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Distance antimagic labelings of Cartesian product of graphs

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ABSTRACT

Let $G=(V,E)$ be a graph of order n. Let $f:V\to\{1,2,\dots,n\}$ be a bijection. The weight $w(v)$ of a vertex v with respect to the labeling f is defined by $w(v)=\sum_{u\in N(v)} f(u)$, where $N(v)$ is the open neighborhood of v. The labeling f is called a distance antimagic labeling if $w(v_1)\neq w(v_2)$ for any two distinct vertices v_1, v_2 in V. In this paper we investigate the existence of distance antimagic labeling for the Cartesian product $G\Box H$ where the graphs G and H are cycles or complete graphs.

KEYWORDS
Distance magic labeling; distance antimagic labeling; Cartesian product

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1. Introduction

By a graph $G=(V,E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Vilfred [8] in his doctoral thesis introduced the concept of sigma labelings. Acharya et al., [1] further studied the concept under the name of neibourhood magic graphs. The same concept was introduced by Miller et al., [6] under the name 1-vertex magic vertex labeling. Sugeng et al., [7] introduced the term distance magic labeling for this concept.

Definition 1.1. [6] A *distance magic labeling* of a graph G of order n is a bijection $f:V\to\{1,2,\dots,n\}$ with the property that there is a positive integer k such that $\sum_{y\in N(x)} f(y)=k$ for every $x\in V$. The constant k is called the *magic constant* of the labeling *f*.

The sum $\sum_{y\in N(x)} f(y)$ is called the *weight* of the vertex x and is denoted by $w(x)$.

For a recent survey and open problems on distance magic graphs we refer to Arumugam et al. [2].

Let G be a distance magic graph of order n with labeling f and magic constant k. Then $\sum_{u\in N_{G^c}(v)} f(u)=\frac{n(n+1)}{2}-k-f(v)$, and hence the set of all vertex weights in G^c is $\left\{\frac{n(n+1)}{2}-k-i : 1\le i\le n\right\}$, which is an arithmetic progression with first term $a=\frac{n(n+1)}{2}-k-n$ and common difference $d=1$.

Motivated by this observation, Arumugam and Kamatchi [3] introduced the following concept of $(a,\ d)$-distance antimagic graph.

Definition 1.2. [3] A graph G is said to be $(a,\ d)$- *distance antimagic* if there exists a bijection $f:V\to\{1,2,\dots,n\}$ such that the set of all vertex weights is $\{a,a+d,a+2d,\dots,a+(n-1)d\}$ and any graph which admits such a labeling is called an $(a,\ d)$- *distance antimagic graph.*

Thus the complement of every distance magic graph is an $(a,1)$-distance antimagic graph.

We observe that if a graph G is $(a,\ d)$-distance antimagic with $d>0$, then for any two distinct vertices u and v we have $w(u)\neq w(v)$. This observation naturally leads to the concept of distance antimagic labeling.

Definition 1.3. [5] Let $G=(V,E)$ be a graph of order n. Let $f:V\to\{1,2,\dots,n\}$ be a bijection. If $w(x)\neq w(y)$ for any two distinct vertices x and y in V, then f is called a *distance antimagic labeling*. Any graph G which admits a distance antimagic labeling is called a *distance antimagic graph.*

Definition 1.4. The *Cartesian product* of G and H, written $G\Box H$, is the graph with vertex set $V(G\Box H)=\{(u,v):u\in$

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$V(G) and v \in V(H)\}$ and edge set $E(G\square H) = \{(u,v)(u',v') : u = u' and v\, v' \in E(H) or v = v' and u u' \in E(G)\}$.

Kamatchi and Arumugam [5] posed the following problem.

Problem 1.5. *If G is distance antimagic, is it true that the graphs $G + K_1, G + K_2$, the Cartesian product $G\square K_2$ are distance antimagic?*

In this paper we discuss the existence of distance antimagic labeling for the Cartesian product $K_n\square K_n$ and $K_3\square C_k$.

2. Main results

In the following theorem we prove the existance of distance antimagic labeling for $G = K_n\square K_n$.

Theorem 2.1. *The graph $G = K_n\square K_n$ is distance antimagic if and only if $n \neq 2$.*

Proof. Let $G = K_n\square K_n$ and suppose $n \neq 2$. Let $V(K_n) = \{v_1, v_2, ..., v_n\}$. We denote the vertex (v_i, v_j) in $K_n\square K_n$ by v_{ij}. Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be the copies of K_n in G.

Define $f : V(G) \to \{1, 2, ..., n^2\}$ by $f(v_{ij}) = (i-1)n + j$.

Then $w(X_i) = \frac{n}{2}[n(2i-1)+1]$ and $w(Y_j) = \frac{n}{2}[2j + n(n-1)]$. Hence

$$
\begin{aligned}
w(v_{ij}) &= w(X_i) + w(Y_j) - 2f(v_{ij}) \\
&= \frac{n^3}{2} + n^2\left(i - 1 + \frac{1}{2n}\right) + 2n(1-i) + j(n-2). \quad (1)
\end{aligned}
$$

Now let v_{ij} and v_{rs} be two distinct vertices in G. Suppose $w(v_{ij}) = w(v_{rs})$.

We consider three cases.

Case 1. $i = r$ and $j \neq s$.

Since $w(v_{ij}) = w(v_{rs})$, it follows from (1) that $j(n-2) = s(n-2)$. Since $n \neq 2$, it follows that $j = s$ and so $v_{ij} = v_{rs}$, which is a contradiction.

Case 2. $i \neq r$ and $j = s$.

Since $w(v_{ij}) = w(v_{rs})$, it follows from (1) that $(n^2 - 2n)i = (n^2 - 2n)r$. Since $n \neq 2$, it follows that $i = r$ and so $v_{ij} = v_{rs}$, which is a contradiction.

Case 3. $i \neq r$ and $j \neq s$.

Since $w(v_{ij}) = w(v_{rs})$, it follows from (1) that $(i-r)n = s - j$. Hence $s = (i-r)n + j$. If $i - r$ is positive, then $s > n$ and if $i - r$ is negative, then s is negative, which is a contradiction. Hence $w(v_{ij}) \neq w(v_{rs})$.

Thus G is distance antimagic when $n \neq 2$.

If $n = 2$, then $G = C_4$ which is not distance antimagic. □

In the following theorem we investigate the existence of distance antimagic labeling of the graph $G = K_3\square C_k$.

Theorem 2.2. *For any odd integer $k \geq 3$, the graph $G = K_3\square C_k$ is distance antimagic.*

Proof. Let $V(K_3) = \{v_1, v_2, v_3\}$ and let $C_k = (u_1, u_2, ..., u_k, u_1)$. We denote the vertex (v_i, u_j) by v_{ij}. If $k = 3$, the result follows from Theroem 2.1. Suppose $k \geq 5$.

Define $f : V(G) \to \{1, 2, ..., 3k\}$ by $f(v_{ij}) = (i-1)k + j$. Then

$$
w(v_{ij}) = \begin{cases} k(i+3)+4 & \text{if } j = 1 \\ k(i+5)+2 & \text{if } j = k \\ k(i+2)+4j & \text{if } 1 < j < k \end{cases} \quad (2)
$$

Suppose there exist two distinct vertices v_{ij} and v_{rs} in G such that

$$
w(v_{ij}) = w(v_{rs}) \quad (3)
$$

We consider three cases.

Case 1. $j = s$.

If $j = s = 1$, then from (1) and (2) we get $k(i+3) + 4 = k(r+3) + 4$.

If $j = s = k$, then $k(i+5) + 2 = k(r+5) + 2$.

If $j = s$ and $1 < j < k$, then $k(i+2) + 4j = k(r+2) + 4j$.

In all these cases $i = r$ and hence $v_{ij} = v_{rs}$ which is a contradiction.

Case 2. $i = r$.

If $j = 1$ and $s = k$, then from (1) and (2) we get $k(i+3) + 4 = k(i+5) + 2$. Hence $k = 1$ which is a contradiction.

If $j = 1$ and $s < k$, then $k(i+3) + 4 = k(i+2) + 4s$. Hence $k = 4(s-1)$. Thus k is even which is a contradiction.

If $1 < j < k$ and $1 < s < k$, then $k(i+2) + 4j = k(i+2) + 4s$. Hence $j = s$ and so $v_{ij} = v_{rs}$, which is again a contradiction.

Case 3. $i \neq r$ and $j \neq s$.

If $j = 1$ and $s = k$, then proceeding as in Case 1, we get $k(r - i + 2) = 2$ which is a contradiction since $k \geq 5$.

If $j = 1$ and $s < k$, then $k(i - r + 1) = 4(s-1)$.

If $1 < j < k$ and $1 < s < k$, then $k(i-r) = 4(s-j)$.

Hence k divides $4(s-1)$ or $4(s-j)$. Since k is odd, it follows that k divides $s - 1$ or k divides $s - j$. Since $s \leq k$, this again leads to a contradiction.

Hence it follows that $w(v_{ij}) \neq w(v_{rs})$. Thus f is a distance antimagic labeling of G. □

3. Conclusion and scope

The labeling defined in Theorem 2.2 is not a distance antimagic labeling for $G = K_3\square C_4$. Hence the following problem arises naturally.

Problem 3.1. *Is $G = K_3\square C_k$ distance antimagic when k is even?*

The investigation of the existence of distance antimagic labeling for $G\square H$ for other graphs G and H is another direction for further research.

Conflict of interest

No conflicts of interest have been reported by the author(s).

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