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ABSTRACT

The crossing number of a graph G, $Cr(G)$ is the minimum number of edge crossings overall good drawings of G. Among the well-known four standard graph products namely Cartesian product, Kronecker product, strong product and lexicographic product, the one that is most difficult to deal with is the Kronecker product. P.K. Jha and S. Devishetty have analyzed the upper bounds for crossing number of Kronecker product of two cycles in, "Orthogonal Drawings and the Crossing Numbers of the Kronecker product of two cycles", *J. Parallel Distrib. Comput.* 72 (2012), 195–204. For any graph G except $K_{1,1,2}$ and K_4 of order at most four, the graph $G \wedge P_n$ is planar. In this paper, we establish the crossing number of Kronecker product of a complete tripartite graph $K_{1,1,3}$ with path and as a corollary, we show that its rectilinear crossing number is same as its crossing number. Also, we give the open problems on the crossing number of above mentioned graphs.

KEYWORDS

Drawing; crossing number; Kronecker product; path; rectilinear crossing number

1. Introduction

A drawing ϕ of a (undirected simple) graph $G=(V,E)$ is a mapping f that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. In addition, a drawing, ϕ is said to be good if: (1) no edge crosses itself, (2) adjacent edges do not cross, (3) no three edges have an interior point in common, (4) if two edges share an interior point p, then they cross at p, and (5) any two edges of a drawing cross atmost once (common interior point). The total number of crossings in ϕ is denoted by $Cr_\phi(E)$ or $Cr_\phi(G)$. A good drawing ϕ of G is said to be optimal if it exhibits the least possible number of crossings, that is $Cr_\phi(E)=Cr(G)$. The crossing number $Cr(G)$ of a graph G is the number of crossings in any optimal drawing of G in the plane. The drawings considered in this paper are all good.

The crossing number of the complete bipartite graph $K_{m,n}$ was given by Zarankiewicz [23] and it has been long conjectured that $Cr(K_{m,n})=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. Kleitman [13] established Zarankiewicz's conjecture for every n and $1\le m\le 6$. Woodall [22] verified that for $7\le m\le 8$ and $7\le n\le 10$, the crossing number of $K_{m,n}$ equals Zarankiewicz number $Z_{m,n}=\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$.

Let C_n be the cycle of length n, P_n be the path on n vertices, nK_1 denote the graph on n isolated vertices and S_n be the star isomorphic to $K_{1,n}$. Of all graph products, the Cartesian product has received maximum attention and there are several exact results on the crossing number of Cartesian product of graphs with paths, cycles, and stars. The Crossing number of $G\times P_n$ is known for all graphs G of order at most five, see [11, 14, 15, 17] and for several graphs G of order six is given in [4, 19]. The crossing number of $G\times C_n$ is established for all graphs G with at most four vertices in [1, 10] and for some graphs on five or six vertices [6, 14, 18]. The crossing number of $G\times S_n$ for all graphs G of orders three or four is determined in [11, 14, 15] and for some graphs of order five is given in [10, 16, 17, 20]. Jha and Devishetty have analyzed the upper bounds for crossing number of Kronecker product of two cycles in [12].

Definition 1.1. The Kronecker product $G_1\wedge G_2$ of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is said to be $G=(V,E)$ where $V=V_1\times V_2$ and for any two typical vertices $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ with $u_i\in V_1$ and $v_i\in V_2$, we have $w_1w_2\in E$ iff $u_1u_2\in E_1$ and $v_1v_2\in E_2$.

This product is variously known as direct product, Tensor product, cardinal product, cross product and graph conjunction. In this paper, we initiate the study of determining the crossing number for Kronecker product of graphs and establish the crossing number of Kronecker product of a complete tripartite graph $K_{1,1,3}$ and P_n in theoretical approach. We use a term "region" in nonplanar drawings, in such case, crossings are considered to be vertices of the "map".

Wagner [21] and Fary [7] proved independently that every graph that can be drawn in the plane without crossings can be so drawn in such a way that all the edges are straight lines. Following from this, Harary and Hill [9] defined the straight-line crossing number $\overline{Cr}(K_n)$ (later called the linear or rectilinear crossing number) to be the

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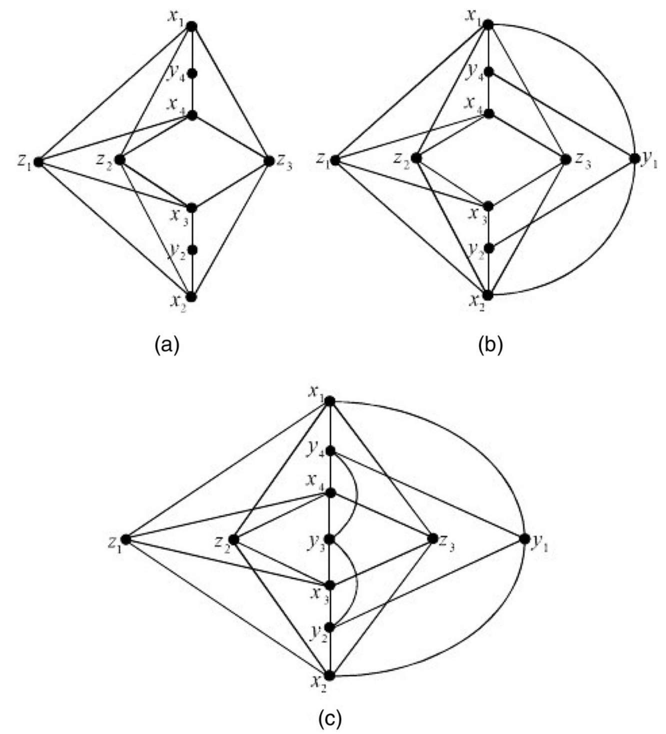

Figure 1. A good drawing of G, G^*, and G^{*^*}.

smallest possible number of crossings needed when the complete graph is drawn with straight lines in the plane.

Definition 1.2. The rectilinear crossing number of a graph G is the minimum number of crossings in a drawing of G in the plane with straight edges and nodes in general position.

Always $Cr(G) \leq \overline{Cr}(G)$. Extending the result of Wagner and Fary mentioned above, Bienstock and Dean [2, 3] showed that if $Cr(G) \leq 3$, then $Cr(G) = \overline{Cr}(G)$. With complete graphs K_n, these values are indeed equal for $n \leq 7$ and for $n = 9$, but for $n = 8$, we have $Cr(K_8) = 18$ (Guy [8]) and $\overline{Cr}(K_8) = 19$. For $n = 10$, $Cr(K_{10}) = 60$. Brodsky et al. [5] proved that $\overline{Cr}(K_{10}) = 62$. It is interesting to observe that the rectilinear crossing number of $K_{1,1,3} \wedge P_n$ is same as its crossing number.

2. Main results

Consider a graph $K_{1,1,3} \wedge P_n$ obtained as the Kronecker product of $K_{1,1,3}$ and P_n. Let $t_{i,j}$ represent a vertex (u_i, v_j) of $K_{1,1,3} \wedge P_n$ with $u_i \in V(K_{1,1,3})$ and $v_j \in V(P_n)$. From definition 1.1, the Kronecker product of $K_{1,1,3}$ and P_n is a simple connected graph $K_{1,1,3} \wedge P_n$ with a vertex set $V = \cup_{j=1}^{n} V_j$ and an edge set $E = \cup_{j=1}^{n-1} E_j$ where, $V_j = \{t_{1,j}, t_{2,j}, t_{3,j}, t_{4,j}, t_{5,j}\}$ and $E_j = \{t_{i,j}t_{2,j+1}, t_{i,j}t_{4,j+1} : i = 1,3,5\} \cup \{t_{2,j}t_{i,j+1} : i = 1,3,4,5\}$ $\cup \{t_{4,j}t_{i,j+1} : i = 1,2,3,5\}$. Then we have, $|V(K_{1,1,3} \wedge P_n)| = 5n$; $|E(K_{1,1,3} \wedge P_n)| = 14(n-1)$; $\delta(K_{1,1,3} \wedge P_n) = 2$ and $\Delta(K_{1,1,3} \wedge P_n) = 8$ such that, the vertices $t_{2,j}, t_{4,j} (j = 2, 3, \ldots, n-1)$ holds a maximum degree eight; the vertices $t_{i,1}, t_{i,n} (i = 1,3,5)$ has a minimum degree two and all other vertices receives a degree four. One can find that the vertices $t_{1,j}, t_{3,j}$ and $t_{5,j}$ are adjacent to only $t_{2,j\pm1}$ and $t_{4,j\pm1}$ in $K_{1,1,3} \wedge P_n$, for all $j = 2, 3, \ldots, n-1$. We write $E(t_{i,j})$ for the set of all edges incident to $t_{i,j}$. Let $H_j = E(t_{1,j}) \cup E(t_{3,j}) \cup E(t_{5,j})$ be such a subgraph of $K_{1,1,3} \wedge P_n$, for $2 \leq j \leq n-1$.

Let G be a graph homeomorphic to $3K_1 + 2K_2$, where $3K_1$ and $2K_2$ are induced on the vertices $\{z_1, z_2, z_3\}$ and $\{x_1, x_2, x_3, x_4\}$ respectively, such that the edges x_1x_4 and x_2x_3 are subdivided through y_4 and y_2 respectively, as shown in Figure 1(a). For $y_i \notin V(G)$, let G^* be a graph obtained by adding a vertex y_i and the edges joining y_i to a pair of non-adjacent vertices x_i and x_{i+1} in G, followed by adding two edges y_iy_2 and y_iy_4, where $i \in \{1,3\}$. That is, $V(G^*) = V(G) \cup \{y_i\}$ and $E(G^*) = E(G) \cup \{x_iy_i, x_{i+1}y_i, y_2y_i, y_4y_i\}$ where $i \in \{1,3\}$.

Thus by performing (*) operation for $i = 1$, we obtain a graph G^* from G, and similarly for another value $i = 3$, the graph G^{*^*} is obtained from G^*. The Figure 1(a–c) represents a good drawing of the graphs G, G^*, and G^{*^*} respectively. At first, we find the crossing number of the graphs G^* and G^{*^*} in terms of the crossing number of G and G^* respectively.

Lemma 2.1. $Cr(G^*) = Cr(G) + 2$ and $Cr(G^{*^*}) = Cr(G^*) + 2$.

Proof. The graph G contains $K_{4,3}$ as its subgraph and has a good drawing with two crossings as in Figure 1(a), shows $Cr(G) = 2$. The graph G^* is obtained from G by adding a vertex $y_1 \notin V(G)$ and the edges x_1y_1, x_2y_1, y_1y_2 and y_1y_4. In Figure 1(b), there is a decomposition into two edge-disjoint subgraphs $K_{4,3}$ and $E(y_1) \cup E(y_2) \cup E(y_4)$ in the drawing of a graph G^*, which has two crossings among the edges of $\cup_1^3 E(z_i) \cong K_{4,3}$, (a subgraph of G), and two crossings between the edges of $E(y_1) \cup E(y_2) \cup E(y_4)$ and the edges of $K_{4,3}$, implies $Cr(G^*) \leq 4 = Cr(G) + 2$. Let W be a subgraph of G^* induced by the edges of $E(y_1) \cup E(y_2) \cup E(y_4)$. Since $\cup_1^3 E(z_i) \cong K_{4,3}$, we have $Cr_\phi(\cup_1^3 E(z_i)) \geq 2$. If the edges of W are crossed at least twice either within or with the edges of $\cup_1^3 E(z_i)$, the result is trivial. Assume that the edges of W cross at most once. Let $S = \{x_1, x_2, x_3, x_4\}$ be a subset of $V(G^*)$.

Case (i). If the edges of W cross $E(z_i)$, say, $E(z_1)$, then clearly, $Cr_\phi(W, E(z_1)) = 1$ and there must be no crossing in W. Since W is an outerplanar graph, it follows that either every vertices in S are placed in a same region, or any edge incident to a pendent vertex can be moved into a triangular region bounded by its adjacent edges as shown in Figure 2.

Now in $W \cup E(z_1)$, there exist no region with all the vertices in S on its boundary, and with the fact that $W \cup E(z_1) \cup E(z_i)$ contains a subgraph homeomorphic to $K_{4,3}$, implies, for one of z_i (except z_1), say z_2, either $Cr_\phi(W, E(z_2)) \geq 1$ or $Cr_\phi(E(z_1), E(z_2)) \geq 1$. If $Cr_\phi(W, E(z_2)) \geq 1$, then clearly we have $Cr_\phi(W, E(z_1) \cup E(z_2)) \geq 2$, a contradiction. If $Cr_\phi(E(z_1), E(z_2)) \geq 1$ with $Cr_\phi(W, E(z_2)) = 0$ then in $W \cup E(z_1) \cup E(z_2)$, there is no region having more than two vertices in S on its boundary, confirms at least two crossings between the edges of $E(z_3)$ and $W \cup E(z_1) \cup E(z_2)$.

Case (ii). If the edges of W do not cross the edges of $E(z_i)$, then all the vertices in S are placed in the unique region of the subdrawing W. Without loss of generality,

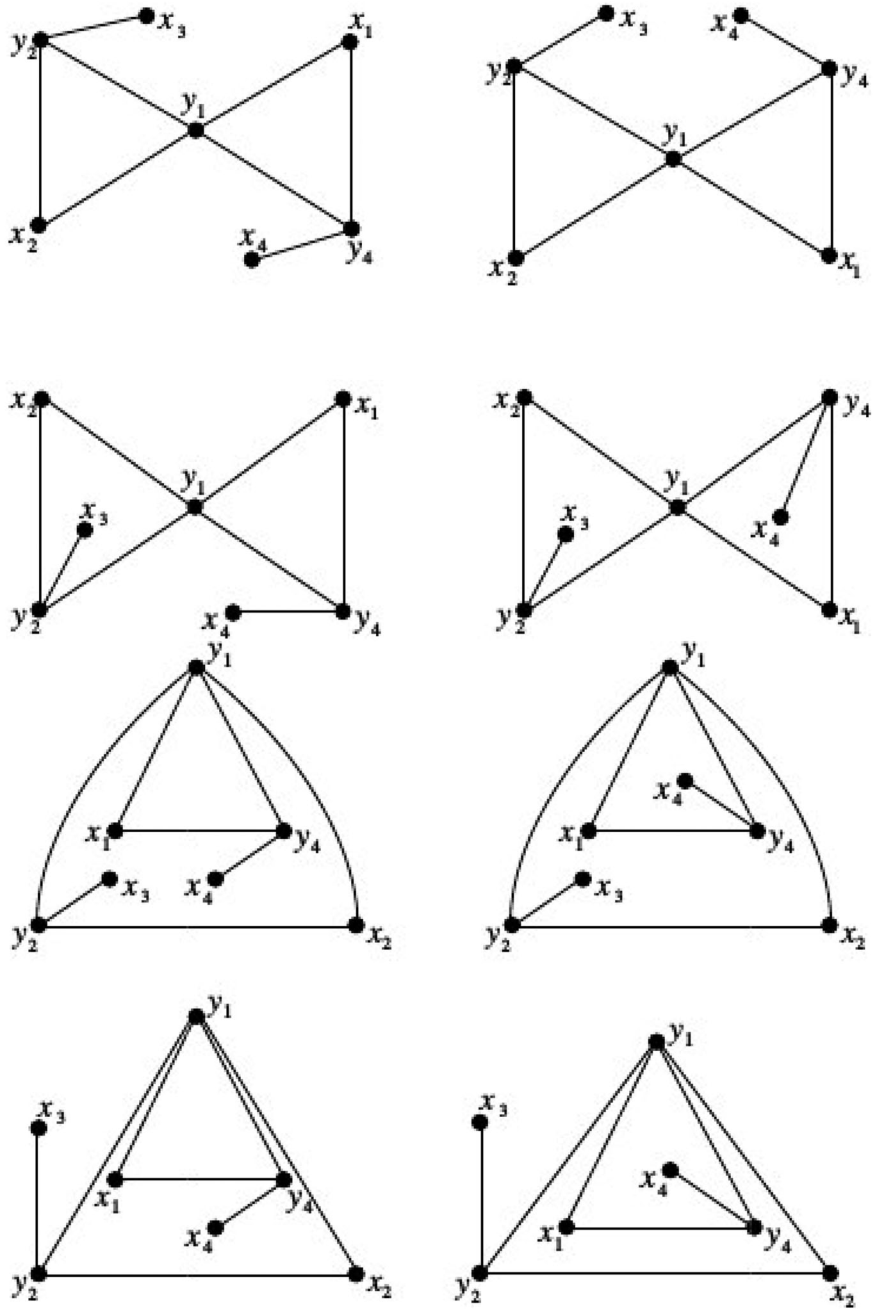

Figure 2. The possible drawings of W with zero crossing.

assume that $Cr_\phi(W, E(z_1)) = 0$. From Figure 3, $E(z_1)$ divides the plane such that exactly two vertices in S lies on the boundary of every region, yields at least four crossings between the edges of $W \cup E(z_1)$ and $E(z_2) \cup E(z_3)$. Thus $Cr(G^*) \geq 4 = Cr(G) + 2$, implies $Cr(G^*) = Cr(G) + 2$.

Similarly, for a graph G^{**} obtained from G^* by applying $(*)$ operation for $i = 3$, we have $Cr(G^{**}) = Cr(G^*) + 2$. It follows that, $Cr(G^*) = 4$ and $Cr(G^{**}) = 6$. □

Lemma 2.2. *In any drawing of $K_{1,1,3} \wedge P_n (n \geq 3)$, there is at least four crossings on the edges of every H_j for $2 \leq j \leq n-1$.*

Proof. Let ϕ be a drawing of $K_{1,1,3} \wedge P_n$. Since the crossing number of $K_{4,3}$ is known to be two and $H_j \cong K_{4,3}$, implicates $Cr_\phi(H_j) \geq 2$. Suppose there exist a pair of subgraphs H_j and H_k, such that $Cr_\phi(H_j, H_k) > 0$, for all $j, k = 2, 3, ..., n-1, j \neq k$. Since $K_{4,3}$ is 3-connected and for each $j = 2, 3, ..., n-1$, we have $H_j = E(t_{1,j}) \cup E(t_{3,j}) \cup E(t_{5,j}) \cong K_{4,3}$, implies $Cr_\phi(H_j, H_k) \geq 3$. Then there appear at least five crossings on the edges of H_j. Suppose there is no crossing between any pair of subgraphs H_j and H_k, for $j, k = 2, 3, ..., n-1, j \neq k$. In a drawing of $K_{1,1,3} \wedge P_n$, along with the edges of H_j, we have the vertices $t_{2,j\pm1} (t_{4,j\pm1})$ are connected to $t_{4,j} (t_{2,j})$ and there is a vertex $t_{5,j-1}$ adjacent to both $t_{2,j}$ and $t_{4,j}$, as shown in Figure 4. Moreover $Cr_\phi(H_j, H_{j-1}) = 0$ implies, by contracting a path $t_{2,j} t_{5,j-1} t_{4,j}$ to a vertex, we obtain a resulting graph isomorphic to $K_{4,4}$. Since $Cr(K_{4,4}) = 4$, we have $Cr_\phi(H_j) \geq 4$. □

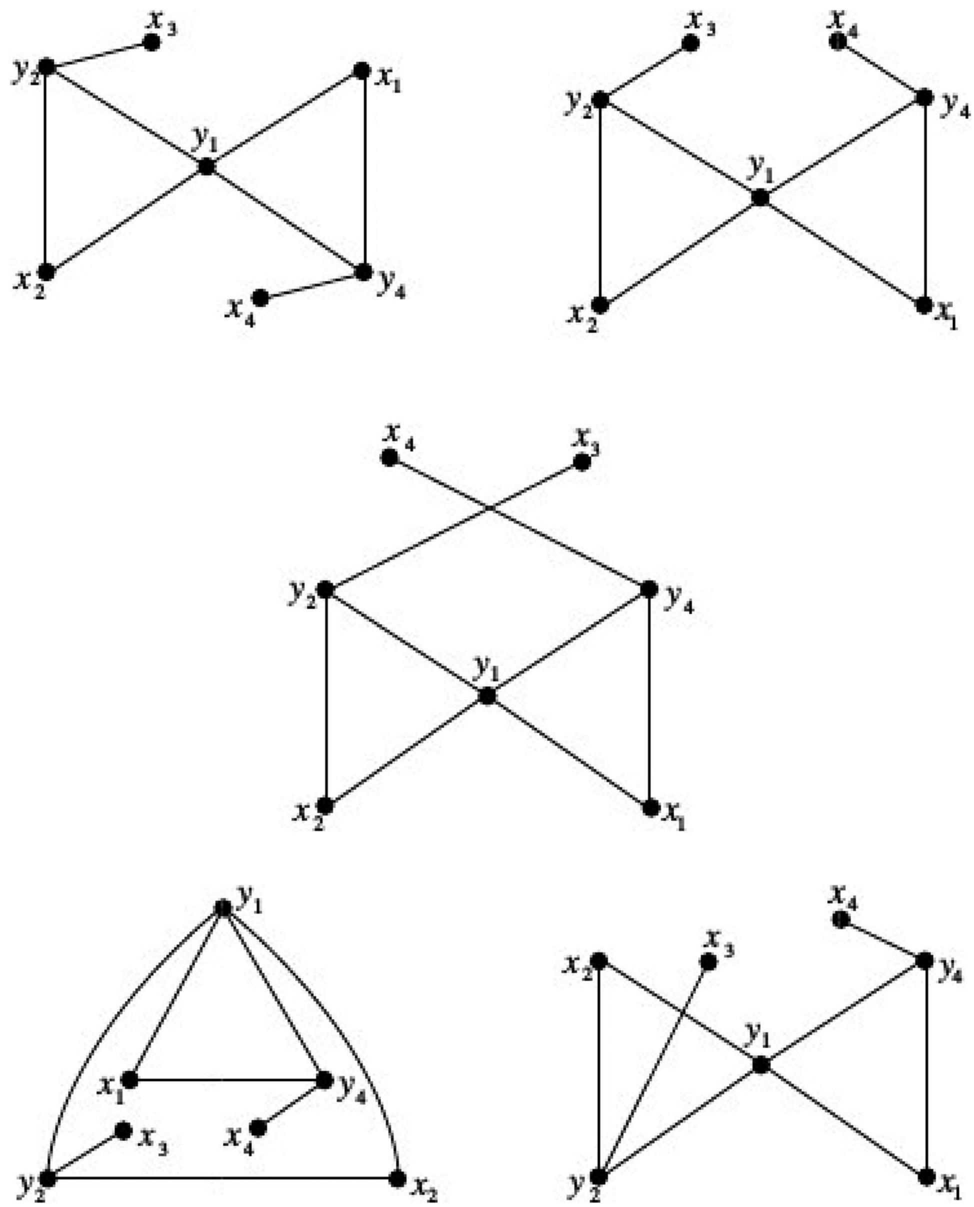

Figure 3. The possible drawings of W with every $v \in S$ placed in a same region.

Lemma 2.3. $Cr(K_{1,1,3} \wedge P_n) = 4(n-2)$, for n = 2, 3, 4.

Proof. The graph $K_{1,1,3} \wedge P_2$ is planar implies $Cr(K_{1,1,3} \wedge P_2) = 0$. Since the graph $K_{1,1,3} \wedge P_3$ has a good drawing with four crossings as shown in Figure 5(a), we have $Cr(K_{1,1,3} \wedge P_3) \leq 4$. Then by Lemma 2.2, $Cr(K_{1,1,3} \wedge P_3) \geq 4$ yields $Cr(K_{1,1,3} \wedge P_3) = 4$. Also, there exist a good drawing for the graph $K_{1,1,3} \wedge P_4$ with eight crossings as in Figure 5(b) and by Lemma 2.2, it follows that $Cr(K_{1,1,3} \wedge P_4) = 8$. □

Lemma 2.4. *If ϕ is a drawing of $K_{1,1,3} \wedge P_n$ and every H_j $(2 \leq j \leq n-1)$ has at most five crossings, then every vertices in a set $S_j = \{t_{2,j\pm1}, t_{4,j\pm1}\}$ are placed in the unique region of a subdrawing $(K_{1,1,3} \wedge P_n)$ H_j.*

Proof. Suppose not all the vertices in a set S_j are placed in a same region. Consider a region with at most three vertices in S_j on its boundary. Since all the three vertices $t_{1,j}, t_{3,j}$ and $t_{5,j}$ are connected to each vertex in S_j, we have at least three crossings appear between the edges of H_j and the edges on a boundary of the considered region. Also $E(t_{1,j}) \cup E(t_{3,j}) \cup E(t_{5,j}) \cong K_{4,3}$, implies $Cr_\phi(E(t_{1,j}) \cup E(t_{3,j}) \cup E(t_{5,j})) \geq 2$. Suppose that for $2 \leq k, j \leq n-1, k \neq j$, there exist H_k $(k \neq j)$, such that, $Cr_\phi(H_j, H_k) > 0$. Then $Cr_\phi(H_j, H_k) \geq 3$, implies there is at least eight crossings on the edges of H_j, which contradicts the assumption of a Lemma. Thus, no edge of H_j is crossed with an edge of H_k, for $2 \leq k, j \leq n-1, k \neq j$. Since the vertices $t_{1,j}, t_{3,j}$ and $t_{5,j}$ are connected to each vertex in S_j and the vertices $t_{2,j\pm1}(t_{4,j\pm1})$ are adjacent to $t_{4,j}(t_{2,j})$ with $t_{5,j-1}$ joined to both $t_{2,j}$ and $t_{4,j}$ implies, by contracting a path $t_{2,j}t_{5,j-1}t_{4,j}$ to a vertex, we obtain a resulting graph isomorphic to $K_{4,4}$. Thus $Cr_\phi(H_j) \geq 4$. Then there are at least seven crossings on the edges of H_j. Again, a contradiction confirms that, in a subgraph $(K_{1,1,3} \wedge P_n)$ H_j, all the vertices in S_j are placed in a same region. □

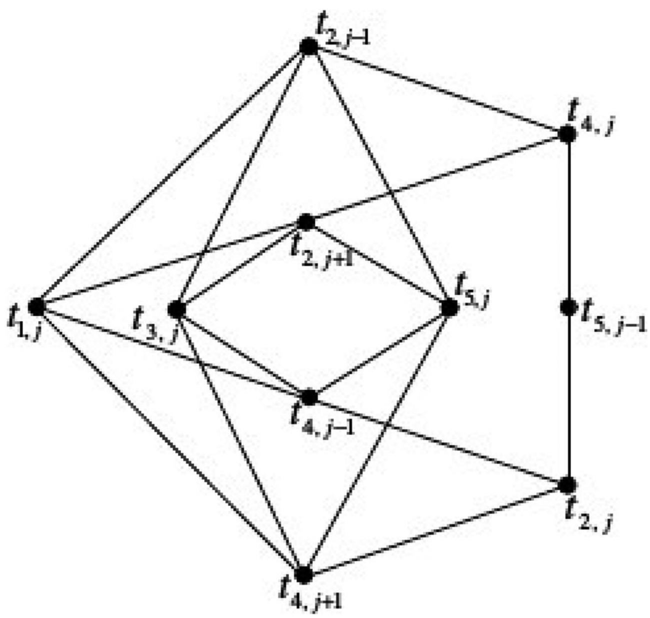

Figure 4. A subdrawing induced by the edges of $H_j \cup \{t_{4,j}t_{2,j\pm1}, t_{2,j}t_{4,j\pm1}, t_{2,j}t_{5,j-1}, t_{4,j}t_{5,j-1}\}$.

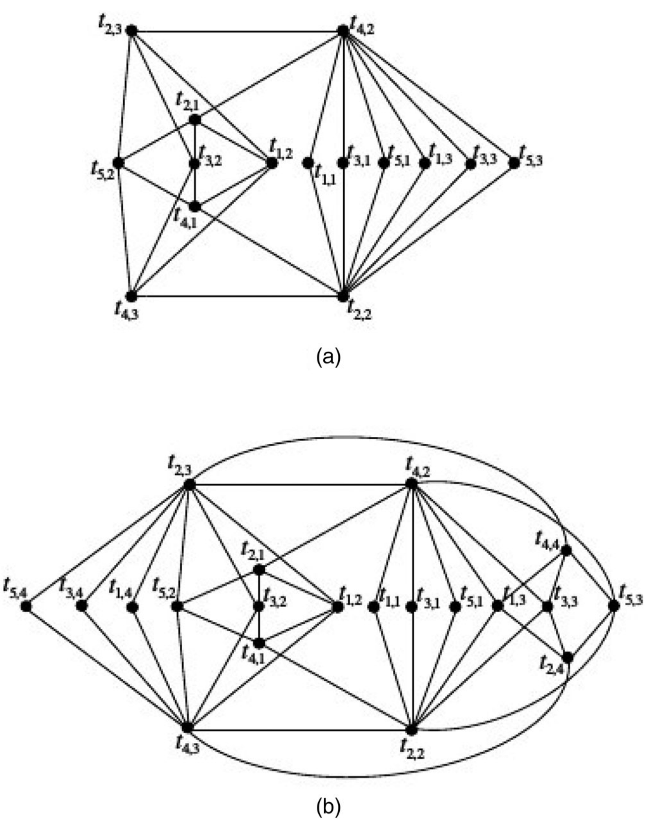

Figure 5. A good drawing of $K_{1,1,1}\wedge P_3$ and $K_{1,1,1}\wedge P_4$.

Lemma 2.5. *If ϕ is a drawing of $K_{1,1,3}\wedge P_n (n \geq 4)$, where every subgraph H_j $(2 \leq j \leq n-1)$ has at most five crossings, then $Cr_\phi(H_j, H_k) = 0$, for $1 \leq k, j \leq n, k \neq j$.*

Proof. Suppose that $Cr_\phi(H_j, H_k) \neq 0$. Let us consider two following cases:

Case (i). Suppose H_j crosses at least two different subgraphs H_k and H_l $(j \neq k \neq l)$. Since $Cr_\phi(H_j, H_k) \geq 3$ and $Cr_\phi(H_j, H_l) \geq 3$, then there are at least six crossings on the edges of H_j, which is a contradiction.

Case (ii). Suppose H_j cross only one subgraph H_k $(k \neq j)$. If H_1 cross H_2, then we consider H_2. If H_{n-1} cross H_n, then we consider H_{n-1}. Since there exist two adjacent copies H_{j-1} and H_{j+1} of H_j, we have at least one adjacent copy, say H_{j-1}, which does not cross H_j. Then with the edges $\{t_{i,j}t_{2,j\pm1}, t_{i,j}t_{4,j\pm1} : i = 1,3,5\} \cup \{t_{2,j}t_{4,j\pm1}, t_{4,j}t_{2,j\pm1}, t_{2,j}t_{5,j-1}, t_{4,j}t_{5,j-1}\}$, by contracting a path $t_{2,j}t_{5,j-1}t_{4,j}$ to a vertex, we obtain a resulting graph isomorphic to $K_{4,4}$. Thus $Cr_\phi(H_j) \geq 4$ and $Cr_\phi(H_j, H_k) \geq 3$, shows there exist at least seven crossings on the edges of H_j. This is a contradiction to the assumption of the Lemma. □

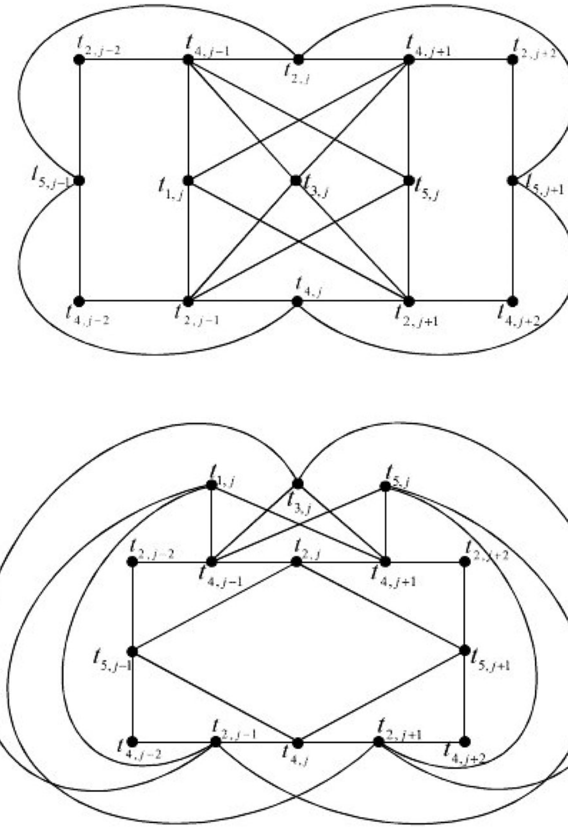

Figure 6. The possible drawings of $H_j' \cup H_j$ with $Cr_\phi(H_j') = 0$ and $Cr_\phi(H_j', H_j) = 0$.

From Lemma 2.4 and Lemma 2.5, we can state the following proposition:

Proposition 2.1. *If ϕ is a drawing of $K_{1,1,3}\wedge P_n (n \geq 4)$, where every subgraphs H_j $(2 \leq j \leq n-1)$ has at most five crossings, then there is no crossing on the edges incident with two degree vertices and on the edges of the paths $t_{4,2}t_{2,3}t_{4,4}\ldots t_{r_1,n-1}$ and $t_{2,2}t_{4,3}t_{2,4}\ldots t_{r_2,n-1}$ respectively. If n is even, $r_1 = 2$ and $r_2 = 4$; and if n is odd, $r_1 = 4$ and $r_2 = 2$.*

Theorem 2.1. *If ϕ is a drawing of $K_{1,1,3}\wedge P_n (n \geq 4)$, where every H_j $(2 \leq j \leq n-1)$ has at most five crossings, then ϕ has at least $6(n-3)+2$ crossings.*

Proof. Consider a subgraph $H_j' : 2 \leq j \leq n-1$ induced on the vertices $\{t_{2,j}, t_{2,j\pm1}, t_{2,j\pm2}, t_{4,j}, t_{4,j\pm1}, t_{4,j\pm2}, t_{5,j\pm1}\}$ of $K_{1,1,3}\wedge P_n$. From Figure 6, clearly, for each $3 \leq j \leq n-2$, H_j' contains the cycles $C_1 : t_{2,j}t_{5,j-1}t_{4,j}t_{5,j+1}t_{2,j}$ and C_2 :

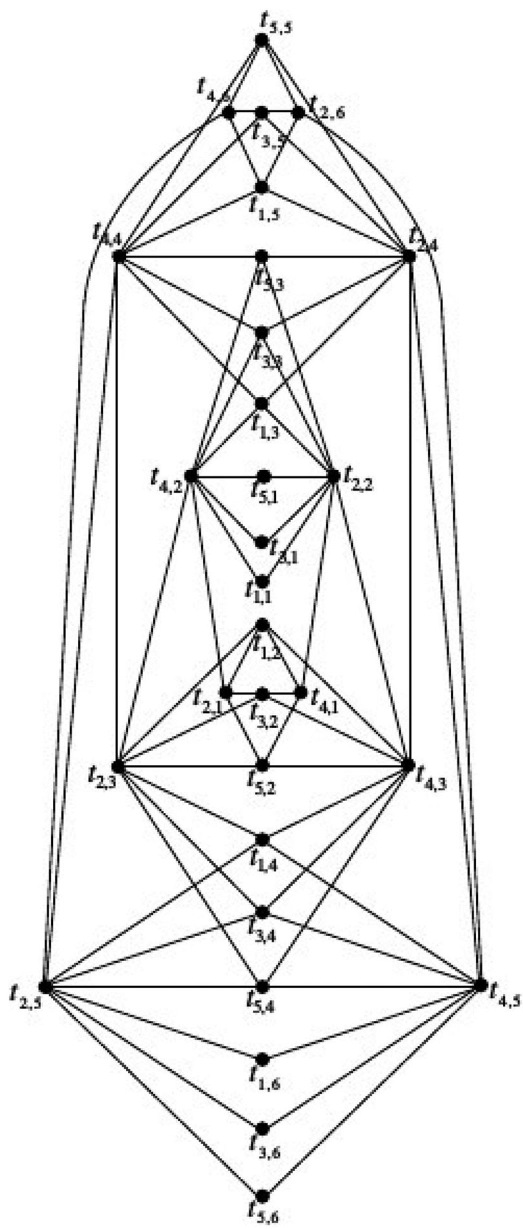

Figure 7. A good drawing of $K_{1,1,3} \wedge P_6$.

$t_{2,j}t_{4,j-1}t_{2,j-2}t_{5,j-1}t_{4,j-2}t_{2,j-1}t_{4,j}t_{2,j+1}t_{4,j+2}t_{5,j+1}t_{2,j+2}t_{4,j+1}t_{2,j}$, shows that $H_j' \cup H_j$ is homeomorphic to G^{*^*}. By Lemma 2.5, $Cr_\phi(H_j, H_k) = 0$ (for $2 \le k, j \le n-1, k \ne j$), implies from Lemma 2.4 and Proposition 2.1, all the vertices $t_{2,j\pm1}$ and $t_{4,j\pm1}$ are to be placed in the unique region of the subdrawing of $(K_{1,1,3} \wedge P_n) \backslash H_j$ in ϕ, such that, there must be no crossing among the edges of H_j', and $Cr_\phi(H_j', H_j) = 0$, for $3 \le j \le n-2$. Then each vertex $t_{1,j}, t_{3,j}$ and $t_{5,j}$ has to be placed in this considered region as shown in Figure 6, either inside or outside the cycle C_2. It is clear that, by Lemma 2.1, the edges of H_j $(j = 3, 4, \ldots, n-2)$ has at least six crossings. Moreover each H_j and H_k are edge-disjoint subgraphs of $K_{1,1,3} \wedge P_n$ for $2 \le j, k \le n-1, j \ne k$, implies the crossings on the edges of each H_j must be distinct. Hence at least $6(n-4)$ crossings appear on the edges of $\cup_3^{n-2} H_j$. By Lemma 2.2, we have at least four crossings on the edges of H_2 and H_{n-1}, respectively. Therefore, there exist at least $6(n-3)+2$ crossings in ϕ. □

Theorem 2.2. $Cr(K_{1,1,3} \wedge P_n) = 6(n-3) + 2$, for $n \ge 4$.

Proof. Consider a good drawing ϕ of $K_{1,1,3} \wedge P_n$ such that,

(i) $\phi(t_{1,1}) = (0.5, 0)$, $\phi(t_{2,1}) = (-1, 0.5)$, $\phi(t_{3,1}) = (1,0)$, $\phi(t_{4,1}) = (-1, -0.5)$, $\phi(t_{5,1}) = (2,0)$, $\phi(t_{2,n}) = ((-1)^n (1 + 3\lfloor\frac{n-2}{2}\rfloor), (-1)^{n-1} 0.5)$, $\phi(t_{4,n}) = ((-1)^n (1+3 \lfloor\frac{n-2}{2}\rfloor), (-1)^n 0.5)$.

(ii) for $2 \le j \le n, \phi(t_{i,j}) = ((-1)^{j+1}\left(\lfloor\frac{i-1}{2}\rfloor + 3\lfloor\frac{j-1}{2}\rfloor\right), 0)$, where $i = 1, 3, 5$;

(iii) for $2 \le j \le n-1, \phi(t_{2,j}) = ((-1)^j (2 + 3\lfloor\frac{j-2}{2}\rfloor), (-1)^{j+1} \lfloor\frac{j+1}{2}\rfloor)$; and $\phi(t_{4,j}) = ((-1)^j (2 + 3\lfloor\frac{j-2}{2}\rfloor), (-1)^j \lfloor\frac{j+1}{2}\rfloor)$.

(iv) the image of each edge is a straight-line segment except the edges $t_{2,n-1}t_{4,n}$ and $t_{4,n-1}t_{2,n}$.

By above construction, a good drawing of $K_{1,1,3} \wedge P_n$ obtained with $6(n-3)+2$ crossings, shows that, $Cr(K_{1,1,3} \wedge P_n) \le 6(n-3) + 2$. For illustration, we show a good drawing of $K_{1,1,3} \wedge P_6$ (rotated vertically) in Figure 7. For convenience, let us denote a vertex or an edge in the drawing ϕ simply by x instead of the image of x, $\phi(x)$. Let us prove the reverse inequality by induction on $n \ge 4$. From Lemma 2.3, we have $Cr(K_{1,1,3} \wedge P_4) = 8$. Thus, the result is true for $n = 4$.

Assume that $Cr_\phi(K_{1,1,3} \wedge P_k) \ge 6(k-3) + 2 (k \ge 4)$, and suppose there is a drawing of $K_{1,1,3} \wedge P_{k+1}$ with fewer than $6(k-2)+2$ crossings. Then by Theorem 2.1, some H_j $(j = 2, 4, \ldots, n-1)$ has crossed at least six times and $Cr_\phi(K_{1,1,3} \wedge P_{k+1}) - Cr_\phi(K_{1,1,3} \wedge P_k) < 6$. Moreover by the removal of V_j, $j \in \{1, k+1\}$ from $K_{1,1,3} \wedge P_{k+1}$, we obtain a graph isomorphic to $K_{1,1,3} \wedge P_k$ implies, the graph $K_{1,1,3} \wedge P_{k+1}$ contains $K_{1,1,3} \wedge P_k$ as its subgraph which has a drawing with fewer than $6(k-3)+2$ crossings. This contradiction to the induction hypothesis, shows that in any drawing, $Cr_\phi(K_{1,1,3} \wedge P_n) \ge 6(n-3)+2$. □

Since from Figure 7, one can easily find a rectilinear drawing of $K_{1,1,3} \wedge P_n$, by placing the vertices $t_{2,n-1}$ and $t_{4,n-1}$ appropriately so as to draw the edges $t_{2,n-1}t_{4,n}$ and $t_{4,n-1}t_{2,n}$ in straight line segments, with exactly $6(n-3)+2$ crossings. Also, for any graph G, $\overline{Cr}(G) \ge Cr(G)$ implies the next Corollary 2.1 can be stated as follows.

Corollary 2.1. $\overline{Cr}(K_{1,1,3} \wedge P_n) = 6(n-3) + 2$.

3. Conclusion

Among all the graphs of order at most four, we find only the graphs $K_{1,1,2}$ and K_4, for which their Kronecker product with the path is non-planar and we state their crossing number as the open problems:

Open Problem 1. $Cr(K_{1,1,2} \wedge P_n) = 2(n-2)$, for $n > 2$.

Open Problem 2. $Cr(K_4 \wedge P_n) = \begin{cases} 4, & \text{for } n = 3; \\ 4(n-1), & \text{for } n \ge 4. \end{cases}$

In future, we extend our work to determine the exact value of crossing number for graphs obtained from different graph operations.

Disclosure statement

No potential conflict of interest was reported by the authors. There are many more applications for the crossing number of graphs and also more results on the Cartesian product of graphs motivated us to initiate the study on determining the crossing number for Kronecker product of graphs. In future, we plan to extend our work to compute the exact value of crossing number for graphs obtained from different graph operations

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