AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

On regular subgraphs of augmented cubes

Amruta Shinde & Y. M. Borse

To cite this article: Amruta Shinde & Y. M. Borse (2020): On regular subgraphs of augmented cubes, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2019.12.007

To link to this article: https://doi.org/10.1016/j.akcej.2019.12.007

Published online: 27 Apr 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2019.12.007

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS Check for updates

On regular subgraphs of augmented cubes

Amruta Shinde and Y. M. Borse

Department of Mathematics, Savitribai Phule Pune University, Pune, India

ABSTRACT
The n-dimensional augmented cube AQ_n is a variation of the hypercube Q_n. It is a $(2n-1)$-regular and $(2n-1)$-connected graph on 2^n vertices. One of the fundamental properties of AQ_n is that it is pancyclic, that is, it contains a cycle of every length from 3 to 2^n. In this paper, we generalize this property to k-regular subgraphs for $k=3$ and $k=4$. We prove that the augmented cube AQ_n with $n \geq 4$ contains a 4-regular, 4-connected and pancyclic subgraph on l vertices if and only if $8 \leq l \leq 2^n$. Also, we establish that for every even integer l from 4 to 2^n, there exists a 3-regular, 3-connected and pancyclic subgraph of AQ_n on l vertices.

KEYWORDS
Augmented cube; hypercube; regular subgraph; pancyclicity; connectivity

1. Introduction

The interconnection networks play an important role in parallel computing and communication systems. The underlying topology of the interconnection network is represented by a graph. The hypercube is a popular network topology because of its good properties, such as strong connectivity, small diameter, symmetry, relatively small degree, bipancyclicity and regularity. Choudum and Sunitha [4] proposed a new variation of the hypercube Q_n called augmented cube AQ_n of n-dimension as an improvements over hypercubes. The augmented cube AQ_n is a $(2n-1)$-regular and $(2n-1)$-connected graph with 2^n vertices and it has diameter $\lceil n/2 \rceil$-diameter. Several results available in literature shows that the augmented cube is a good candidate for computer network topology design; see [4–6, 8, 10, 11, 15].

A graph G is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$, whereas G is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$. Moreover, G is *nearly pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ except possibly for one value. Cycles are fundamental networks for parallel and distributed computing as they are suitable for designing simple algorithms with low communication cost [7]. Pancyclicity of a network is an important factor in determining whether the network topology can simulate cycles of various lengths. Connectivity is a crucial parameter for interconnection networks as it measures the stability of a network. Pancyclicity and connectivity properties for augmented cubes are studied in [6, 8, 15].

The augmented cube AQ_n is pancyclic. Therefore cycles of every length from 3 to 2^n can be embedded into it. Thus AQ_n contains a 2-regular and 2-connected subgraph on l vertices for every integer l with $3 \leq l \leq 2^n$. It is natural to think of generalizing this fundamental property of the augmented cubes to the existence of k-regular, k-connected and pancyclic subgraphs. This will be useful to get subgraphs of AQ_n with less number of vertices which retain the important properties of AQ_n such as regularity, pancyclicity and high connectivity. For hypercubes, this problem is studied in [2, 3, 14].

Mane and Waphare [12] investigated for the existence of a k-regular, k-connected and bipancyclic subgraph of the hypercube Q_n with 2^n vertices for given k. Lu et al. [9] considered the similar problem for the Cartesian product of cycles. Ramras [14] proved that the hypercube Q_n contains a 3-regular subgraph with l vertices for even integer l from 8 to 2^n except 10. Borse and Shaikh [2] improved this result by proving that for such values of l there exists a 3-regular subgraph of Q_n with l vertices which is 3-connected and bipancyclic too. Similar results for the classes of the Cartesian product of cycles and the Cartesian product of paths are obtained in [1] and [13], respectively. For the existence of 4-regular subgraphs, Borse and Shaikh [3] established that there exists a 4-regular, 4-connected and bipancyclic subgraph on l vertices in the hypercube Q_n if and only if $l=16$ or l is an even integer with $24 \leq l \leq 2^n$.

In this paper, we generalize the property of pancyclicity to the existence of 3-regular subgraphs and 4-regular subgraphs in augmented cubes. Since AQ_n is simple and the total degree of a 3-regular graph is even, the number of vertices of every 3-regular subgraph of AQ_n is even and at least 4.

The following are the main results of the paper.

Theorem 1.1. *Let $n \geq 2$ and l be integers such that l is even and $4 \leq l \leq 2^n$. Then there exists a 3-regular, 3-connected*

CONTACT Amruta Shinde samruta421@gmail.com Department of Mathematics, Savitribai Phule Pune University, Pune, India.

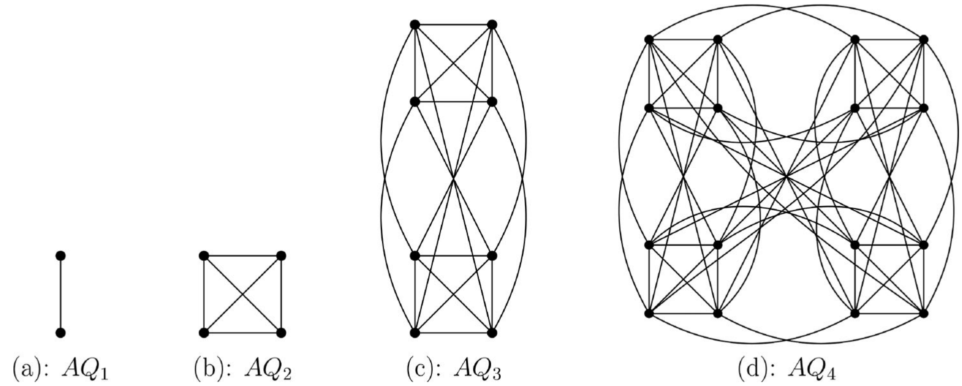


Figure 1. Augmented cubes.

and nearly pancyclic subgraph of the augmented cube AQ_n with l vertices.

Theorem 1.2. *Let $n \geq 4$ and l be integers. Then the augmented cube AQ_n contains a 4-regular, 4-connected and pancyclic subgraph with l vertices if and only if $8 \leq l \leq 2^n$.*

The paper is organized as follows. In Section 2, we obtain some lemmas which are used in the proofs in the subsequent sections. Theorem 1.1 is proved in Section 3. The proof of Theorem 1.2 is divided in two sections, Section 4 deals with the case $27 \leq l \leq 2^n$, whereas the remaining cases are covered in Section 5. Also, in Section 5, we prove a result about the non-existence of 4-regular subgraphs with $l < 7$ vertices.

2. Preliminaries

In this section, we provide the definition of the augmented cube AQ_n, and obtain some results regarding pancyclicity, connectivity and the existence of particular types of cycles in AQ_n which are used in the subsequent sections.

For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. By a k-cycle, we mean a cycle of length k, denoted by C_k. A path with vertices $a_1, a_2, ..., a_n$ in order is written as $\langle a_1, a_2, ..., a_n \rangle$ and cycles are also written similarly. The Cartesian product $G \Box H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$, where any two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. The n-dimensional hypercube Q_n is the Cartesian product of n copies of the complete graph K_2.

An n-dimensional augmented cube, for $n \geq 1$, denoted by AQ_n, contains 2^n vertices, each labeled by an n-bit binary string $a_n a_{n-1} \cdots a_1$. We define $AQ_1 = K_2$. For $n \geq 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding 2^n edges between the two as follows: Let $V(AQ^0_{n-1}) = \{0a_{n-1} \cdots a_1 : a_i = 0 \text{ or } 1\}$ and $V(AQ^1_{n-1}) = \{1b_{n-1} \cdots b_1 : b_i = 0 \text{ or } 1\}$. A vertex $0a_{n-1} \cdots a_1$ of AQ^0_{n-1} is adjoined to a vertex $1b_{n-1} \cdots b_1$ of AQ^1_{n-1} iff for every i, $1 \leq i \leq n-1$, either

(i) $a_i = b_i$ or
(ii) $a_i = \overline{b_i}$.

Type (i) edges are *hypercube edges* while Type (ii) are *complement edges*, and their sets are denoted by E_h and E_c, respectively. Thus, we have $AQ_n = AQ^0_{n-1} \cup AQ^1_{n-1} \cup E_h \cup E_c$. Note that E_h and E_c are perfect matchings between AQ^0_{n-1} and AQ^1_{n-1} and further, $AQ_n - E_c$ is isomorphic to $AQ_{n-1} \Box K_2$. The augmented cubes of dimension 1, 2, 3 and 4 are illustrated in Figure 1.

We need the following results.

Lemma 2.1. [16] *Let H_i be an n_i-regular and n_i-connected graph for $i = 1, 2$. Then the graph $H_1 \Box H_2$ is $(n_1 + n_2)$-regular and $(n_1 + n_2)$-connected.*

Lemma 2.2. [8] *For $n \geq 2$, the augmented cube AQ_n is edge-pancyclic.*

Lemma 2.3. [12] *If P and Q are non-trivial paths and one of them has even number of vertices, then $P \Box Q$ is bipancyclic.*

Corollary 2.4. *If C_1 and C_2 are two cycles and one of them has even length, then $C_1 \Box C_2$ is bipancyclic.*

Corollary 2.5. *If C is a cycle, then $C \Box K_2$ is bipancyclic.*

We obtain the following two results about pancyclicity of particular types of graphs.

Lemma 2.6. *If C is an odd cycle of length m, then $C \Box K_2$ is bipancyclic and m-pancyclic.*

Proof. Let $G = C \Box K_2$. Then the graph G contains two vertex-disjoint cycles, say $C = \langle a_1, a_2, ..., a_{m-1}, a_m, a_1 \rangle$ and $C' = \langle b_1, b_2, ..., b_{m-1}, b_m, b_1 \rangle$ such that a_i is adjacent to b_i for $i = 1, 2, ..., m$ (see Figure 2(a)). By Corollary 2.5, G is bipancyclic. The graph G contains a cycle C of length m. Replacing the edge $\langle a_1, a_2 \rangle$ of C by the path $\langle a_1, b_1, b_2, a_2 \rangle$ we get a cycle C_{m+2} in G of length $m + 2$ as shown in Figure 2(b). We continue replacing the edge $\langle a_{2i-1}, a_{2i} \rangle$ of the cycle $C_{m+2(i-1)}$ by the path $\langle a_{2i-1}, b_{2i-1}, b_{2i}, a_{2i} \rangle$ of length three to get a new cycle C_{m+2i}, for $i = 1, 2, ..., (m-1)/2$. Thus G contain cycles of all odd length from m to $2m - 1$. Hence G is bipancyclic and m-pancyclic. □

We prove below that adding a path of length two to adjacent vertices of a ladder gives a pancyclic graph.

Lemma 2.7. *Let $m \geq 4$ be an integer and let C be an m-cycle. Suppose the graph H is obtained from the ladder $C \Box K_2$ by*

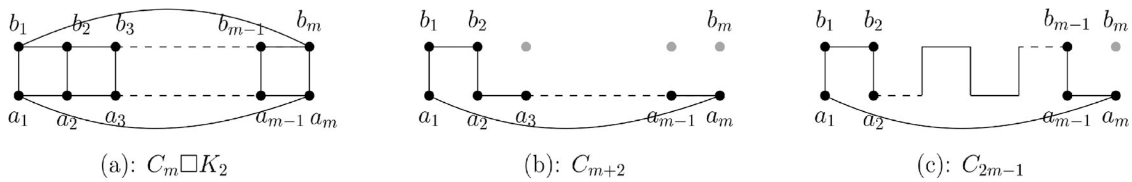


Figure 2. Odd cycles in $C_m \square K_2$.

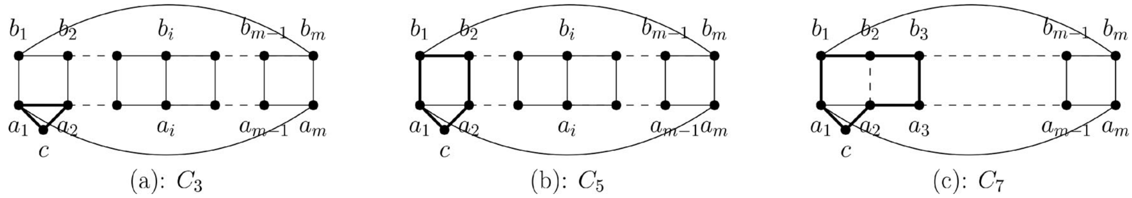


Figure 3. Odd cycles in H.

identifying the end vertices of a path of length two to a pair of adjacent vertices of the cycle C. Then H is pancyclic.

Proof. We can write $C\square K_2 = C \cup C' \cup M$, where $C = \langle a_1, a_2, ..., a_{m-1}, a_m, a_1\rangle$ and $C' = \langle b_1, b_2, ..., b_{m-1}, b_m, b_1\rangle$ and $M = \{\langle a_i, b_i\rangle : i = 1, 2, ..., m\}$. Due to symmetry in $C\square K_2$, we may assume that $H = (C\square K_2) \cup P$, where P is a path $\langle a_1, c, a_2\rangle$, (see Figure 3(a)). By Corollary 2.5, $C\square K_2$ is bipancyclic. Therefore H contains a cycle of every even length from 4 to $2m$. Clearly, $\langle a_1, c, a_2, a_1\rangle$ is a 3-cycle C_3 in H. On replacing the edge $\langle a_1, a_2\rangle$ of C_3 by the path $\langle a_1, b_1, b_2, a_2\rangle$, we get a 5-cycle C_5. To get a 7-cycle C_7 from C_5, we replace the edge $\langle a_2, b_2\rangle$ by a path P of length 3 where $P = \langle a_2, a_3, b_3, b_2\rangle$. We continue replacing an edge $\langle a_i, b_i\rangle$ of an odd cycle C_r by a path $\langle a_i, a_{i+1}, b_{i+1}, b_i\rangle$ to get a cycle of length $r+2$. This procedure gives cycles in H of all odd length from 3 to $2m+1$. Hence H is pancyclic. □

The next two results are about the existence of special types of cycles in AQ_n, and they are used in the construction of 3-regular and 4-regular subgraphs in AQ_n in the subsequent sections.

Lemma 2.8. *Let $n \geq 3$ and l be integers such that $7 \leq l \leq 2^n - 1$. Then there exists a cycle $C = \langle u_1, u_2, ..., u_l, u_1\rangle$ in AQ_n and a vertex $v \in V(AQ_n) - V(C)$ such that*

(i) *v is adjacent to u_1, u_2, u_3, u_k for some $4 < k < l$.*
(ii) *$\langle u_1, u_3\rangle$ and $\langle u_2, u_k\rangle$ are chords of C.*

Proof. We prove the result by induction on n. A cycle of length 7 in AQ_3 satisfying the properties (i) and (ii) is shown in Figure 4(a), whereas such a cycle of length 8 in AQ_4 is shown in the Figure 4(b). Hence the result holds for $n = 3$, and also it holds for $l = 7$ and $l = 8$ when $n \geq 4$.

Suppose $n \geq 4$ and $l \geq 9$. Assume that the result holds for $n-1$. Write $AQ_n = AQ^0_{n-1} \cup AQ^1_{n-1} \cup E_h \cup E_c$.

If $7 \leq l \leq 2^{n-1} - 1$, then by induction, AQ^0_{n-1} and so AQ_n contains a cycle of length l as desired. Suppose $2^{n-1} \leq l \leq 2^n - 1$. Then $l = 2^{n-1} - 1 + k$ for some k with $3 \leq k \leq 2^{n-1}$. Let C be a cycle on $l = 2^{n-1} - 1$ vertices satisfying the properties (i) and (ii), choose an edge $f = \langle a, b\rangle$ of C such that $a, b \notin \{u_1, u_2, u_3\}$. Let $f' = \langle a', b'\rangle$ be the corresponding edge in AQ^1_{n-1}. Then $\langle a, a'\rangle$ and $\langle b, b'\rangle$ are hypercube edges of AQ_n. Clearly, $(C - f) \cup \{f', \langle a, a'\rangle, \langle b, b'\rangle\}$ is a cycle of length $2^{n-1} + 1$ satisfying (i) and (ii). By Lemma 2.2, AQ^1_{n-1} contains a cycle C_k of length k containing the edge f'. Therefore $(C - f) \cup (C_k - f') \cup \{\langle a, a'\rangle, \langle b, b'\rangle\}$ is a cycle in AQ_n of length $2^{n-1} - 1 + k = l$ satisfying (i) and (ii). (See Figure 4(c).)

Suppose $l = 2^{n-1}$. As $l \geq 9$, we have $n \geq 5$. By induction, AQ^0_{n-1} contains a cycle Z of length $l = 2^{n-1} - 2$ satisfying (i) and (ii). Choose an edge $g = \langle x, y\rangle$ of Z such that $x, y \notin \{u_1, u_2, u_3\}$. If $g' = \langle x', y'\rangle$ is the edge in AQ^1_{n-1} corresponding to g, then $(Z - g) \cup \{g', \langle x, x'\rangle, \langle y, y'\rangle\}$ is a cycle of length l satisfying (i) and (ii). This completes the proof. □

Similarly, we get the following result.

Lemma 2.9. *Let n and l be integers such that $4 \leq l \leq 2^n$. Then there exists a cycle C on l vertices in AQ_n with a chord e that forms a triangle with two edges of C.*

Proof. Clearly, $n \geq 2$. We proceed by induction on n. The result obviously holds for $n = 2$ as $AQ_2 = K_4$ contains a 4-cycle with a chord. Suppose $n \geq 3$. Since AQ_2 is a subgraph of AQ_n, the result holds for $l = 4$. For $l \in \{5, 6\}$, a required cycle of length l in AQ_3 and so in AQ_n, is shown in Figure 5.

Suppose $7 \leq l \leq 2^n - 1$. By Lemma 2.8, we get cycle C of length l with a chord e which forms a triangle with two edges of C. Suppose $l = 2^n$. Let Z be a cycle in AQ^0_{n-1} of length 2^{n-1} containing two adjacent edges e_1 and e_2 that forms a triangle with a chord e of Z. Since $n \geq 3$, there is an edge $f = \langle x, y\rangle$ of Z different from e_1 and e_2. Let Z' be a cycle of AQ^1_{n-1} corresponding to Z and let $f' = \langle x', y'\rangle$ be the edge of AQ^1_{n-1} corresponding to f. Then $(Z - f) \cup (Z' - f') \cup \{\langle x, x'\rangle, \langle y, y'\rangle\}$ is a cycle of length 2^n with a chord e forming a triangle with two edges of this cycle. Hence the result holds for $l = 2^n$. This completes the proof. □

The next two results are about connectivity.

Lemma 2.10. *Let G_i be a k_i-connected graph for $i = 1$, 2 with $V(G_1) = \{a_1, a_2, ..., a_p\}$ and $V(G_2) = \{b_1, b_2, ..., b_p\}$. Then*

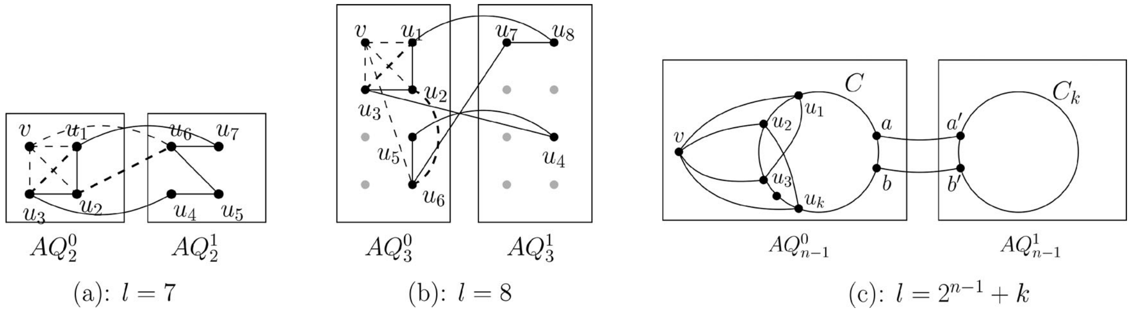


Figure 4. Cycles of length l.

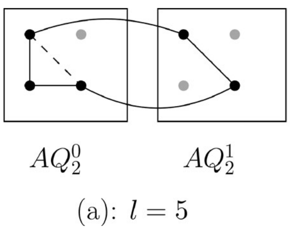


Figure 5. Cycles of length 5 and 6.

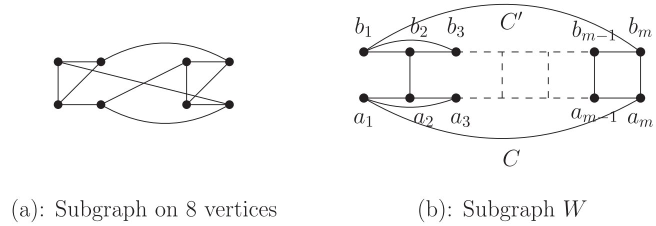


Figure 7. 3-regular subgraphs of AQ_n.

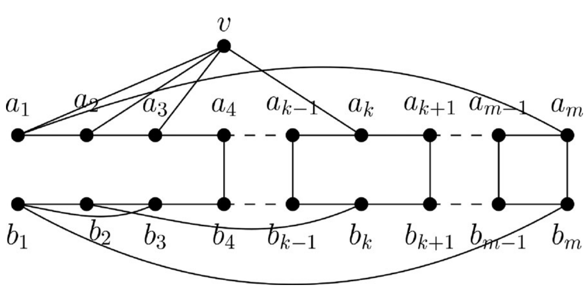


Figure 6. The graph G.

the graph $G = G_1 \cup G_2 \cup M$ is $(k+1)$-connected, where $M = \{\langle a_i, b_i\rangle : i = 1, 2, ..., p\}$ and $k = min\{k_1, k_2\}$.

Proof. Since G_i is k_i-connected, it is k-connected and has at least $k_i + 1 \geq k + 1$ vertices. It is sufficient to prove that deletion of any k vertices from G leaves a connected graph. Let $S \subseteq V(G)$ with $|S| = k$. We prove that $G - S$ is connected. Suppose S is a subset of $V(G_1)$ or $V(G_2)$. If $S \subseteq V(G_1)$, then $G - S$ connected as every component of $G_1 - S$ has a neighbour in the connected graph G_2. Similarly, $G - S$ is connected if $S \subseteq V(G_2)$. Suppose S intersects both $V(G_1)$ and $V(G_2)$. Let $S_i \cap V(G_i)$ for $i = 1, 2$. As $1 \leq |S_1|, |S_2| < k$, both $G_1 - S_1$ and $G_2 - S_2$ are connected and they are joined to each other by at least $p - k \geq 1$ edges of the matching M. Hence $G - S$ is connected. □

Lemma 2.11. *For $m \geq 7$, let H be a 3-regular graph on $2m$ vertices consisting of two cycles $C = \langle a_1, a_2, ..., a_m, a_1\rangle$ and $C' = \langle b_1, b_2, ..., b_m, b_1\rangle$ and a perfect matching $M = \{\langle a_i, b_i\rangle : i = 1, 2, ..., m\}$. Let N be a graph obtained from H by adding a vertex v and four new edges $\{\langle v, a_1\rangle, \langle v, a_2\rangle, \langle v, a_3\rangle, \langle v, a_k\rangle\}$ for some k with $4 < k < m$. Then the graph $G = (N - \{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \langle a_3, b_3\rangle, \langle a_k, b_k\rangle\}) \cup \{\langle b_1, b_3\rangle, \langle b_2, b_k\rangle\}$ is 3-connected. (See Figure 6.)*

Proof. Let $S \subseteq V(G)$ with $|S| = 2$. It suffices to prove that $G - S$ is connected. There are at least $m - 4 - 2 \geq 1$ edges between C and C' in $G - S$. Let G_1 and G_2 be the subgraphs of G induced by $V(C) \cup \{v\}$ and $V(C')$, respectively. Then G_1 and G_2 are 2-connected.

Suppose S intersects both $V(G_1)$ and $V(G_2)$. Let $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. Then $|S_1| = |S_2| = 1$. Hence $G_1 - S_1$ and $G_2 - S_2$ are connected and are joined to each other by at least one edge, giving $G - S$ connected.

Suppose $S \subseteq V(G_2)$. If $G_2 - S$ is connected, then $G - S$ is connected as $G_2 - S$ has a neighbour in the connected graph G_1. Suppose $G_2 - S$ is disconnected. Then it has two components. Let $B = \{b_1, b_2, b_3, b_k\}$. Note that the degree of every member of the set $B - S$ is at least one in $G_2 - S$. Since $m \geq 7$, it follows that every component of $G_2 - S$ contains a vertex from $V(C) - B$ and so has a neighbour in G_1. This shows that $G - S$ is connected. Similarly, if $S \subseteq V(G_1)$, then every component of $G_1 - S$ has a neighbour in the connected graph G_2. This completes the proof. □

3. Existence of 3-regular subgraphs in AQ_n

In this section, we prove Theorem 1.1.

The total degree of a graph is an even integer, the number of vertices of a 3-regular graph is always an even integer. Obviously, a 3-regular subgraph of the simple graph AQ_n has at least 4 vertices.

Since AQ_2 is a complete graph K_4, it is obviously a 3-regular, 3-connected and pancyclic subgraph of AQ_n, for $n \geq 2$. Clearly, $C_3 \square K_2$ is a 3-regular subgraph of AQ_3 on 6 vertices, where C_3 is a 3-cycle. This graph is 3-connected by Lemma 2.1 and pancyclic by Lemma 2.6 and it is also a subgraph AQ_n for $n \geq 3$. Moreover, by Lemma 2.1 and Corollary 2.5, $C_4 \square K_2$ is a 3-regular, 3-connected and bipancyclic subgraph of AQ_3 on 8 vertices, where C_4 is a 4-cycle.

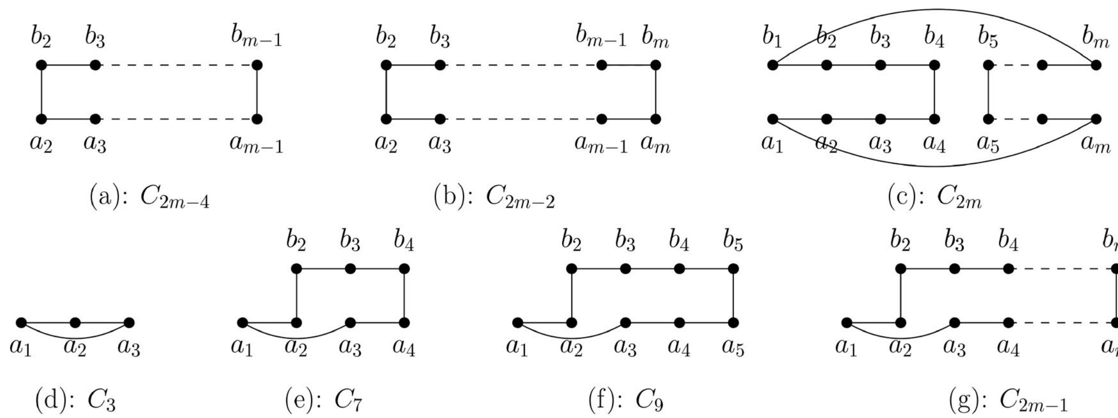


Figure 8. Cycles in W.

We now prove Theorem 1.1 which is restated below for convenience.

Theorem 3.1. *Let $n \geq 2$ and l be integers such that l is even and $4 \leq l \leq 2^n$. Then there exists a 3-regular, 3-connected and nearly pancyclic subgraph of the augmented cube AQ_n on l vertices.*

Proof. As seen above, the result holds for $l=4$ and $l=6$. A 3-regular subgraph of AQ_3 on 8 vertices is shown in Figure 7(a). Clearly, this graph is pancyclic. Further deletion of any two edges from it leaves a connected graph. Hence it is 3-edge connected and so, it is 3-connected. Thus the result holds for $l=8$.

Suppose $10 \leq l \leq 2^n$. Then $l=2m$ for some integer m with $5 \leq m \leq 2^{n-1}$. Write $AQ_n = AQ^0_{n-1} \cup AQ^1_{n-1} \cup E_h \cup E_c$. By Lemma 2.9, AQ^0_{n-1} contains a cycle $C = \langle a_1, a_2, \ldots, a_m, a_1 \rangle$, on m vertices such that $e = \langle a_1, a_3 \rangle$ is a chord of C. Let $C' = \langle b_1, b_2, \ldots, b_m, b_1 \rangle$ be the corresponding cycle in AQ^1_{n-1}. Then a_i is adjacent to b_i for each i, $i = 1, 2, \ldots, m$. Let $W = C \cup C' \cup D$, where $D = \{\langle a_2, b_2 \rangle, \langle a_4, b_4 \rangle, \langle a_5, b_5 \rangle, \ldots, \langle a_m, b_m \rangle\}$. Then W is a 3-regular subgraph of AQ_n on $2m$ vertices as shown in Figure 7(b).

Claim 1. W is nearly pancyclic.

We first show that W is bipancyclic. Note that $W - \{a_1, a_2, a_3, b_1, b_2, b_3\}$ is a ladder on $2m-6$ vertices and so, by Lemma 2.5, it is bipancyclic. Hence W contains a cycle of every even length from 4 to $2m-6$. Cycles C_{2m-4}, C_{2m-2} and C_{2m} in W of lengths $2m-4, 2m-2$ and $2m$, respectively are shown in Figure 8(a), (b) and (c).

We now prove the existence of odd length cycles in W. The cycles C_3 of length 3 and C_7 of length 7 are shown in Figure 8(d) and 8(e), respectively. Replacing the edge $\langle a_4, b_4 \rangle$ of C_7 by the path $\langle a_4, a_5, b_5, b_4 \rangle$ of length 3 gives a 9-cycle C_9 in W. Continuing this process of replacing an edge of a cycle by a path of length three, we get cycles of all odd length from 7 to $2m-1$. Note that W does not contain a 5-cycle. Thus W is nearly pancyclic.

Claim 2. W is 3-connected.

Let $S \subseteq V(W)$ with $|S| = 2$ vertices. We prove that $W - S$ is connected. There are at least $m - 4 \geq 1$ edges between C and C' in $W - S$. Suppose S intersects both $V(C)$ and $V(C')$. Let $S_1 = V(C) \cap S$ and let $S_2 = V(C') \cap S$. Then both $C - S_1$ and $C' - S_2$ are connected, giving $W - S$ connected. Suppose $S \subseteq V(C')$. Let G_1 be the subgraph of W induced by $V(C')$. Suppose $G_1 - S$ has a component D with no neighbour in C. Then $V(D) \subseteq \{b_1, b_3\}$. The degree of each of b_1 and b_3 is 3 in G_1, the minimum degree of D is at least one as $|S| = 2$. Hence D consists of only one edge $\langle b_1, b_3 \rangle$. This shows that S contains the neighbours b_2, b_4 and b_m of b_1 and b_3 in G_1. Hence $|S| \geq 3$ as $m \geq 5$, a contradiction. Thus every component of $G_1 - S$ has a neighbour in the connected graph C and hence, $W - S$ is connected. Similarly, $W - S$ is connected if $S \subseteq V(C)$. Hence W is 3-connected. This proves the claim.

Thus W is a 3-regular, 3-connected and nearly pancyclic subgraph of AQ_n on l vertices. This completes the proof. □

4. Existence of 4-regular subgraphs of AQ_n

In this section, we prove Theorem 1.2 for $28 \leq l \leq 2^n$ by constructing a 4-regular, 4-connected and pancyclic subgraphs of AQ_n on l vertices. We use the following notation in our proofs.

Notation: We write $AQ_n = AQ^0_{n-1} \cup AQ^1_{n-1} \cup E_h \cup E_c$. Then write $AQ^0_{n-1} = AQ^{00}_{n-2} \cup AQ^{01}_{n-2} \cup E'_h \cup E'_c$ and $AQ^1_{n-1} = AQ^{10}_{n-2} \cup AQ^{11}_{n-2} \cup E''_h \cup E''_c$. For convenience, we denote AQ^{00}_{n-2}, AQ^{10}_{n-2}, AQ^{11}_{n-2} and AQ^{01}_{n-2} by AQ^0_{n-2}, AQ^1_{n-2}, AQ^2_{n-2} and AQ^3_{n-2}, respectively. Then $AQ^0_{n-1} = AQ^0_{n-2} \cup AQ^3_{n-2} \cup E'_h \cup E'_c$ and $AQ^1_{n-1} = AQ^1_{n-2} \cup AQ^2_{n-2} \cup E''_h \cup E''_c$. Note that the subgraph $AQ^0_{n-2} \cup AQ^1_{n-2} \cup AQ^2_{n-2} \cup AQ^3_{n-2} \cup E_h \cup E'_h \cup E''_h$ of AQ_n is isomorphic to $AQ_{n-2} \square C_4$ where C_4 is a cycle of length 4.

We select a copy of a cycle from each of the four copies of AQ_{n-2} in AQ_n and use them to construct a 4-regular subgraph (Figure 9).

Proposition 4.1. *Let n and l be integers such that $n \geq 5$ and $28 \leq l \leq 2^n$. Then there exists a 4-regular, 4-connected and pancyclic subgraph H of AQ_n on l vertices.*

Proof. As $28 \leq l \leq 2^n$, we have $l = 4m$ with $7 \leq m \leq 2^{n-2}$, or $l = 4m+1, 4m+2$ or $4m+3$ with $7 \leq m \leq 2^{n-2}-1$. In each of these four cases, we construct a 4-regular subgraph of AQ_n.

Case (i). $l = 4m$

As in the above notation, express AQ_n into four copies of AQ_{n-2}. Then $AQ_{n-2} \square C_4$ is a spanning subgraph of AQ_n.

As $7 \leq m \leq 2^{n-2}$, by Lemma 2.9, there exists a cycle $Z^0 = \langle x_1, x_2, ..., x_m, x_1\rangle$ in AQ^0_{n-2} on m vertices with a chord $\langle x_1, x_3\rangle$. Let $Z^1 = \langle y_1, y_2, ..., y_m, y_1\rangle$; $Z^2 = \langle z_1, z_2, ..., z_m, z_1\rangle$; $Z^3 = \langle w_1, w_2, ..., w_m, w_1\rangle$ be the corresponding cycles in AQ^1_{n-2}, AQ^2_{n-2} and AQ^3_{n-2}, respectively. Then $\langle x_i, y_i\rangle, \langle y_i, z_i\rangle, \langle z_i, w_i\rangle$ and $\langle w_i, x_i\rangle$ are hypercube edges in AQ_n. Let

$$H_0 = Z^0 \cup Z^1 \cup Z^2 \cup Z^3 \cup \{\langle x_i, y_i, z_i, w_i, x_i\rangle : i = 1, 2, ..., m\}.$$

Then H_0 is isomorphic to $Z^0 \square C_4$, where C_4 is a 4-cycle. Hence H_0 is a 4-regular, 4-connected and bipancyclic subgraph of AQ_n on $4m$ vertices. We modify H_0 to get a pancyclic subgraph H_1 as follows. Let

$$H_1 = H_0 - \{\langle x_1, w_1\rangle, \langle x_3, w_3\rangle\} \cup \{\langle x_1, x_3\rangle, \langle w_1, w_3\rangle\}.$$

(See Figure 10(a).) Clearly, H_1 is a 4-regular subgraph of AQ_n with $4m$ vertices.

We now prove that H_1 is 4-connected. Let L and R be the subgraphs of H_1 induced by $V(Z^0 \cup Z^3)$ and by $V(Z^1 \cup Z^2)$, respectively. Since R is isomorphic to $Z^2 \square K_2$, it is 3-connected. Also, as in proof of Theorem 1.1, L is 3-connected. There is a perfect matching in H_1 joining L and R. Hence, by Lemma 2.10, H_1 is 4-connetced.

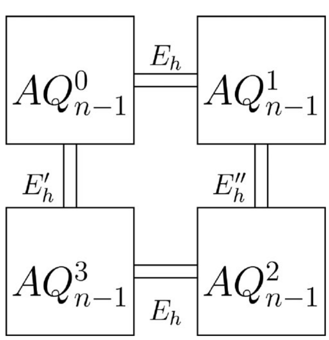


Figure 9. $AQ_{n-2} \square C_4$.

To prove pancyclicity of H_1, let $P = (Z^0 - \langle x_1, x_m\rangle) \cup (Z^1 - \langle w_1, w_m\rangle) \cup \{\langle x_m, d_m\rangle\}$ be a path on $2m$ vertices in L and let Q be the corresponding path in R. Then $W = P \cup Q \cup M$ is a ladder on $4m$ vertices, where $M = \{\langle x_i, y_i\rangle, \langle w_i, z_i\rangle : i = 1, 2, ..., m\}$. Let C_3 be the triangle $\langle x_1, x_2, x_3, x_1\rangle$ in H_1. Then, by Lemma 2.7, $W \cup C_3$ is a pancyclic graph. Since $W \cup C_3$ is a spanning subgraph of H_1, the graph H_1 is also pancyclic.

Thus we have constructed a 4-regular, 4-connected and pancyclic subgraph in Case (i).

Suppose $l = 4m+1, 4m+2$ or $4m+3$ with $7 \leq m \leq 2^{n-2}-1$. As in Case (i), express AQ_n into four copies AQ^0_{n-2}, AQ^1_{n-2}, AQ^2_{n-2} and AQ^3_{n-2} of AQ_{n-2}. Since $7 \leq m \leq 2^{n-2}-1$, by Lemma 2.9, there exists a cycle $C^0 = \langle a_1, a_2, ..., a_m, a_1\rangle$ in AQ^0_{n-2} of length m which has two chords $\langle a_1, a_3\rangle$ and $\langle a_2, a_k\rangle$ for some $4 < k < m$ and there is a vertex $v_0 \in V(AQ^0_{n-2}) - V(C^0)$ with four neighbours a_1, a_2, a_3 and a_k on C^0. Let $C^1 = \langle b_1, b_2, ..., b_m, b_1\rangle$, $C^2 = \langle c_1, c_2, ..., c_m, c_1\rangle$ and $C^3 = \langle d_1, d_2, ..., d_m, d_1\rangle$ be the corresponding cycles in AQ^1_{n-2}, AQ^2_{n-2} and AQ^3_{n-2}, respectively. Let v_i be the vertex of AQ^i_{n-2} corresponding to v_0 for $i = 1, 2$.

Let

$$H = C^0 \cup C^1 \cup C^2 \cup C^3 \cup \{\langle a_i, b_i, c_i, d_i, a_i\rangle : \; i = 1, 2, ..., m\}.$$

Then H is a 4-regular, 4-connected and bipancyclic subgraph of AQ_n on $4m$ vertices. We use this graph to construct 4-regular subgraphs on $4m+1, 4m+2$ and $4m+3$ vertices.

Case (ii). $l = 4m+1$

Let M_1 be the subgraph of AQ_n with vertex set $\{v_0, a_1, a_2, a_3, a_k\}$, with $4 < k < m$ and edge set $\{\langle v_0, a_1\rangle, \langle v_0, a_2\rangle, \langle v_0, a_3\rangle, \langle v_0, a_k\rangle\}$. Define

$$H_2 = H \cup M_1 - \{\langle a_1, d_1\rangle, \langle a_2, d_2\rangle, \langle a_3, d_3\rangle, \langle a_k, d_k\rangle\} \cup \{\langle d_1, d_3\rangle, \langle d_2, d_k\rangle\}.$$

(See Figure 10(b).) Then H_2 is a 4-regular subgraph of AQ_n on $4m+1 = l$ vertices.

Case (iii). $l = 4m+2$

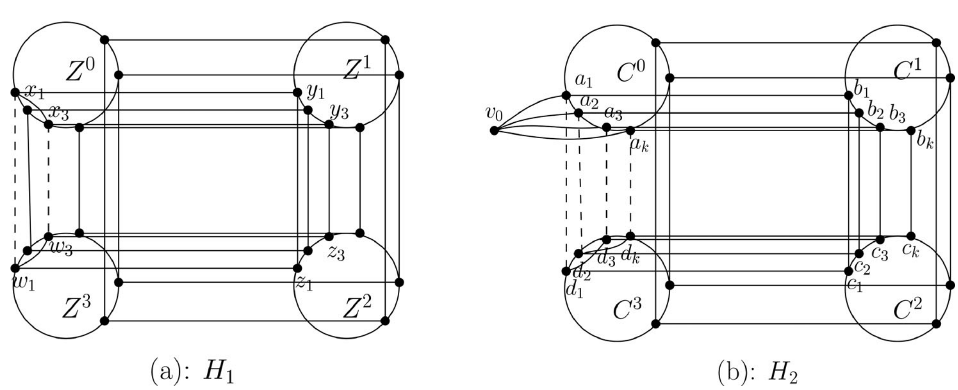
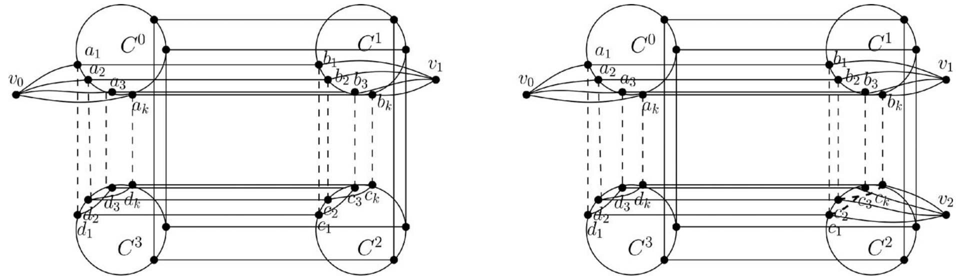
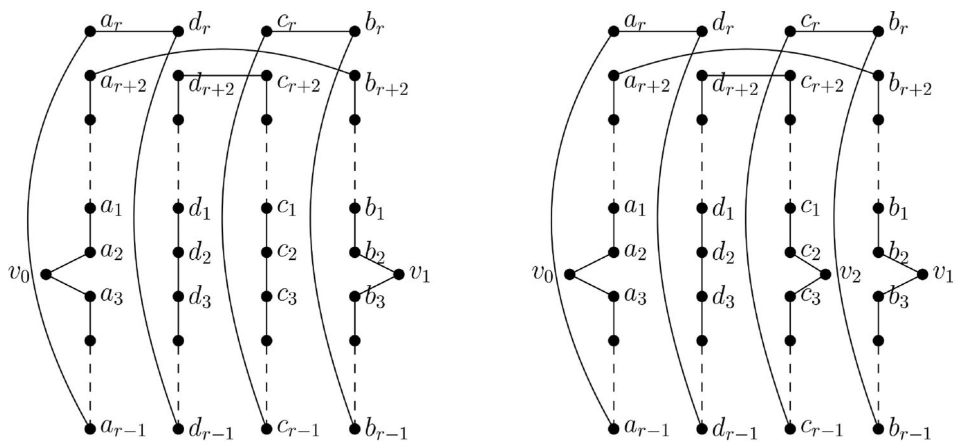


Figure 10. 4-regular subgraphs of AQ_n.



(a): H_3 (b): H_4

Figure 11. 4-regular subgraphs of AQ_n.



(a): $l = 4m + 2$ (b): $l = 4m + 3$

Figure 12. Cycles in H_3 and H_4.

Consider the subgraph H_2 of Case (ii). Let M_2 be the graph with vertex set $\{v_1, b_1, b_2, b_3, b_k\}$ and edge set $\{\langle v_1, b_1\rangle, \langle v_1, b_2\rangle, \langle v_1, b_3\rangle, \langle v_1, b_k\rangle\}$. Let

$$H_3 = H_2 \cup M_2 - \{\langle b_1, c_1\rangle, \langle b_2, c_2\rangle, \langle b_3, c_3\rangle, \langle b_k, c_k\rangle\} \cup \{\langle c_1, c_3\rangle, \langle c_2, c_k\rangle\}.$$

(See Figure 11(a).) Clearly, H_3 is a 4-regular subgraph of AQ_n on $4m + 2$ vertices.

Case (iv). $l = 4m + 3$

We construct a graph H_4 on l vertices from the graph H_3 of Case (iii) and the graph M_3 whose vertex set $\{v_2, c_1, c_2, c_3, c_k\}$ and edge set $\{\langle v_2, c_1\rangle, \langle v_2, c_2\rangle, \langle v_2, c_3\rangle, \langle v_2, c_k\rangle\}$. Let

$$H_4 = H_3 \cup M_3 - \{\langle c_1, c_3\rangle, \langle c_2, c_k\rangle\}.$$

(See Figure 11(b).) Clearly, H_4 is a 4-regular subgraph of AQ_n on $4m + 3$ vertices.

We now prove that the graphs H_2, H_3 and H_4 constructed above are pancyclic and 4-connected.

Claim 1. The graphs H_2, H_3 and H_4 are pancyclic.

Let $i \in \{2, 3, 4\}$. Since $m \geq 7$, there exists a vertex a_r of C^0 such that $a_r \notin \{a_1, a_2, a_3, a_k\}$. Hence $\langle a_r, d_r\rangle, \langle b_r, c_r\rangle \in E(H_i)$. The graph H_i contains the ladder $L = P \cup Q \cup M$, where

$$P = (C^0 - \langle a_r, a_{r+1}\rangle) \cup (C^3 - \langle d_r, d_{r+1}\rangle) \cup \{\langle a_r, d_r\rangle\},$$
$$Q = (C^1 - \langle b_r, b_{r+1}\rangle) \cup (C^2 - \langle c_r, c_{r+1}\rangle) \cup \{\langle b_r, c_r\rangle\}$$
$$M = \{\langle a_i, b_i\rangle, \langle c_i, d_i\rangle : \ i = 1, 2, ..., m\}.$$

Then $L \cup C_3$ is a subgraph of H_i on $4m + 1$ vertices, where $C_3 = \langle a_1, v_0, a_2, a_1\rangle$. By Lemma 2.7, $L \cup C_3$ is pancyclic. Hence H_i contain a cycle of every length from 3 to $4m + 1$. A cycle C_{4m+2} of length $4m + 2$ in H_3 and so in H_4 is shown in Figure 12(a) while a cycle in H_4 of length $4m + 3$ is shown in Figure 12(b). Thus H_2, H_3 and H_3 are pancyclic.

Claim 2. The graphs H_2, H_3 and H_4 are 4-connected.

For $i \in \{2, 3, 4\}$, let

$$L_i = H_i \cap (AQ^0_{n-2} \cup AQ^3_{n-2} \cup E_h) \text{ and}$$
$$R_i = H_i \cap (AQ^1_{n-2} \cup AQ^2_{n-2} \cup E_h).$$

By Lemma 2.11, the graphs L_2, L_3, L_4 and R_3 are 3-connected. Since R_2 is isomorphic to $C_m \square K_2$, it is also 3-connected by Lemma 2.1. Further, as in proof of Lemma 2.11, the graph R_4 is 3-connected.

Let $S \subseteq V(H_i)$ with $|S| = 3$. We prove that $H_i - S$ is connected. In $H_i - S$, there are at least $2m - |S| \geq 2m - 3 \geq 10$ edges between L_i and R_i. Suppose S intersects with both $V(L_i)$ and $V(R_i)$. Let $S_1 = S \cap V(L_i)$ and $S_2 = S \cap V(R_i)$. Then both $L_i - S_1$ and $R_i - S_2$ are connected and are joined

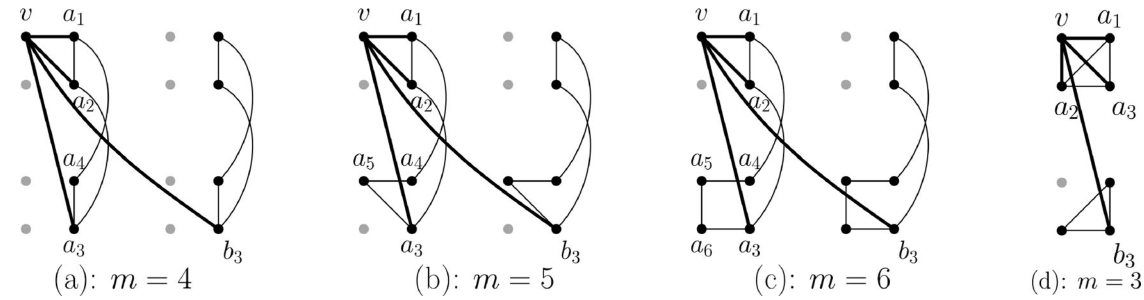


Figure 13. Cycles in AQ_4.

to each other and so, $H_i - S$ is connected. Suppose $S \subseteq V(L_i)$. As the degree of v_0 in L_i is 4, it is not an isolated vertex in $L_i - S$. Therefore, every component of $L_i - S$ has a neighbour in the connected graph R_i. Therefore $H_i - S$ is connected. Similarly, every component of $R_i - S$ has a neighbour in the connected graph L_i if $S \subseteq V(R_i)$ and hence, $H_i - S$ is connected. Thus H_i is 4-connected. This proves the claim.

Thus, in each case, we have constructed a 4-regular, 4-connected and pancyclic subgraph of AQ_n on l vertices. □

5. 4-Regular subgraphs of smaller size

We complete the proof of Theorem 1.2 in this section. In the previous section, we constructed 4-regular subgraphs of AQ_n on l vertices for $28 \le l \le 2^n$. Now in this section, we construct such subgraphs of AQ_n on l vertices for $7 \le l \le 27$. We obtain the following lemmas to construct 4-regular subgraphs in AQ_4 and AQ_5.

Lemma 5.1. *Write $AQ_4 = AQ_3^0 \cup AQ_3^1 \cup E_h \cup E_c$. Suppose $m \in \{4,5,6\}$. Then there exists an m-cycle C containing a path $\langle a_1, a_2, a_3\rangle$ in AQ_3^0 and a vertex $v \in V(AQ_3^0) - V(C)$ such that*

(i) *v is adjacent to a_1, a_2, a_3;*
(ii) *v is adjacent to the vertex b_3 of AQ_3^1 corresponding to a_3.*

Proof. In AQ_4, the vertex set of AQ_3^0 is $\{0000, 0001, 0011, 0111, 0101, 0100, 0110, 0010\}$. Let $v = 0000$, $a_1 = 0001$, $a_2 = 0011$, $a_3 = 0111$, $a_4 = 0101$, $a_5 = 0100$ and $a_6 = 0110$. Then $\langle a_1, a_2, a_3\rangle$ is a path in AQ_3^0 and v is adjacent to a_1, a_2, a_3. The vertex v is also adjacent to the vertex $b_3 = 1111$ of AQ_3^1 corresponding to a_3. Observe that $\langle a_1, a_2, a_3, a_4, a_1\rangle$, $\langle a_1, a_2, a_3, a_5, a_4, a_1\rangle$ and $\langle a_1, a_2, a_3, a_6, a_5, a_4, a_1\rangle$ are cycles in AQ_3^0 of length 4, 5 and 6, respectively, and avoids the vertex v. (See Figures 13(a), (b), (c).) Then C contains the path $\langle a_1, a_2, a_3\rangle$ and it is a required cycle. This completes the proof. □

We get a similar result for AQ_3 and $m = 3$. (See Figure 13(d).)

Lemma 5.2. *Write $AQ_3 = AQ_2^0 \cup AQ_2^0 \cup E_h \cup E_c$. Then there exists a triangle $C = \langle a_1, a_2, a_3, a_1\rangle$ in AQ_2^0 and $v \in V(AQ_2^0) - V(C)$ such that*

(i) *v is adjacent to a_1, a_2, a_3;*
(ii) *v is adjacent to the vertex b_3 of AQ_2^1 corresponding to a_3.*

Lemma 5.3. *Let l be an integer with $13 \le l \le 27$. Then there exists a 4-regular, 4-connected and pancyclic subgraph on l vertices (i) in AQ_4 if $l \le 16$ (ii) in AQ_5 if $l > 16$.*

Proof. If l is a multiple of 4, then, as in Case (i) of Proposition 4.1, there exists a 4-regular, 4-connected and pancyclic subgraph of AQ_4 with l vertices. Suppose l is not a multiple of 4. Then $l = 4m + 1$, $4m + 2$ or $4m + 3$ for some integer m such that $3 \le m \le 6$.

Let $n = 4$ if $m = 3$, and $n = 5$ if $4 \le m \le 6$. As in the notation of Section 4, write $AQ_n = AQ_{n-1}^0 \cup AQ_{n-1}^0 \cup E_h \cup E_c$ with $AQ_{n-1}^0 = AQ_{n-2}^0 \cup AQ_{n-2}^3 \cup E_h' \cup E_c'$ and $AQ_{n-1}^1 = AQ_{n-2}^1 \cup AQ_{n-2}^2 \cup E_h'' \cup E_c''$. By Lemmas 5.1 and 5.2, there exists a cycle C^0 of length m in AQ_{n-1}^0 containing a path $\langle a_1, a_2, a_3\rangle$ and a vertex $v_0 \in V(AQ_{n-2}^0) - V(C^0)$ such that v_0 is adjacent to a_1, a_2, a_3 and d_3, where d_3 is the vertex of AQ_{n-2}^3 corresponding to a_3. Let C^1, C^2 and C^3 be the cycles in AQ_{n-2}^1, AQ_{n-2}^2 and AQ_{n-2}^3 respectively, corresponding to C^0. Similarly, let b_i, c_i, d_i be the vertices of AQ_{n-2}^1, AQ_{n-2}^2, AQ_{n-2}^3, respectively, corresponding to a_i for $i = 1, 2, 3$. Also, let the vertices of AQ_{n-2}^1 and AQ_{n-2}^2 corresponding to v_0 be v_1 and v_2, respectively. Let

$$H = C^0 \cup C^1 \cup C^2 \cup C^3 \cup E_h \cup E_h' \cup E_h''.$$

Then H is isomorphic to $C^0 \square C_4$, where C_4 is a cycle of length 4. Hence H is a 4-regular, 4-connected and bipancyclic subgraph of AQ_n on $4m$ vertices.

Case (i). $l = 4m + 1$ with $3 \le m \le 6$.

We construct a 4-regular graph H_1 on $4m + 1$ vertices by adding a vertex to the above graph H. Let M be the subgraph of AQ_n with vertex set $\{v_0, a_1, a_2, a_3, d_3\}$ and edge set $\{\langle v_0, a_1\rangle, \langle v_0, a_2\rangle, \langle v_0, a_3\rangle, \langle v_0, d_3\rangle\}$. Let

$$H_1 = H \cup M - \{\langle a_1, a_2\rangle, \langle a_3, d_3\rangle\}.$$

(See Figure 14(a).) Then H_1 is a 4-regular subgraph of AQ_n with l vertices.

Case (ii). $l = 4m + 2$ with $3 \le m \le 6$.

We construct a 4-regular graph by adding a vertex to the graph H_1. Suppose N is the subgraph of AQ_n with vertex set $\{v_1, b_1, b_2, b_3, c_3\}$ and edge set $\{\langle v_1, b_1\rangle, \langle v_1, b_2\rangle, \langle v_1, b_3\rangle, \langle v_1, c_3\rangle\}$. Define

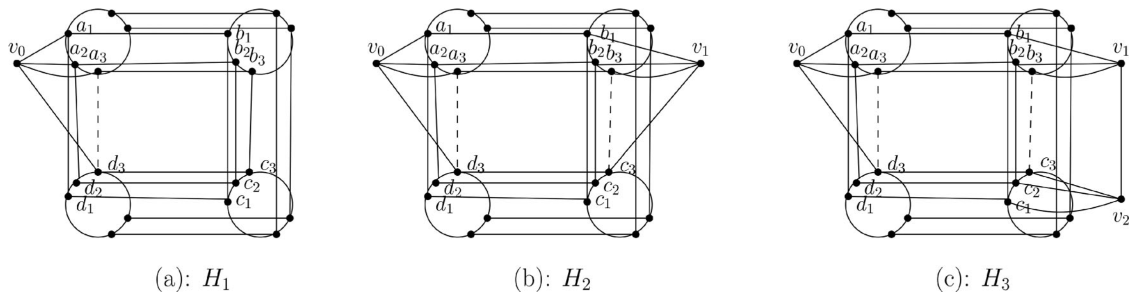


Figure 14. 4-regular subgraphs of AQ_5.

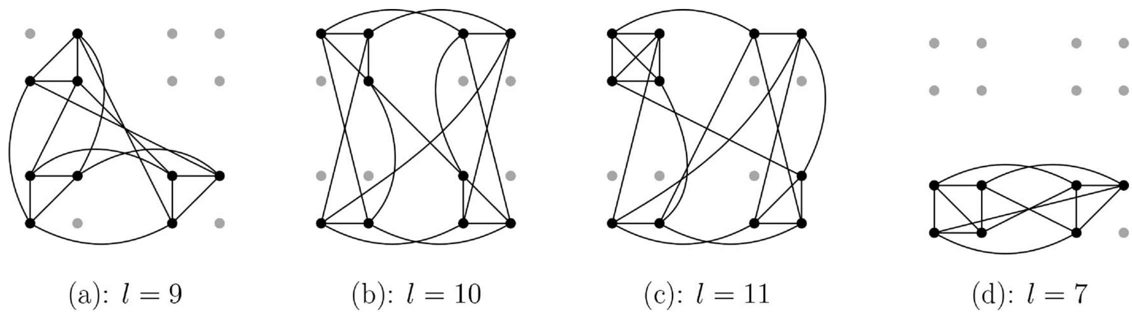


Figure 15. 4-regular subgraphs on l vertices.

$$H_2 = H_1 \cup N - \{\langle b_1, b_2\rangle, \langle b_3, c_3\rangle\}.$$

(See Figure 14(b).) Then the subgraph H_2 of AQ_n is 4-regular with l vertices.

Case (iii). $l = 4m + 3$ with $3 \le m \le 6$.

Let L be the graph with vertex set $\{v_2, c_1, c_2, c_3\}$ and edge set $\{\langle v_2, c_1\rangle, \langle v_2, c_2\rangle, \langle v_2, c_3\rangle\}$ and let

$$H_3 = (H_2 \cup L \cup \{\langle v_1, v_2\rangle\}) - \{\langle v_1, c_3\rangle\}.$$

(See Figure 14(c).) Clearly, the subgraph H_3 of AQ_n is 4-regular with l vertices.

Claim 1. The graphs H_1, H_2 and H_3 are pancyclic.

Let $i \in \{1, 2, 3\}$. We prove that H_i is pancyclic. Let

$$P = (C^0 - \langle a_1, a_2\rangle) \cup (C^3 - \langle d_1, d_2\rangle) \cup \{\langle a_1, d_1\rangle\}$$

and let

$$Q = (C^1 - \langle c_1, c_2\rangle) \cup (C^2 - \langle b_1, b_2\rangle) \cup \{\langle c_1, b_1\rangle\}.$$

Then P and Q are vertex-disjoint paths in AQ_n of length $2m$ each. The perfect matching between them gives a ladder L on $4m$ vertices. Clearly, L contains one edge of the triangle $\langle v_0, a_2, a_3, v_0\rangle$ in H_i. Hence, by Lemma 2.7, H_i is pancyclic.

Claim 2. The graphs H_1, H_2 and H_3 are 4-connected.

For $i \in \{1, 2, 3\}$. We prove that the graph H_i is 4-connected.

$$L_i = H_i \cap (AQ^0_{n-2} \cup AQ^3_{n-2} \cup E'_h) \text{ and}$$
$$R_i = H_i \cap (AQ^1_{n-2} \cup AQ^2_{n-2} \cup E''_h).$$

By modifying Lemma 2.11, the graphs L_1, L_2 and L_3 and R_2 are 3-connected. Since R_1 is isomorphic to $C_m \square K_2$, it is also 3-connected by Lemma 2.1. Further, as in proof of Lemma 2.11, the graph R_3 is 3-connected.

As in Claim 2 of Proposition 4.1, the H_i is 4-connected. This proves the claim.

Hence we get 4-regular, 4-connected and pancyclic subgraph of augmented cube on l vertices whenever $13 \le l \le 27$. □

Lemma 5.4. *For an integer l with $8 \le l \le 11$, there exists a 4-regular, 4-connected and pancyclic subgraph of AQ_4 on l vertices.*

Proof. As in Case (i) of Proposition 4.1, there exists a 4-regular, 4-connected and pancyclic subgraph of AQ_4 with l vertices if l is a multiple of 4. Suppose l is not multiple of 4. Then $l \in \{9, 10, 11\}$. Figure 15(a), (b) and (c) give 4-regular subgraphs of AQ_4 on 9, 10 and 11 vertices, respectively. It follows from Lemma 2.7 that these graphs are pancyclic. Also, it is easy to verify that they are 4-connected. □

We now prove that no augmented cube contains a 4-regular subgraph with number of vertices less than 7, however, there is unique 4-regular subgraph of an augmented cube on 7 vertices.

Proposition 5.5. *For any integer l with $1 \le l \le 6$, there does not exist a 4-regular subgraph with l vertices in an augmented cube.*

Proof. Assume that there is an augmented cube AQ_n containing a 4-regular subgraph H with l vertices. Choose smallest n such that AQ_n contains H. Clearly, $n \ge 3$, and $l \ge 5$ as H is simple. Now write $AQ_n = AQ^0_{n-1} \cup AQ^1_{n-1} \cup E_h \cup E_c$. Then H is not a subgraph of AQ^i_{n-1} for $i = 0, 1$. Let $H_i = H \cap AQ^i_{n-1}$ for $i = 0, 1$. Then $H_i \ne \emptyset$ and so its minimum degree is at least two. Therefore H_i contains a cycle and thus has at least 3 vertices for $i = 0, 1$. This shows that $l = 6$ and both H_0 and H_1 are triangles. Hence each vertex of H_0 has two neighbours in H_1 and vice versa.

Let $H_0 = \langle 0a, 0b, 0c, 0a \rangle$. Then $\langle 0a, 1a \rangle$, $\langle 0b, 1b \rangle$, $\langle 0c, 1c \rangle \in E_h$. Hence $H_1 = \langle 1a, 1b, 1c, 1a \rangle$. Further, each vertex of H_0 has one more neighbor in H_1. Without loss of generality, we may assume that $\langle 0a, 1b \rangle$, $\langle 0b, 1c \rangle$ and $\langle 0c, 1a \rangle$ are edges in H. Therefore these three edges belong to E_c. This shows that $\overline{0a} = 1b$, $\overline{0c} = 1a$ and $\overline{0b} = 1c$. Therefore $\bar{a} = b$ and $\bar{b} = c$ giving $a = c$. This is a contradiction. Hence the result holds. □

Lemma 5.6. *Every 4-regular subgraph of AQ_n with 7 vertices is isomorphic to the graph shown in Figure 15(d).*

Proof. Let n be the smallest integer such that the augmented cube AQ_n contains a 4-regular subgraph H with 7 vertices. Write $AQ_n = AQ^0_{n-1} \cup AQ^1_{n-1} \cup E_h \cup E_c$ and let $H_i = H \cap AQ^i_{n-1}$ for $i = 0, 1$. Then the minimum degree of H_i is at least two and therefore H_i contains a cycle. Hence we may assume that H_1 is a triangle and H_0 contains a 4-cycle. Consequently, every vertex of H_1 has two neighbours in H_0. Thus there are exactly six edges between H_0 and H_1. It follows that H_1 is a 4-cycle with one chord. This implies that H is isomorphic to the graph of Figure 15(d). □

Now, we complete the proof of Theorem 1.2 formally. We restate this theorem here for convenience.

Theorem 5.7. *Let $n \geq 4$ and l be integers. Then the augmented cube AQ_n contains a 4-regular, 4-connected and pancyclic subgraph with l vertices if and only if $8 \leq l \leq 2^n$.*

Proof. Suppose AQ_n contains a 4-regular subgraph on l vertices. Then, by Proposition 5.5, $l \geq 7$. Also, by Lemma 5.6, every 4-regular subgraph of AQ_n on 7 vertices is isomorphic to the graph shown in Figure 15(d). However, this graph is pancyclic, 3-connected but not 4-connected. Hence $8 \leq l \leq |V(AQ_n)| = 2^n$. Converse follows from Proposition 4.1. □

Funding

The first author acknowledges Council of Scientific and Industrial Research, India, for the award of Junior Research Fellowship, file number 09/137(0572)/2017-EMR-I. The second author is supported by DST-SERB, Government of India through the project MTR/2018/000447.

References

[1] Borse, Y. M, Saraf, J. B. Existence of 3-regular subgraphs in Cartesian product of cycles. *AKCE Int. J. Graphs Comb.* 16 (3):332–342. doi:10.1016/j.akcej.2018.07.001.
[2] Borse, Y. M, Shaikh, S. R. (2015). On 3-regular bipancyclic subgraphs of hypercubes. *Int. J. Comb.* 2015:1–4.
[3] Borse, Y. M, Shaikh, S. R. (2017). On 4-regular 4-connected bipancyclic subgraphs of hypercubes. *Discrete Math. Algorithm. Appl.* 09(03):1750032.
[4] Choudum, S. A, Sunitha, V. (2002). Augmented cubes. *Networks* 40(2):71–84.
[5] Choudum, S. A, Sunitha, V. (2008). Automorphisms of augmented cubes. *Int. J. Comput. Math.* 85(11):1621–1627.
[6] Hsieh, S.-Y, Shiu, J.-Y. (2007). Cycle embedding of augmented cubes. *Appl. Math. Comput* 191(2):314–319.
[7] Leighton, F. T. (1992). I*ntroduction to Parallel Algorithms and Architectures: Arrays Trees, Hypercubes*. San Mateo, California: M. Kaufmann Publishers Inc.,
[8] Liu, H, Liu, Q. Edge-pancyclicity and edge-hamiltonian of augmented hypercube. In: *Proc Fourth International Conference on Computational and Information Sciences*, Chongqing, China, 2012,
[9] Lu, M., Zhang, S, Yang, W. (2018). Regular connected bipancyclic spanning subgraphs of torus networks. *Parallel Process. Lett.* 28(04):1850013. Article 1850013, pages.
[10] Ma, M., Liu, G, Xu, J.-M. (2008). The super connectivity of augmented cubes. *Inf. Process. Lett* 106(2):59–63.
[11] Ma, M., Song, Y, Xu, J.-M. (2013). Fault Tolerance of Augmented Cubes. *AKCE Int. J. Graphs Comb.* 101:37–55.
[12] Mane, S. A, Waphare, B. N. (2011). Regular connected bipancyclic spanning subgraphs of hypercubes. *Comput. Math. Appl.* 62(9):3551–3554.
[13] Miao, L, Yang, W. (2018). On 3-regular subgraphs in Cartesian products of paths. *J. Inter. Net.* 18(02n03):1850009–1850012.
[14] Ramras, M. (1999). Regular subgraphs of hypercubes. *Ars Combin* 52:21–32.
[15] Wang, H.-L., Wang, J.-W, Xu, J.-M. (2009). Fault-tolerant panconnectivity of augmented cubes. *Front. Math. China* 4(4):697–719.
[16] Xu, M., Xu, J.-M, Hou, X. (2005). Fault diameter of Cartesian product graphs. *Inf. Process. Lett.* 93(5):245–248.