}$$

Note for each i, $\mathrm{wt}(v_i)$ are even for the first n vertices, while for the last n vertices, $\mathrm{wt}(v_i)$ are odd. For all $i \ne j$, it is obvious that $\mathrm{wt}(v_i) \ne \mathrm{wt}(v_j)$. We can conclude that the weights are all distinct. Thus, $\mathrm{dis}_2^0(\mathbb{L}_n) = \lceil \frac{2n+4}{9} \rceil$. □

From the labeling of graph $\mathbb{L}_n$ with $n \equiv 5 \pmod 9$, we constructed the labeling for other value of $n \pmod 9$. If we delete v_1 and v_{2n} from the graph $\mathbb{L}_n$, we obtain a triangular ladder graph G' with

$$V(G') = V(\mathbb{L}_n) - \{v_1, v_{2n}\}$$
$$= \{v_i' : v_i' = v_{i+1} \in V(\mathbb{L}_n)\}.$$

It is easy to see that the graph G' is isomorphic with graph $\mathbb{L}_{n-1}$, simply by defined an injective function $V(G') \to V(\mathbb{L}_{n-1})$ with $v_i' \mapsto v_i$ where $v_i' \in V(G')$ and $v_i \in V(\mathbb{L}_{n-1})$. We can conclude that after the deletion of v_1 and v_{2n} from the graph $\mathbb{L}_n$, we obtain the graph $\mathbb{L}_{n-1}$.

Based on this observation, we construct labeling for graph $\mathbb{L}_m$ with $m \not\equiv 5 \pmod 9$. To construct those labeling, we did the deletion and reduction processes as follows. Let $\mathbb{L}_n$ be a triangular ladder graph with labeling defined as $\lambda : V(\mathbb{L}_n) \to \{1, ..., k\}$.

Deletion Process: Delete v_1 and v_{2n} from the graph $\mathbb{L}_n$ and all the incident edges of these vertices. In consequence of this deletion, the weight of vertices in $N_d(v_1)$, those are v_i with $2 \le i \le 2d+1$, will reduce by $\lambda(v_1)$ and in $N_d(v_{2n})$, those are v_i with $2n - 2d \le i \le 2n - 1$, will reduce by $\lambda(v_{2n})$. As a result of this process, we obtain graph $\mathbb{L}_{n-1}$ where the label of vertex $v_i \in V(\mathbb{L}_{n-1})$ equals to the label of vertex $v_{i+1} \in V(\mathbb{L}_n)$. It means the weight of vertex $v_i \in V(\mathbb{L}_{n-1})$, with $2d+1 \le i \le 2(n-1) - 2d$, equals to the weight of vertex $v_{i+1} \in V(\mathbb{L}_n)$.

Reduction Process A: Assume that $\lambda(v_{2n}) > 1$. Reduce $\lambda(v_{2n})$ by 1, then $\mathrm{wt}(v_i)$ for all $v_i \in N_d[v_{2n}]$ will reduce by 1.

Reduction Process B: Assume that $\lambda(v_i) > 1$ for all i. Reduce each of $\lambda(v_i)$ by 1, then every $\mathrm{wt}(v_i)$ will reduce by $|N_d(v_i)| + 1$.

We apply deletion process and reduction process (A or B) to construct labeling for graph $\mathbb{L}_n$ with $n \not\equiv 5 \pmod 9$. As a result, we found the exact value of $\mathrm{dis}_2^0(\mathbb{L}_n)$ for some cases and upper bound for the rest.

Lemma 2.1.2. Let $\mathbb{L}_m$ be a triangular ladder graph with $m \ge 5$ and $m \equiv 4 \pmod 9$. Then

$$\mathrm{dis}_2^0(\mathbb{L}_m) = \left\lceil \frac{2m+4}{9} \right\rceil.$$

Proof. Lemma 2.1 gives $\mathrm{dis}_2^0(\mathbb{L}_m) \ge \lceil \frac{2m+4}{9} \rceil$. Now, we will construct the labeling for this graph and prove that the largest label used is $\lceil \frac{2m+4}{9} \rceil$. Consider the labeling defined for the graph $\mathbb{L}_n$ with $n \equiv 5 \pmod 9$.

Step 1. Apply deletion process to the graph $\mathbb{L}_n$. The graph obtained is $\mathbb{L}_{n-1}$ with the same labels and the same vertex weights as in $\mathbb{L}_n$, except for $\mathrm{wt}(v_i)$ where $v_i \in N_2(v_1) \cup N_2(v_{2n})$. The new weights for those vertices are $\mathrm{wt}(v_2) = \mathrm{wt}(v_{2n-1}) = 10$, $\mathrm{wt}(v_3) = \mathrm{wt}(v_{2n-2}) = 12$, $\mathrm{wt}(v_4) = \mathrm{wt}(v_{2n-3}) = 14$ and $\mathrm{wt}(v_5) = \mathrm{wt}(v_{2n-4}) = 16$.

Step 2. Apply reduction process A to the graph obtained from Step 1. The new weights for $v_i \in N_2[v_{2n-2}]$ are $\mathrm{wt}(v_{2n-1}) = 9$, $\mathrm{wt}(v_{2n-2}) = 11$, $\mathrm{wt}(v_{2n-3}) = 13$, $\mathrm{wt}(v_{2n-4}) = 15$ and $\mathrm{wt}(v_{2n-5}) = 18$.

From both steps, there are nine vertices that have new weight. Those are $\mathrm{wt}(v_2) = 10, \mathrm{wt}(v_3) = 12, \mathrm{wt}(v_4) = 14, \mathrm{wt}(v_5) = 16, \mathrm{wt}(v_{2n-5}) = 18, \mathrm{wt}(v_{2n-4}) = 15, \mathrm{wt}(v_{2n-3}) = 13, \mathrm{wt}(v_{2n-2}) = 11$ and $\mathrm{wt}(v_{2n-1}) = 9$. Meanwhile, the rest of vertices still have the same weight,

$$\mathrm{wt}(v_i) = \begin{cases} 2i + 8, & 6 \le i \le n; \\ 4n - 2i + 9, & n+1 \le i \le 2n - 6. \end{cases}$$

It is easy to see that all of the vertex weights are distinct.

Based on these processes, we can define an injective function $V(\mathbb{L}_{n-1}) \to V(\mathbb{L}_n)$ with $v_i' \mapsto v_{i+1}$, where $v_i' \in V(\mathbb{L}_{n-1})$, so that the labeling for the graph $\mathbb{L}_{n-1}$ can be defined as a function $\lambda' : V(\mathbb{L}_{n-1}) \to \{1, ..., \lceil \frac{2n+4}{9} \rceil\}$ with

$$\lambda'(v_i') = \begin{cases} \lambda(v_{i+1}), & 1 \le i \le 2n - 3; \\ \lambda(v_{2n-1}) - 1, & i = 2n - 2. \end{cases}$$

From this definition of the labeling λ', we obtain distinct weights for all vertices in $\mathbb{L}_{n-1}$. The result for the first four vertices are $\mathrm{wt}(v_1') = 10, \mathrm{wt}(v_2') = 12, \mathrm{wt}(v_3') = 14, \mathrm{wt}(v_4') = 16$ and for the last five vertices are $\mathrm{wt}(v_{2n-6}') = 18$, $\mathrm{wt}(v_{2n-5}') = 15$, $\mathrm{wt}(v_{2n-4}') = 13$, $\mathrm{wt}(v_{2n-3}') = 11$, $\mathrm{wt}(v_{2n-2}') = 9$, while for the other vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 10, & 5 \le i \le n - 1; \\ 4n - 2i + 7, & n \le i \le 2n - 7. \end{cases}$$

The largest label used is the same as the largest label used in the proof of Lemma 2.1.1, that is $\lceil \frac{2n+4}{9} \rceil$. Since, for

$n \equiv 5 \pmod 9$, $\lceil \frac{2n+4}{9} \rceil = \lceil \frac{2(n-1)+4}{9} \rceil$, we can conclude that $\mathrm{dis}_2^0(\mathbb{L}_{n-1}) = \lceil \frac{2(n-1)+4}{9} \rceil$. Set $n - 1 = m$, then $\mathrm{dis}_2^0(\mathbb{L}_m) = \lceil \frac{2m+4}{9} \rceil$ for $m \equiv 4 \pmod 9$. $\qquad \square$

Proofs of our result in Lemmas 2.1.3 throughout 2.1.9 are obtained in similar process as the proof in Lemma 2.1.2. But, in order to obtain graph $\mathbb{L}_m$ with other $m \pmod 9$, we applied deletion process i times with $2 \le i \le 8$ to the graph $\mathbb{L}_n$ with $n \equiv 5 \pmod 9$.

Lemma 2.1.3. Let $\mathbb{L}_m$ be a triangular ladder graph with $m \ge 5$ and $m \equiv 3 \pmod 9$. Then

$$\mathrm{dis}_2^0(\mathbb{L}_m) = \left\lceil \frac{2m+4}{9} \right\rceil.$$

Proof. To obtain labeling for the graph $\mathbb{L}_{n-2}$, we applied deletion process twice and reduction process A. Let $v_i' \in V(\mathbb{L}_{n-2})$, then labeling for the graph $\mathbb{L}_{n-2}$ is defined as $\lambda' : V(\mathbb{L}_{n-2}) \to \left\{ 1, ..., \lceil \frac{2n+4}{9} \rceil \right\}$ with

$$\lambda'(v_i') = \begin{cases} \lambda(v_{i+2}), & 1 \le i \le 2n - 5; \\ \lambda(v_{2n-2}) - 1, & i = 2n - 4. \end{cases}$$

The vertex weights obtained are $\mathrm{wt}(v_1') = 10, \mathrm{wt}(v_2') = 12, \mathrm{wt}(v_3') = 14, \mathrm{wt}(v_4') = 18, \mathrm{wt}(v_{2n-8}') = 20, \mathrm{wt}(v_{2n-7}') = 16, \mathrm{wt}(v_{2n-6}') = 13, \mathrm{wt}(v_{2n-5}') = 11, \mathrm{wt}(v_{2n-4}') = 9$ and for the rest vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 12, & 5 \le i \le n - 2; \\ 4n - 2i + 5, & n - 1 \le i \le 2n - 9. \end{cases}$$

Thus, $\mathrm{dis}_2^0(\mathbb{L}_{n-2}) = \lceil \frac{2(n-2)+4}{9} \rceil$. Set $m = n - 2$, then we have $\mathrm{dis}_2^0(\mathbb{L}_m) = \lceil \frac{2m+4}{9} \rceil$ for $m \equiv 3 \pmod 9$. $\qquad \square$

Lemma 2.1.4. Let $\mathbb{L}_m$ be a triangular ladder graph with $m \ge 5$ and $m \equiv 0 \pmod 9$. Then

$$\mathrm{dis}_2^0(\mathbb{L}_m) = \left\lceil \frac{2m+4}{9} \right\rceil.$$

Proof. We performed deletion process five times and reduction process B to obtain labeling for graph $\mathbb{L}_{n-5}$. Let $v_i' \in V(\mathbb{L}_{n-5})$, then labeling for the graph $\mathbb{L}_{n-5}$ is defined as $\lambda' : V(\mathbb{L}_{n-5}) \to \left\{ 1, ..., \lceil \frac{2n+4}{9} \rceil \right\}$ with

$$\lambda'(v_i') = \lambda(v_{i+5}) - 1.$$

The weights of the vertices are $\mathrm{wt}(v_1') = 7, \mathrm{wt}(v_2') = 10, \mathrm{wt}(v_3') = 13, \mathrm{wt}(v_4') = 16, \mathrm{wt}(v_{2n-13}') = 15, \mathrm{wt}(v_{2n-12}') = 12, \mathrm{wt}(v_{2n-11}') = 9, \mathrm{wt}(v_{2n-10}') = 6$, and for the other vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 9, & 5 \le i \le n - 5; \\ 4n - 2i - 10, & n - 4 \le i \le 2n - 14. \end{cases}$$

Therefore, $\mathrm{dis}_2^0(\mathbb{L}_{n-5}) = \lceil \frac{2(n-5)+4}{9} \rceil$. If we set $m = n - 5$, then we obtain $\mathrm{dis}_2^0(\mathbb{L}_m) = \lceil \frac{2m+4}{9} \rceil$ for $m \equiv 0 \pmod 9$. $\qquad \square$

Lemma 2.1.5. Let $\mathbb{L}_m$ be a triangular ladder graph with $m \ge 5$ and $m \equiv 8 \pmod 9$. Then

$$\mathrm{dis}_2^0(\mathbb{L}_m) = \left\lceil \frac{2m+4}{9} \right\rceil.$$

Proof. The labeling for graph $\mathbb{L}_{n-6}$ is obtained by using deletion process six times and reduction process B. Let $v_i' \in V(\mathbb{L}_{n-6})$, then labeling for the graph $\mathbb{L}_{n-6}$ is defined as $\lambda' : V(\mathbb{L}_{n-6}) \to \left\{ 1, ..., \lceil \frac{2n+4}{9} \rceil \right\}$ with

$$\lambda'(v_i') = \lambda(v_{i+6}) - 1.$$

The weights of the vertices are $\mathrm{wt}(v_1') = 9, \mathrm{wt}(v_2') = 12, \mathrm{wt}(v_3') = 15, \mathrm{wt}(v_4') = 18, \mathrm{wt}(v_{2n-15}') = 17, \mathrm{wt}(v_{2n-14}') = 14, \mathrm{wt}(v_{2n-13}') = 11, \mathrm{wt}(v_{2n-12}') = 8$, and for the other vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 11, & 5 \le i \le n - 6; \\ 4n - 2i - 12, & n - 5 \le i \le 2n - 16. \end{cases}$$

Hence, $\mathrm{dis}_2^0(\mathbb{L}_{n-6}) = \lceil \frac{2(n-6)+4}{9} \rceil$. We obtain $\mathrm{dis}_2^0(\mathbb{L}_m) = \lceil \frac{2m+4}{9} \rceil$ for $m \equiv 0 \pmod 9$ by setting $m = n - 6$. $\qquad \square$

Lemma 2.1.6. Let $\mathbb{L}_m$ be a triangular ladder graph with $m \ge 5$ and $m \equiv 2 \pmod 9$. Then

$$\left\lceil \frac{2m+4}{9} \right\rceil \le \mathrm{dis}_2^0(\mathbb{L}_m) \le \left\lceil \frac{2m+4}{9} \right\rceil + 1.$$

Proof. With applying deletion process three times and reduction process A, we obtain labeling for the graph $\mathbb{L}_{n-3}$. Let $v_i' \in V(\mathbb{L}_{n-3})$, then labeling for the graph $\mathbb{L}_{n-3}$ is defined as $\lambda' : V(\mathbb{L}_{n-3}) \to \left\{ 1, ..., \lceil \frac{2n+4}{9} \rceil \right\}$ with

$$\lambda'(v_i') = \begin{cases} \lambda(v_{i+3}), & 1 \le i \le 2n - 7; \\ \lambda(v_{2n-3}) - 1, & i = 2n - 6. \end{cases}$$

The weights of the vertices are $\mathrm{wt}(v_1') = 10, \mathrm{wt}(v_2') = 12, \mathrm{wt}(v_3') = 16, \mathrm{wt}(v_4') = 20, \mathrm{wt}(v_{2n-10}') = 22, \mathrm{wt}(v_{2n-9}') = 18, \mathrm{wt}(v_{2n-8}') = 14, \mathrm{wt}(v_{2n-7}') = 11, \mathrm{wt}(v_{2n-6}) = 9$, and for the rest vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 14, & 5 \le i \le n - 3; \\ 4n - 2i + 3, & n - 2 \le i \le 2n - 11. \end{cases}$$

Since $\lceil \frac{2n+4}{9} \rceil = \lceil \frac{2(n-3)+4}{9} \rceil + 1$ for $n \equiv 5 \pmod 9$, then $\lceil \frac{2(n-3)+4}{9} \rceil \le \mathrm{dis}_2^0(\mathbb{L}_{n-3}) \le \lceil \frac{2(n-3)+4}{9} \rceil + 1$. Set $m = n - 3$, so that we have $\lceil \frac{2m+4}{9} \rceil \le \mathrm{dis}_2^0(\mathbb{L}_m) \le \lceil \frac{2m+4}{9} \rceil + 1$ for $m \equiv 2 \pmod 9$. $\qquad \square$

Lemma 2.1.7. Let $\mathbb{L}_m$ be a triangular ladder graph with $m \ge 5$ and $m \equiv 1 \pmod 9$. Then

$$\left\lceil \frac{2m+4}{9} \right\rceil \le \mathrm{dis}_2^0(\mathbb{L}_m) \le \left\lceil \frac{2m+4}{9} \right\rceil + 1.$$

Proof. We obtain labeling for the graph $\mathbb{L}_{n-4}$ by applying deletion process four times and reduction process A. Let $v_i' \in V(\mathbb{L}_{n-4})$, then defined the labeling as $\lambda' : V(\mathbb{L}_{n-4}) \to \left\{ 1, ..., \lceil \frac{2n+4}{9} \rceil \right\}$ with

$$\left\lceil \frac{2m+4}{9} \right\rceil \leq \mathrm{dis}_2^0(\mathbb{L}_m) \leq \left\lceil \frac{2m+4}{9} \right\rceil + 1.$$

Proof. We applied deletion process eight times and followed by reduction process B to obtain labeling for the graph $\mathbb{L}_{n-8}$ which can be defined as follows. Let $v_i' \in V(\mathbb{L}_{n-7})$, then defined $\lambda' : V(\mathbb{L}_{n-8}) \rightarrow \left\{1, ..., \left\lceil \frac{2n+4}{9} \right\rceil\right\}$ with

$$\lambda'(v_i') = \lambda(v_{i+8}) - 1.$$

The weights of the vertices are $\mathrm{wt}(v_1') = 13, \mathrm{wt}(v_2') = 16, \mathrm{wt}(v_3') = 19, \mathrm{wt}(v_4') = 22, \mathrm{wt}(v_{2n-19}') = 21, \mathrm{wt}(v_{2n-18}') = 18, \mathrm{wt}(v_{2n-17}') = 15, \mathrm{wt}(v_{2n-16}') = 12,$ and for the rest vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 15, & 5 \leq i \leq n - 8; \\ 4n - 2i - 16, & n - 7 \leq i \leq 2n - 20. \end{cases}$$

Similar as in the proof of Lemma 2.1.6, if we set $m = n - 8$, then we obtain $\left\lceil \frac{2m+4}{9} \right\rceil \leq \mathrm{dis}_2^0(\mathbb{L}_m) \leq \left\lceil \frac{2m+4}{9} \right\rceil + 1$ for $m \equiv 6 \pmod 9$. □

Consider Lemmas 2.1.1 throughout 2.1.9. From these results, we conclude the value of $\mathrm{dis}_2^0(\mathbb{L}_m)$ for $m \geq 5$ and state it in Theorem 2.1.1 as follows.

Theorem 2.1.1. *Let $\mathbb{L}_n$ be a triangular ladder graph with $n \geq 5$. Then*

$$\mathrm{dis}_2^0(\mathbb{L}_n) = \begin{cases} 3, & n = 7; \\ \left\lceil \frac{2n+4}{9} \right\rceil, & n \equiv 0, 3, 4, 5, 8 \pmod 9; \end{cases}$$

while for $n \equiv 1, 2, 6, 7 \pmod 9$ and $n \neq 7$,

$$\left\lceil \frac{2n+4}{9} \right\rceil \leq \mathrm{dis}_2^0(\mathbb{L}_n) \leq \left\lceil \frac{2n+4}{9} \right\rceil + 1.$$

2.2. Path

Let P_n with $n \geq 2$ be a path with n vertices. We denote the vertices and edges of this graph as follows

$$V(P_n) = \{u_i : i = 1, 2, ..., n\},$$
$$E(P_n) = \{u_i u_{i+1} : i = 1, 2, ..., n - 1\}.$$

Consider our observation result on triangular ladder graph and path which is stated in Observation 2.2.1.

Observation 2.2.1. *Given graph $\mathbb{L}_n$ and P_{2n} with $|V(\mathbb{L}_n)| = |V(P_{2n})| = 2n$. Let $v_i \in V(\mathbb{L}_n)$ and $u_i \in V(P_{2n})$, then we can define a bijective function $V(\mathbb{L}_n) \rightarrow V(P_{2n})$ with $v_i \mapsto u_i$ such that $N_d(v_i) = N_{2d}(u_i)$.*

Based on this observation, we obtain our first result on path as follows.

Theorem 2.2.1. *Let $\mathbb{L}_n$ has an inclusive vertex irregular d-distance labeling with $\mathrm{dis}_d^0(\mathbb{L}) = k$. Then P_{2n} has an inclusive vertex irregular 2d-distance labeling with $\mathrm{dis}_{2d}^0(P_{2n}) = \mathrm{dis}_d^0(\mathbb{L}_n)$.*

Proof. Let $\lambda : V(\mathbb{L}_n) \rightarrow \{1, ..., k\}$ be an inclusive vertex irregular d-distance labeling for the graph $\mathbb{L}_n$. Then, we can define a labeling with distance equals to $2d$ for P_{2n} as β with $\beta(u_i) = \lambda(v_i)$, where $u_i \in V(P_{2n})$ and $v_i \in V(\mathbb{L}_n)$. Weight
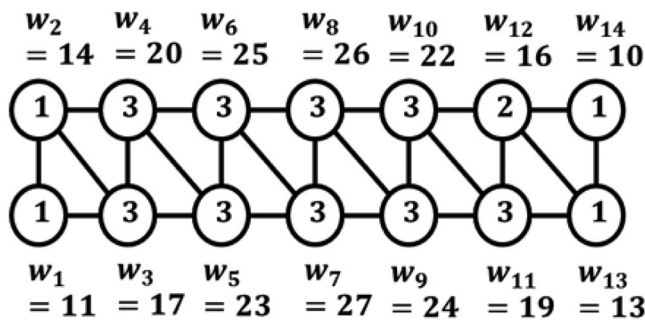

Figure 3. Illustration of inclusive vertex irregular 2-distance labeling for $\mathbb{L}_7$.

$$\lambda'(v_i') = \begin{cases} \lambda(v_{i+4}), & 1 \leq i \leq 2n - 9; \\ \lambda(v_{2n-4}) - 1, & i = 2n - 8. \end{cases}$$

The weights of the vertices are $\mathrm{wt}(v_1') = 10, \mathrm{wt}(v_2') = 14, \mathrm{wt}(v_3') = 18, \mathrm{wt}(v_4') = 22, \mathrm{wt}(v_{2n-12}') = 24, \mathrm{wt}(v_{2n-11}') = 20, \mathrm{wt}(v_{2n-10}') = 16, \mathrm{wt}(v_{2n-9}') = 12, \mathrm{wt}(v_{2n-8}') = 9,$ and for the rest vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 16, & 5 \leq i \leq n - 4; \\ 4n - 2i + 1, & n - 3 \leq i \leq 2n - 13. \end{cases}$$

Similar as in the proof of Lemma 2.1.6, we can have $\left\lceil \frac{2m+4}{9} \right\rceil \leq \mathrm{dis}_2^0(\mathbb{L}_m) \leq \left\lceil \frac{2m+4}{9} \right\rceil + 1$ for $m \equiv 1 \pmod 9$ by setting $m = n - 4$. □

Lemma 2.1.8. *Let $\mathbb{L}_m$ be a triangular ladder graph with $m \geq 5$ and $m \equiv 7 \pmod 9$. Then*

$$\left\lceil \frac{2m+4}{9} \right\rceil \leq \mathrm{dis}_2^0(\mathbb{L}_m) \leq \left\lceil \frac{2m+4}{9} \right\rceil + 1.$$

Proof. The labeling for the graph $\mathbb{L}_{n-7}$ is obtained by performing deletion process seven times and reduction process B. Let $v_i' \in V(\mathbb{L}_{n-7})$, then define the labeling as $\lambda' : V(\mathbb{L}_{n-7}) \rightarrow \left\{1, ..., \left\lceil \frac{2n+4}{9} \right\rceil\right\}$ with

$$\lambda'(v_i') = \lambda(v_{i+7}) - 1.$$

The weights of the vertices are $\mathrm{wt}(v_1') = 11, \mathrm{wt}(v_2') = 14, \mathrm{wt}(v_3') = 17, \mathrm{wt}(v_4') = 20, \mathrm{wt}(v_{2n-17}') = 19, \mathrm{wt}(v_{2n-16}') = 16, \mathrm{wt}(v_{2n-15}') = 13, \mathrm{wt}(v_{2n-14}') = 10,$ and for the other vertices are

$$\mathrm{wt}(v_i') = \begin{cases} 2i + 13, & 5 \leq i \leq n - 7; \\ 4n - 2i - 14, & n - 6 \leq i \leq 2n - 18. \end{cases}$$

Similar as in the proof of Lemma 2.1.6, we can conclude $\left\lceil \frac{2m+4}{9} \right\rceil \leq \mathrm{dis}_2^0(\mathbb{L}_m) \leq \left\lceil \frac{2m+4}{9} \right\rceil + 1$ for $m \equiv 7 \pmod 9$ by setting $m = n - 7$. □

Note that for graph $\mathbb{L}_m$ with $m = 7$, it is impossible to have $\left\lceil \frac{2m+4}{9} \right\rceil = 2$ as the largest label. Labeling illustration for this graph is shown in Figure 3.

If we used 2 as the largest label for $\mathbb{L}_7$, then there will be at least two vertices with the same weight. Therefore, for the case $m = 7$, we concluded $\mathrm{dis}_2^0(\mathbb{L}_7) = \left\lceil \frac{2m+4}{9} \right\rceil + 1 = 3$.

Lemma 2.1.9. *Let $\mathbb{L}_m$ be a triangular ladder graph with $m \geq 5$ and $m \equiv 6 \pmod 9$. Then*

for every vertex u_i can be obtained as follows.

$$\text{wt}(u_i) = \beta(u_i) + \sum_{u_j \in N_{2d}(u_i)} \beta(u_j) = \lambda(v_i) + \sum_{v_j \in N_d(v_i)} \lambda(v_j)$$
$$= \text{wt}(v_i).$$

It is obvious that all $\text{wt}(u_i)$ are distinct. Thus, β is an inclusive vertex irregular $2d$-distance labeling for path P_{2n} with the largest used label k. So that, $\text{dis}_{2d}^0(P_{2n}) \leq k$.

Suppose $\beta' : V(P_{2n}) \to \{1, ..., K\}$ be an inclusive vertex irregular $2d$-distance labeling for graph P_{2n} with $K < k$. Then, we can obtain an inclusive vertex irregular d-distance labeling for $\mathbb{L}_n$, that is λ' with $\lambda'(v_i) = \beta'(u_i)$. The largest label used is $K < k$. It is contradiction with the fact that $\text{dis}_d^0(\mathbb{L}_n) = k$. So, we have $\text{dis}_{2d}^0(P_{2n}) \geq k$.

Since $\text{dis}_{2d}^0(P_{2n}) \leq k$ and $\text{dis}_{2d}^0(P_{2n}) \geq k$, we can conclude that $\text{dis}_{2d}^0(P_{2n}) = k$. Therefore, $\text{dis}_{2d}^0(P_{2n}) = \text{dis}_d^0(\mathbb{L}_n)$. $\qquad \square$

Combining Theorems 1.2, 2.1.1, and 2.2.1, we obtain $\text{dis}_2^0(P_n)$ and $\text{dis}_4^0(P_n)$ for some n as follows.

Corollary 2.2.1. *Let P_n be a path with $n \geq 6, n \equiv 0, 2, 4, 6 \pmod{10}$ and $n = 8$. Then,*

$$\text{dis}_2^0(P_n) = \begin{cases} 3, & n = 8; \\ \lceil \frac{n+2}{5} \rceil, & otherwise. \end{cases}$$

Corollary 2.2.2. *Let P_n be a path with $n \geq 10, n \equiv 0, 6, 8, 10, 16 \pmod{18}$ and $n = 14$. Then,*

$$\text{dis}_4^0(P_n) = \begin{cases} 3, & n = 14; \\ \lceil \frac{n+4}{9} \rceil, & otherwise. \end{cases}$$

Our next result is about the value of $\text{dis}_2^0(P_n)$ for some odd numbers n. We derived this result by constructing labeling with $d = 2$ for P_n in similar process as the construction of labeling with $d = 2$ for $\mathbb{L}_n$ with $n \not\equiv 5 \pmod 9$. However, we only apply deletion process which is stated as follows in order to obtain the labeling for P_n. Let $\beta : V(P_n) \to \{1, ..., k\}$ be an inclusive vertex irregular d-distance labeling for P_n.

Deletion Process A: Delete vertex u_1 and its incident edge from P_n. This process will reduce weight of vertices in $N_d(u_1)$, those are u_i with $2 \leq i \leq 1 + d$, by $\beta(u_1)$. As a result, we obtain the graph P_{n-1} where the label of vertex $u_i \in V(P_{n-1})$ equals to the label of vertex $u_{i+1} \in V(P_n)$. It means the weight of vertex $u_i \in P_{n-1}$, with $1 + d \leq i \leq n - 1$, equals to the weight of vertex $u_{i+1} \in V(P_n)$.

Deletion Process B: Delete vertex u_n and its incident edge from P_n. In consequence of this deletion, the weight of vertices in $N_d(u_n)$, those are u_i with $n - d \leq i \leq n - 1$, will be reduced by $\beta(u_n)$. From this process, we obtain the graph P_{n-1} where the label of $u_i \in V(P_{n-1})$ equals to the label of vertex $u_i \in V(P_n)$. So that the weight of vertex $u_i \in P_{n-1}$, with $1 \leq i \leq (n-1) - d$, equals to the weight of vertex $u_i \in V(P_n)$.

With applying the deletion process A or B to the graph P_n, we obtained labeling for the graph P_{n-1}. Since labeling

with distance $2d$ of the graph P_n can be obtained from the labeling with distance d of graph $\mathbb{L}_m$ with $n = 2m$, labeling for the graph $\mathbb{L}_{n-1}$ also can be obtained from $\mathbb{L}_m$. We derived our result on $\text{dis}_2^0(P_n)$ for some odd numbers n based on Theorems 1.2 and 2.2.1. In the proof of Lemmas 2.2.1–2.2.3, the proof of Theorem 1.2 is used for defining vertex label of graph $\mathbb{L}_m$ and the weight of these vertices. The labels of vertices of P_{2m} are constructed using the proof of Theorem 2.2.1.

Lemma 2.2.1. *Let P_n be a path with $n \geq 5$ and $n \equiv 1 \pmod{10}$. Then*

$$\text{dis}_2^0(P_n) = \left\lceil \frac{n+2}{5} \right\rceil.$$

Proof. The graph $\mathbb{L}_m$ with $m \equiv 1 \pmod 5$ has inclusive vertex irregular 1-distance labeling as follows. Let $v_i \in V(\mathbb{L}_m)$, the labeling for $\mathbb{L}_m$ can be defined as $\lambda : V(\mathbb{L}_m) \to \left\{ 1, ..., \lceil \frac{2m+2}{5} \rceil \right\}$ with $\lambda(v_{2m}) = \lambda(v_{2m-1}) = \lambda(v_{2m-2}) = \lambda(v_{2m-4}) = 1, \lambda(v_{2m-3}) = \lambda(v_{2m-5}) = 2$, while for the rest vertices are

$$\lambda(v_i) = \begin{cases} (2i+13)/5, & 1 \leq i \leq m-1, i \equiv 1 \pmod 5; \\ 2(2m-i-1)/5, & m \leq i \leq 2m-6, i \equiv 1 \pmod 5; \\ \lambda(v_{i-1}) - 1, & m \leq i \leq 2m-6, i \equiv 2 \pmod 5; \\ \lambda(v_{i-1}), & otherwise; \end{cases}$$

and the weights of vertices are $\text{wt}(v_1) = 9, \text{wt}(v_2) = 12, \text{wt}(v_{2m-7}) = 13, \text{wt}(v_{2m-6}) = 11, \text{wt}(v_{2m-5}) = 10, \text{wt}(v_{2m-4}) = 8, \text{wt}(v_{2m-3}) = 7, \text{wt}(v_{2m-2}) = 6, \text{wt}(v_{2m-1}) = 5, \text{wt}(v_{2m}) = 3$, and for the rest vertices are

$$\text{wt}(v_i) = \begin{cases} 2i+9, & 3 \leq i \leq m-3; \\ 4m-2i-2, & m-2 \leq i \leq 2m-8. \end{cases}$$

So, the inclusive vertex irregular 2-distance labeling for the graph P_{2m} is defined as $\beta : V(P_{2m}) \to \left\{ 1, ..., \lceil \frac{2m+2}{5} \rceil \right\}$ with $\beta(u_i) = \lambda(v_i)$ such that $\text{wt}(u_i) = \text{wt}(v_i)$, where $u_i \in V(P_{2m})$.

With applying deletion process B to the graph P_{2m}, we obtained labeling with $d = 2$ for the graph P_{2m-1} as follows. Let $u_i' \in V(P_{2m-1})$, the labeling for P_{2m-1} can be defined as $\beta' : V(P_{2m-1}) \to \left\{ 1, ..., \lceil \frac{2m+2}{5} \rceil \right\}$ with $\beta'(u_i') = \beta(u_i) = \lambda(v_i)$. Thus, the obtained weights are $\text{wt}(u_i') = \text{wt}(u_i) = \text{wt}(v_i)$ for $1 \leq i \leq 2m - 3$ and $\text{wt}(u_i') = \text{wt}(u_i) - 1 = \text{wt}(v_i) - 1$ for $i = 2m - 1$ and $2m - 2$. It is easy to see that all the weights are distinct.

The largest label used in this labeling is $\lceil \frac{2m+2}{5} \rceil$, while the lower bound value which is obtained from Lemma 2.1 for P_{2m-1} is $\lceil \frac{(2m-1)+2}{5} \rceil$. Since $m \equiv 1 \pmod 5$, then $\lceil \frac{(2m-1)+2}{5} \rceil = \lceil \frac{2m+2}{5} \rceil$. Therefore, $\text{dis}_2^0(P_{2m-1}) = \lceil \frac{(2m-1)+2}{5} \rceil$. By setting $n = 2m - 1$, we have $\text{dis}_2^0(P_n) = \lceil \frac{n+2}{5} \rceil$ with $n \equiv 1 \pmod{10}$. $\qquad \square$

Lemma 2.2.2. *Let P_n be a path with $n \geq 5$ and $n \equiv 5 \pmod{10}$. Then*

$$\text{dis}_2^0(P_n) = \left\lceil \frac{n+2}{5} \right\rceil.$$

Proof. The inclusive vertex irregular 1-distance labeling for the graph $\mathbb{L}_m$ with $m \equiv 3 \pmod 5$ is defined as follows. Let $v_i \in V(\mathbb{L}_m)$, then the labeling is $\lambda : V(\mathbb{L}_m) \to \left\{1, ..., \left\lceil \frac{2m+2}{5} \right\rceil \right\}$ with

$$\lambda(v_i) = \begin{cases} (2i+8)/5, & 1 \le i \le m+2, i \equiv 1 \pmod 5; \\ 2(2m-i)/5+1, & m+3 \le i \le 2m, i \equiv 1 \pmod 5; \\ \lambda(v_{i-1}) - 1, & m+3 \le i \le 2m, i \equiv 2 \pmod 5; \\ \lambda(v_{i-1}), & \text{otherwise}; \end{cases}$$

and the weights of vertices are

$$\text{wt}(v_i) = \begin{cases} 2i+4, & 1 \le i \le m; \\ 4m-2i+5, & m+1 \le i \le 2m. \end{cases}$$

Let $u_i \in V(P_{2m})$. Then, the inclusive vertex irregular 2-distance labeling for the graph P_{2m} is defined as $\beta : V(P_{2m}) \to \left\{1, ..., \left\lceil \frac{2m+2}{5} \right\rceil \right\}$ with $\beta(u_i) = \lambda(v_i)$ such that $\text{wt}(u_i) = \text{wt}(v_i)$.

To obtain the labeling with $d = 2$ for graph P_{2m-1}, we performed deletion process A to the graph P_{2m}. Let $u_i' \in V(P_{2m-1})$, the labeling for P_{2m-1} can be defined as β': $V(P_{2m-1}) \to \left\{1, ..., \left\lceil \frac{2m+2}{5} \right\rceil \right\}$ with $\beta'(u_i') = \beta(u_{i+1}) = \lambda(v_{i+1})$. The weights obtained are $\text{wt}(u_i') = \text{wt}(u_{i+1}) = \text{wt}(v_{i+1})$ for $3 \le i \le 2m-1$ and $\text{wt}(u_i') = \text{wt}(u_i) - 2 = \text{wt}(v_i) - 2$ for $i = 1$ and 2. It is easy to see that all the weights are distinct. Similar with the proof of Lemma 2.2.1, by setting $n = 2m-1$, we can conclude $\text{dis}_2^0(P_n) = \left\lceil \frac{n+2}{5} \right\rceil$ with $n \equiv 5 \pmod{10}$. □

Lemma 2.2.3. *Let P_n be a path with $n \ge 5$ and $n \equiv 9 \pmod{10}$. Then*

$$\text{dis}_2^0(P_n) = \left\lceil \frac{n+2}{5} \right\rceil.$$

Proof. Let $v_i \in V(\mathbb{L}_m)$ with $m \equiv 0 \pmod 5$, then the inclusive vertex irregular 1-distance labeling for the graph $\mathbb{L}_m$ is defined as $\lambda : V(\mathbb{L}_m) \to \left\{1, ..., \left\lceil \frac{2m+2}{5} \right\rceil \right\}$ with

$$\lambda(v_i) = \begin{cases} 1, & i = 1, 2; \\ (2i+9)/5, & 1 \le i \le m+2, i \equiv 3 \pmod 5; \\ 2(2m-i+3)/5, & m+3 \le i \le 2m, i \equiv 3 \pmod 5; \\ \lambda(v_{i-1}) - 1, & m+3 \le i \le 2m, i \equiv 4 \pmod 5; \\ \lambda(v_{i-1}), & \text{otherwise}; \end{cases}$$

and the weights of vertices are $\text{wt}(v_1) = 5, \text{wt}(v_2) = 8, \text{wt}(v_{2m-1}) = 7, \text{wt}(v_{2m}) = 4$, and for the other vertices are

$$\text{wt}(v_i) = \begin{cases} 2i+5, & 3 \le i \le m; \\ 4m-2i+6, & m+1 \le i \le 2m-2. \end{cases}$$

Hence, the inclusive vertex irregular 2-distance labeling for the graph P_{2m} is defined as $\beta : V(P_{2m}) \to \left\{1, ..., \left\lceil \frac{2m+2}{5} \right\rceil \right\}$

with $\beta(u_i) = \lambda(v_i)$ such that $\text{wt}(u_i) = \text{wt}(v_i)$, where $u_i \in V(P_{2m})$.

The labeling with $d = 2$ for the graph P_{2m-1} is obtained by applying deletion process B to the graph P_{2m}. Let $u_i' \in V(P_{2m-1})$, the labeling for P_{2m-1} can be defined as β': $V(P_{2m-1}) \to \left\{1, ..., \left\lceil \frac{2m+2}{5} \right\rceil \right\}$ with $\beta'(u_i') = \beta(u_i) = \lambda(v_i)$. Thus, the weights obtained are $\text{wt}(u_i') = \text{wt}(u_i) = \text{wt}(v_i)$ for $1 \le i \le 2m-3$ and $\text{wt}(u_i') = \text{wt}(u_i) - 1 = \text{wt}(v_i) - 1$ for $i = 2m-2$ and $2m-1$. It is easy to see that all the weights are distinct. Similar with the proof of Lemma 2.2.1, with setting $n = 2m-1$, we can obtain $\text{dis}_2^0(P_n) = \left\lceil \frac{n+2}{5} \right\rceil$ with $n \equiv 9 \pmod{10}$. □

Immediately from Corollary 2.2.1 and Lemmas 2.2.1–2.2.3, we obtained our result on $\text{dis}_2^0(P_n)$ as follows.

Theorem 2.2.2. *Let P_n be a path with $n \ge 5, n \equiv 0, 1, 2, 4, 5, 6, 9 \pmod{10}$ and $n = 8$. Then*

$$\text{dis}_2^0(P_n) = \begin{cases} 3, & n = 8; \\ \left\lceil \frac{n+2}{5} \right\rceil, & \text{otherwise}. \end{cases}$$

Our next result is the value of $\text{dis}_4^0(P_n)$ for some odd numbers n that we derived from Theorems 2.1.1 and 2.2.1. To conclude this result, we constructed labeling with $d = 4$ in similar process with the process of our previous result on Theorem 2.2.2. Yet in the proofs of the next result, we also perform addition process (A or B) which is stated as follows, aside from deletion process (A or B). Let $\beta : V(P_n) \to \{1, ..., k\}$ be an inclusive vertex irregular d-distance labeling for P_n.

Addition Process A: Add vertex u_0 with label 1 and edge $u_0 u_1$ to the graph P_n. Then $\text{wt}(u_0) = 1 + \sum_{i=1}^{d} \beta(u_i)$ and the weight of vertices in $N_d(u_0)$, those are u_i with $1 \le i \le d$, will increase by 1. From this process, we obtain graph P_{n+1} where the label of vertex $u_i \in V(P_{n+1})$ equals to the label of vertex $u_{i-1} \in V(P_n) \cup \{u_0\}$. It means the weight of vertex $u_i \in V(P_{n+1})$, with $d+2 \le i \le n+1$, equals to the weight of vertex $u_{i-1} \in V(P_n) \cup \{u_0\}$.

Addition Process B: Add vertex u_{n+1} with label 2 and edge $u_n u_{n+1}$ to the graph P_n. In consequence of this addition, the weight of vertices in $N_d(u_{n+1})$, those are u_i with $(n+1) - d \le i \le n$, will increase by 2. Meanwhile, the weight of the new vertex is $\text{wt}(u_{n+1}) = 2 + \sum_{i=(n+1)-d}^{n} \beta(u_i)$. As a result, we obtain graph P_{n+1} where the label of vertex $u_i \in V(P_{n+1})$ equals to the label of vertex $u_i \in V(P_n) \cup \{u_{n+1}\}$. So that the weight of vertex $u_i \in V(P_{n+1})$, with $1 \le i \le n-d$, equals to the weight of vertex $u_i \in V(P_n) \cup \{u_{n+1}\}$.

Lemma 2.2.4. *Let P_n be a path with $n \ge 9$ and $n \equiv 1, 17 \pmod{18}$. Then*

$$\text{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil.$$

Proof. Consider the labeling with $d=2$ for graph $\mathbb{L}_m$ with $m \equiv 0 \pmod 9$ (proof of Lemma 2.1.4). Let $v_i \in V(\mathbb{L}_m)$ and λ is the labeling for graph $\mathbb{L}_m$. Then, labeling with $d=4$ for graph P_{2m} can be defined as $\beta(u_i) = \lambda(v_i)$, where $u_i \in V(P_{2m})$, such that $\mathrm{wt}(u_i) = \mathrm{wt}(v_i)$.

- Case 1: $n \equiv 1 \pmod{18}$

  With applying addition process A to the graph P_{2m}, we obtained graph P_{2m+1} and its labeling with $d=4$ as follows. Let $u_i' \in V(P_{2m+1})$, then labeling for the graph P_{2m+1} can be defined as $\beta'(u_1') = 1$ and $\beta'(u_i') = \beta(u_{i-1})$ for $2 \leq i \leq 2m+1$. The obtained vertex weights are $\mathrm{wt}(u_1') = 5$, $\mathrm{wt}(u_2') = 8$, $\mathrm{wt}(u_3') = 11$, $\mathrm{wt}(u_4') = 14$, $\mathrm{wt}(u_5') = 17$, $\mathrm{wt}(u_{2m-2}) = 15$, $\mathrm{wt}(u_{2m-1}') = 12$, $\mathrm{wt}(u_{2m}') = 9$, $\mathrm{wt}(u_{2m+1}') = 6$ and for the rest vertices are

  $$\mathrm{wt}(u_i') = \begin{cases} 2i + 7, & 6 \leq i \leq m+1; \\ 4m - 2i + 12, & m+2 \leq i \leq 2m-3. \end{cases}$$

  It is obvious that all the weights are distinct. Lemma 2.1 gives $\mathrm{dis}_4^0(P_{2m+1}) \geq \left\lceil \frac{(2m+1)+4}{9} \right\rceil$. Meanwhile, the largest label used in this labeling is the same as the largest label for graph P_{2m}, which is $\left\lceil \frac{2m+4}{9} \right\rceil$. Since $\left\lceil \frac{(2m+1)+4}{9} \right\rceil = \left\lceil \frac{2m+4}{9} \right\rceil$ for $m \equiv 0 \pmod 9$, then $\mathrm{dis}_4^0(P_{2m+1}) = \left\lceil \frac{(2m+1)+4}{9} \right\rceil$. Set $n = 2m+1$, then $\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil$ with $n \equiv 1 \pmod{18}$.

- Case 2: $n \equiv 17 \pmod{18}$

  For this case, we applied deletion process B to the graph P_{2m}. We obtained graph P_{2m-1} and its labeling with $d=4$ as follows. Let $u_i' \in V(P_{2m-1})$, then labeling for graph P_{2m-1} can be defined as $\beta'(u_i') = \beta(u_i)$. Thus, the weights are $\mathrm{wt}(u_1') = 7, \mathrm{wt}(u_2') = 10, \mathrm{wt}(u_3') = 13, \mathrm{wt}(u_4') = 16, \mathrm{wt}(u_{2m-4}') = 17, \mathrm{wt}(u_{2m-3}') = 14, \mathrm{wt}(u_{2m-2}') = 11, \mathrm{wt}(u_{2m-1}') = 8$ and for the other vertices are

  $$\mathrm{wt}(u_i') = \begin{cases} 2i + 9, & 5 \leq i \leq m; \\ 4m - 2i + 10, & m+1 \leq 2m-5. \end{cases}$$

  Similar as the Case 1, we can have $\mathrm{dis}_4^0(P_{2m-1}) = \left\lceil \frac{(2m-1)+4}{9} \right\rceil$. Finally set $n = 2m-1$, then $\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil$ for $n \equiv 17 \pmod{18}$. □

Lemma 2.2.5. *Let P_n be a path with $n \geq 9$ and $n \equiv 7 \pmod{18}$. Then*

$$\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil.$$

Proof. Let λ is the labeling with $d=2$ for graph $\mathbb{L}_m$ with $m \equiv 4 \pmod 9$ (proof of Lemma 2.1.2). Then, the labeling with $d=4$ for P_{2m} can be defined as $\beta(u_i) = \lambda(v_i)$, where $u_i \in V(P_{2m})$ and $v_i \in V(\mathbb{L}_m)$, so that $\mathrm{wt}(u_i) = \mathrm{wt}(v_i)$. With performing deletion process A twice and addition process B once to the graph P_{2m}, we obtained graph P_{2m-1} and its labeling with $d=4$ as follows. Let $u_i' \in V(P_{2m-1})$, then the labeling for P_{2m-1} can be defined as $\beta'(u_{2m-1}') = 2$ and $\beta'(u_i') = \beta(u_{i+2})$ for $1 \leq i \leq 2m-2$. Thus, the weights are $\mathrm{wt}(u_1') = 10$, $\mathrm{wt}(u_2') = 12$, $\mathrm{wt}(u_3') = 16$, $\mathrm{wt}(u_4') = 20$,

$\mathrm{wt}(u_{2m-6}') = 18$, $\mathrm{wt}(u_{2m-5}') = 17$, $\mathrm{wt}(u_{2m-4}') = 15$, $\mathrm{wt}(u_{2m-3}') = 13$, $\mathrm{wt}(u_{2m-2}') = 11$, $\mathrm{wt}(u_{2m-1}') = 9$ and for the rest vertices are

$$\mathrm{wt}(u_i') = \begin{cases} 2i + 14, & 5 \leq i \leq m-2; \\ 4m - 2i + 7, & m-1 \leq i \leq 2m-7. \end{cases}$$

With similar argument as in proof of Lemma 2.2.4, we conclude $\mathrm{dis}_4^0(P_{2m-1}) = \left\lceil \frac{(2m-1)+4}{9} \right\rceil$. So that, we can obtain $\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil$ for $n \equiv 7 \pmod{18}$ simply by setting $n = 2m - 1$. □

Lemma 2.2.6. *Let P_n be a path with $n \geq 9$ and $n \equiv 9 \pmod{18}$. Then*

$$\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil.$$

Proof. Consider the labeling with $d=2$ for graph $\mathbb{L}_m$ with $m \equiv 5 \pmod 9$ (proof of Lemma 2.1.1). Let $v_i \in V(\mathbb{L}_m)$ and λ is the labeling for graph $\mathbb{L}_m$. Then, the labeling with $d=4$ for P_{2m} can be defined as $\beta(u_i) = \lambda(v_i)$, where $u_i \in V(P_{2m})$. So that, $\mathrm{wt}(u_i) = \mathrm{wt}(v_i)$. The graph P_{2m-1} and its labeling with $d=4$ are obtained by performing deletion process A to the graph P_{2m}. Let $u_i' \in V(P_{2m-1})$, then labeling for P_{2m-1} can be defined as $\beta'(u_i') = \beta(u_{i+1})$. Thus, the weights are $\mathrm{wt}(u_1') = 10, \mathrm{wt}(u_2') = 12, \mathrm{wt}(u_3') = 14, \mathrm{wt}(u_4') = 16$ and for the other vertices are

$$\mathrm{wt}(u_i') = \begin{cases} 2i + 10, & 5 \leq i \leq m-1; \\ 4m - 2i + 7, & m \leq i \leq 2m-1. \end{cases}$$

We can conclude that $\mathrm{dis}_4^0(P_{2m-1}) = \left\lceil \frac{(2m-1)+4}{9} \right\rceil$ with similar argument as in Lemma 2.2.4. Set $n = 2m - 1$, then we have $\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil$ with $n \equiv 9 \pmod{18}$. □

Lemma 2.2.7. *Let P_n be a path with $n \geq 9$ and $n \equiv 15 \pmod{18}$. Then*

$$\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil.$$

Proof. Let $v_i \in V(\mathbb{L}_m)$ and λ is the labeling with $d=2$ for graph $\mathbb{L}_m$ with $m \equiv 8 \pmod 9$ (proof of Lemma 2.1.5). Then, the labeling with $d=4$ for P_{2m} can be defined as $\beta(u_i) = \lambda(v_i)$, where $u_i \in V(P_{2m})$, so that $\mathrm{wt}(u_i) = \mathrm{wt}(v_i)$. With applying deletion process B to the graph P_{2m}, we obtain graph P_{2m-1} and its labeling with $d=4$ as follows. Let $u_i' \in V(P_{2m-1})$, then the labeling for P_{2m-1} can be defined as $\beta'(u_i') = \beta(u_i)$. Thus, the weights are $\mathrm{wt}(u_1') = 9, \mathrm{wt}(u_2') = 12$, $\mathrm{wt}(u_3') = 15$, $\mathrm{wt}(u_4') = 18$, $\mathrm{wt}(u_{2m-4}') = 19$, $\mathrm{wt}(u_{2m-3}') = 16, \mathrm{wt}(u_{2m-2}') = 13, \mathrm{wt}(u_{2m-1}') = 10$ and for the other vertices are

$$\mathrm{wt}(u_i') = \begin{cases} 2i + 11, & 5 \leq i \leq m; \\ 4m - 2i + 12, & m+1 \leq i \leq 2m-5. \end{cases}$$

So that, with similar argument as in the proof of Lemma 2.2.4, we conclude $\mathrm{dis}_4^0(P_{2m-1}) = \left\lceil \frac{(2m-1)+4}{9} \right\rceil$. By setting $n = 2m - 1$, then we have $\mathrm{dis}_4^0(P_n) = \left\lceil \frac{n+4}{9} \right\rceil$ for $n \equiv 15 \pmod{18}$. □

Immediately from Corollary 2.2.2 and Lemmas 2.2.4 throughout 2.2.7, we conclude our result on $\mathrm{dis}_4^0(P_n)$ as follows.

Theorem 2.2.3. *Let P_n be a path with $n \geq 9, n \equiv 0, 1, 6, 7, 8, 9, 10, 15, 16, 17 \pmod{18}$ and $n = 14$. Then*

$$\mathrm{dis}_4^0(P_n) = \begin{cases} 3, & n = 14 \\ \left\lceil \frac{n+4}{9} \right\rceil, & otherwise. \end{cases}$$

Our further investigation brought us to the result of $\mathrm{dis}_3^0(P_n)$. Different with our previous results on path, this result cannot be obtained by using the relation between the labeling from graph $\mathbb{L}_m$ and P_{2m}. Yet, the labeling idea for the graph $\mathbb{L}_m$ is used in order to construct the labeling with $d = 3$ for the graph P_n with $n \equiv 1 \pmod{7}$. Based on Lemma 2.1, it is easy to see that $\mathrm{dis}_3^0(P_n) = \infty$ for $n < 7$. So, the value of $\mathrm{dis}_3^0(P_n)$ is obtained for $n \geq 7$.

Lemma 2.2.8. *Let P_n be a path with $n \geq 7$ and $n \equiv 1 \pmod{7}$. Then*

$$\mathrm{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil.$$

Proof. Proof of this result is divided into two cases, $n \equiv 8$ and $1 \pmod{14}$. For each case, we defined a labeling with $d = 3$. Then, we show that the weights obtained is distinct for all vertices and we show that the largest label used is equal with the lower bound value which is given by Lemma 2.1, that is $\mathrm{dis}_3^0 \geq \left\lceil \frac{n+3}{7} \right\rceil$.

- Case 1: $n \equiv 8 \pmod{14}$

  Let $u_i \in V(P_n)$. Defined the labeling for P_n as $\beta_1 : V(P_n) \rightarrow \left\{ 1, ..., \left\lceil \frac{n+3}{7} \right\rceil \right\}$ with

  $$\beta_1(u_i) = \begin{cases} 2/7(i+6), & i \equiv 1 \pmod{7}, 1 \leq i \leq n/2 + 3; \\ 2/7(n-i) + 1, & i \equiv 1 \pmod{7}, n/2 + 4 \leq i \leq n; \\ \beta_1(u_{i-1}) - 1, & i \equiv 2 \pmod{7}, n/2 + 4 \leq i \leq n; \\ \beta(u_{i-1}), & otherwise. \end{cases}$$

  The largest label used is $\beta_1(u_{\frac{n}{2}-3}) = \frac{2}{7}\left(\frac{n}{2}+3\right) = \frac{n+6}{7}$ which is equals to $\left\lceil \frac{n+3}{7} \right\rceil$. With this labeling, then we obtain the weights of the vertices as follows.

  $$\mathrm{wt}(u_i) = \begin{cases} 2i + 6, & 1 \leq i \leq n/2; \\ 2(n-i) + 7, & n/2 + 1 \leq i \leq n. \end{cases}$$

  It is easy to see that all the weights are distinct. Thus, we can conclude that $\mathrm{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil$ for $n \equiv 1 \pmod{7}$ with n even.

- Case 2: $n \equiv 1 \pmod{14}$

  For this case, defined the labeling for P_n as $\beta_2 : V(P_n) \rightarrow \left\{ 1, ..., \left\lceil \frac{n+3}{7} \right\rceil \right\}$ with

  $$\beta_2(u_i) = \begin{cases} 1, & 1 \leq i \leq 4; \\ 1/7(2i+11), & i \equiv 5 \pmod{7}, 5 \leq i \leq (n+1)/2 + 3; \\ 2/7(n-i+4), & i \equiv 5 \pmod{7}, (n+1)/2 + 4 \leq i \leq n; \\ \beta_2(u_{i-1}) - 1, & i \equiv 6 \pmod{7}, (n+1)/2 + 4 \leq i \leq n; \\ \beta_2(u_{i-1}), & otherwise. \end{cases}$$

The weights of vertices obtained are $\mathrm{wt}(u_1) = 4, \mathrm{wt}(u_2) = 7, \mathrm{wt}(u_3) = 10, \mathrm{wt}(u_{n-2}) = 11, \mathrm{wt}(u_{n-1}) = 8, \mathrm{wt}(u_n) = 5$ and for other vertices are

$$\mathrm{wt}(u_i) = \begin{cases} 2i + 5, & 4 \leq i \leq (n+1)/2; \\ 2(n-i+4), & (n+1)/2 + 1 \leq i \leq n - 3. \end{cases}$$

The largest label used is $\beta_2(u_{\frac{n+1}{2}-3}) = \left\lceil \frac{n+5}{7} \right\rceil$ which is equals to $\left\lceil \frac{n+3}{7} \right\rceil$. Therefore, we can conclude that $\mathrm{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil$ for $n \equiv 1 \pmod{7}$ with n odd. $\square$

To obtain the exact value of $\mathrm{dis}_3^0(P_n)$ for other value of $n \pmod{7}$, we applied deletion process (A or B) to the labeled graph P_n with $n \equiv 1 \pmod{7}$. As a result from this process, we have the labeling for P_n with $n \equiv 0, 6$ and $5 \pmod{7}$. Based on the labeling obtained, we conclude the exact value of $\mathrm{dis}_3^0(P_n)$ for $n \equiv 0, 6$ and $5 \pmod{7}$ as we state in the following Lemma.

Lemma 2.2.9. *Let P_n be a path with $n \geq 7$ and $n \equiv 0, 6, 5 \pmod{7}$. Then*

$$\mathrm{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil.$$

Proof. For the first case of our proof, consider the labeling with $d = 3$ for P_m with $m \equiv 8 \pmod{14}$, while $m \equiv 1 \pmod{14}$ for the second case. In each cases, we applied the deletion process A and/or B to obtain the labeling for P_n with other value of $n \pmod{14}$.

- Case 1: $n \equiv 7, 6, 5 \pmod{14}$

  With applying the deletion process A j times to the graph P_m with $m \equiv 8 \pmod{14}$, we obtain graph P_{m-j} for $1 \leq j \leq 3$. The labeling for P_{m-j} with $1 \leq j \leq 3$ can be defined as $\beta'(u_i') = \beta_1(u_{i+j})$, where $u_i' \in V(P_{m-j})$. The weights obtained for u_i' with $4 \leq i \leq m - j$ are

  $$\mathrm{wt}(u_i') = \begin{cases} 2(i+j) + 6, & 4 \leq i \leq m/2 - j; \\ 2(m - (i+j)) + 7, & m/2 - j + 1 \leq i \leq m - j. \end{cases}$$

  Meanwhile, the weights obtained for the first three vertices for graph P_{m-1} are $\mathrm{wt}(u_1') = 8, \mathrm{wt}(u_2') = 10, \mathrm{wt}(u_3') = 12$, while for P_{m-2} are $\mathrm{wt}(u_1') = 8, \mathrm{wt}(u_2') = 10, \mathrm{wt}(u_3') = 14$ and for P_{m-3} are $\mathrm{wt}(u_1') = 8, \mathrm{wt}(u_2') = 12, \mathrm{wt}(u_3') = 16$. It is easy to see that for each graph P_{m-j}, the vertex weights obtained are all distinct. Note that Lemma 2.1 gives $\mathrm{dis}_3^0(P_{m-j}) = \left\lceil \frac{m-j+3}{7} \right\rceil$ which is equals to the largest label used, that is P_{m-j} is $\left\lceil \frac{m+3}{7} \right\rceil$, since $m \equiv 8 \pmod{14}$ and $1 \leq j \leq 3$. Thus, we have $\mathrm{dis}_3^0(P_{m-j}) = \left\lceil \frac{m-j+3}{7} \right\rceil$. Set $n = m - j$, then $\mathrm{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil$ with $n \equiv 7, 6, 5 \pmod{14}$.

- Case 2: $n \equiv 0, 13, 12 \pmod{14}$

  Consider graph P_m with $m \equiv 1 \pmod{14}$. Graph P_{m-1} is obtained by applying deletion process A once to graph P_m, while graph P_{m-2} is obtained by applying deletion process A and B once and graph P_{m-3} is obtained by applying deletion process A twice and deletion process B once. The labeling and the vertex weights obtained for

Labeling for graph P_{m-1}:	Labeling for graph P_{m-2}:	Labeling for graph P_{m-3}:
$\beta'(u_i') = \beta_2(u_{i+1})$	$\beta'(u_i') = \beta_2(u_{i+1})$	$\beta'(u_i') = \beta_2(u_{i+2})$
and the weights are:	and the weights are:	and the weights are:
$\text{wt}(u_1') = 6$	$\text{wt}(u_1') = 6$	$\text{wt}(u_1') = 8$
$\text{wt}(u_2') = 9$	$\text{wt}(u_2') = 9$	$\text{wt}(u_2') = 11$
$\text{wt}(u_3') = 12$	$\text{wt}(u_3') = 12$	$\text{wt}(u_3') = 14$
$\text{wt}(u_{m-3}') = 11$	$\text{wt}(u_{m-4}') = 13$	$\text{wt}(u_{m-5}') = 13$
$\text{wt}(u_{m-2}') = 8$	$\text{wt}(u_{m-3}') = 10$	$\text{wt}(u_{m-4}') = 10$
$\text{wt}(u_{m-1}') = 5$	$\text{wt}(u_{m-2}') = 7$	$\text{wt}(u_{m-3}') = 7$
for $4 \leq i \leq (m+1)/2 - 1$:	for $4 \leq i \leq (m+1)/2 - 1$:	for $4 \leq i \leq (m+1)/2 - 2$:
$\text{wt}(u_i') = 2i + 7$	$\text{wt}(u_i') = 2i + 7$	$\text{wt}(u_i') = 2i + 9$
for $(m+1)/2 \leq i \leq m - 4$:	for $(m+1)/2 \leq i \leq m - 5$:	for $(m+1)/2 - 1 \leq i \leq m - 6$:
$\text{wt}(u_i') = 2(m-i+3)$	$\text{wt}(u_i') = 2(m-i+3)$	$\text{wt}(u_i') = 2(m-i+2)$

each graph can be stated as follows. Let $u_i' \in V(P_{m-j})$ with $1 \leq j \leq 3$.

With similar argument as the previous case, we can conclude that $\text{dis}_3^0(P_{m-j}) = \left\lceil \frac{m-j+3}{7} \right\rceil$. Simply by setting $n = m - j$, we have $\text{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil$ for $n \equiv 0, 13, 12 \pmod{14}$.

□

Based on Lemmas 2.2.8 and 2.2.9, we conclude our result of $\text{dis}_3^0(P_n)$ in Theorem 2.2.4.

Theorem 2.2.4. *Let P_n be a path with $n \geq 7$ and $n \equiv 0, 1, 5, 6 \pmod{7}$. Then*

$$\text{dis}_3^0(P_n) = \left\lceil \frac{n+3}{7} \right\rceil.$$

3. Conclusion

In the previous section, we generalized the lower bound value of $\text{dis}_d^0(G)$ and the sufficient condition for $\text{dis}_d^0(G) < \infty$. For triangular ladder graph $\mathbb{L}_n$, we established exact value of $\text{dis}_2^0(\mathbb{L}_n)$ for $n \equiv 0, 3, 4, 5, 8 \pmod{9}$ and $n = 7$, while for the other values of $n \pmod{9}$ we only found the upper bound value. We also found relation between triangular ladder graph $\mathbb{L}_n$ and P_{2n}. Based on this result, we derived that $\text{dis}_d^0(\mathbb{L}_n) = \text{dis}_{2d}^0(P_{2n})$. For our other result on P_n, we derived it from our previous result on relation between triangular ladder graph and path. Thus we obtained the exact values of $\text{dis}_2^0(P_n)$ for $n \equiv 0, 1, 2, 4, 5, 6, 9 \pmod{10}$ and $n = 8$ and $\text{dis}_4^0(P_n)$ for $n \equiv 0, 1, 6, 7, 8, 9, 10, 15, 16, 17 \pmod{18}$ and $n = 14$. Yet, on our ongoing investigation we cannot find the exact value of $\text{dis}_2^0(P_n)$ and $\text{dis}_4^0(P_n)$ for other values of n. Another investigation that we have done is $\text{dis}_d^0(P_n)$ with $d = 3$. We cannot derived this value directly from the result on graph $\mathbb{L}_m$, but we used the idea of labeling on $\mathbb{L}_m$ to construct the labeling with $d = 3$ for graph P_n with $n \equiv 1 \pmod{7}$. As a result, we have the exact value of $\text{dis}_3^0(P_n)$ for $n \equiv 0, 1, 5, 6 \pmod{7}$. Similar with the previous results on path, we cannot find the exact value of $\text{dis}_3^0(P_n)$ for other values of $n \pmod{7}$ yet.

Based on our investigation results, we propose the following open problems.

Problem 1. *For graph $\mathbb{L}_n, n \geq 5, n \equiv 1, 2, 6, 7 \pmod{9}, n \neq 7$, determine the exact value of $\text{dis}_2^0(\mathbb{L}_n)$.*

Problem 2. *For graph $P_n, n \geq 5, n \equiv 3, 7, 8 \pmod{10}, n \neq 8$, determine the exact value of $\text{dis}_2^0(P_n)$.*

Problem 3. *For graph $P_n, n \geq 9, n \equiv 2, 3, 4, 5, 11, 12, 13, 14 \pmod{18}, n \neq 14$, determine the exact value of $\text{dis}_4^0(P_n)$.*

Problem 4. *For graph $P_n, n \geq 7, n \equiv 2, 3, 4 \pmod{7}$, determine the exact value of $\text{dis}_3^0(P_n)$.*

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