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Secure total domination in chain graphs and cographs

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ABSTRACT

Let $G = (V,E)$ be a graph without isolated vertices. A subset D of vertices of G is called a *total dominating set* of G if for every $u \in V$, there exists a vertex $v \in D$ such that $uv \in E$. A total dominating set D of a graph G is called a *secure total dominating set* of G if for every $u \in V \backslash D$, there exists a vertex $v \in D$ such that $uv \in E$ and $(D \backslash \{v\}) \cup \{u\}$ is a total dominating set of G. The *secure total domination number* of G, denoted by $\gamma_{st}(G)$, is the minimum cardinality of a secure total dominating set of G. Given a graph G, the *secure total domination problem* is to find a secure total dominating set of G with minimum cardinality. In this paper, we first show that the secure total domination problem is linear time solvable on graphs of bounded clique-width. We then propose linear time algorithms for computing the secure total domination number of chain graphs and cographs.

KEYWORDS

Total domination; secure total domination; chain graphs; cographs; linear time algorithm

1. Introduction

Let $G = (V,E)$ be a simple and undirected graph without isolated vertices. A subset D of vertices of G is called a *dominating set* (resp. *total dominating set*) of G if for every $u \in V \backslash D$ (resp. $u \in V$), there exists a vertex $v \in D$ such that $uv \in E$. The minimum cardinality of a dominating set (resp. total dominating set) of G is denoted by $\gamma(G)$ (resp. $\gamma_t(G)$). A dominating set D of a graph G is called a *secure dominating set* of G if for every $u \in V \backslash D$, there exists a vertex $v \in D$ such that $uv \in E$ and $(D \backslash \{v\}) \cup \{u\}$ is a dominating set of G. The concept of secure domination in graphs was introduced by Cockayne et al. [3] and studied in the literature [8, 10, 12, 14]. The concept of secure domination in graphs was extended to secure total domination in graphs by Benecke et al. [2]. A total dominating set D of a graph G is called a *secure total dominating set* of G if for every $u \in V \backslash D$, there exists a vertex $v \in D$ such that $uv \in E$ and $(D \backslash \{v\}) \cup \{u\}$ is a total dominating set of G. The *secure total domination number* of G, denoted by $\gamma_{st}(G)$, is the minimum cardinality of a secure total dominating set of G. Given a graph G, the *secure total domination problem* is to find a secure total dominating set of G with minimum cardinality. Klostermeyer and Mynhardt [10] gave several bounds on the secure total domination number of a graph. Further, Duginov [7] has studied the hardness of the approximation of the secure total domination problem in graphs. He also established various bounds on the secure total domination number of a graph.

The reduction used in [7] for showing the NP-completeness of the decision version of the secure total domination problem for chordal bipartite graphs and graphs of separability at most 2 can be used to show that the decision version of the secure total domination problem is NP-complete for undirected path graphs and circle graphs.

In this paper, we first show that the secure total domination problem can be solved in linear time on graphs of bounded clique-width by exploiting the result due to Courcelle et al. [6] on Monadic Second Order Logic. Courcelle et al. [6] do not give any explicit idea to obtain the solution of the problem using the structure of the given graph. By using the structure of chain graphs and cographs, we propose linear time algorithms for computing the secure total domination number of a given chain graph and cograph.

The paper is organized as follows. In Section 2, we present some pertinent definitions and preliminary results. In Section 3, we show the linear time solvability of the secure total domination problem in graphs of bounded clique-width. In Section 4 and Section 5, we present a linear time algorithm to compute the secure total domination number of a given chain graph and cograph, respectively.

2. Preliminaries

Let $G = (V,E)$ be a simple and undirected graph. For any vertex $v \in V$, the *open neighborhood* of v in G is the set $N_G(v) = \{u : uv \in E\}$ and the *closed neighborhood* of v in G is the set $N_G[v] = N_G(v) \cup \{v\}$. For $A \subseteq V$, we define $N_G(A) = \cup_{x \in A} N_G(x)$. The *degree* of a vertex v in G is $|N_G(v)|$ and is denoted by $d_G(v)$. A vertex of G with degree one is said to be a *pendant vertex*. A vertex of G that is adjacent to a pendant vertex is said to be a *support vertex*. The *complement* of the graph G, denoted by G^c, is the graph with the vertex set V and the edge set $\{xy : x, y \in V \text{ and } xy \notin E\}$. For $A \subseteq V$, let $G[A]$ denote the subgraph of G *induced*

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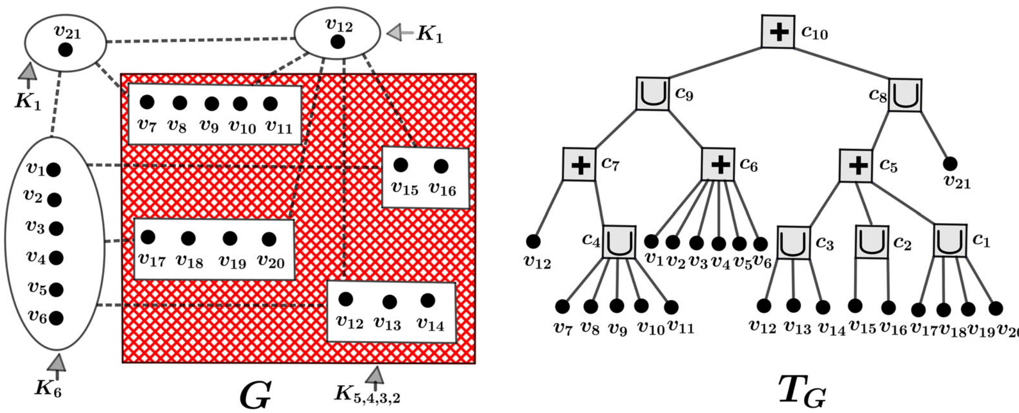

Figure 1. A cograph G and its cotree T_G (Dashed line in G represents each vertex of a set is adjacent to each vertex of another set).

by A. Let G_1 and G_2 be two graphs with disjoint vertex sets. The *disjoint union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. A complete graph G with n vertices is denoted by K_n. A graph G is *complete k-partite graph* if $V(G)$ is the union of k disjoint independent sets, called partite sets of G, and two vertices are adjacent if and only if they are in different partite sets. We denote complete k-partite graph G by $K_{m_1, m_2, \ldots, m_k}$, where m_1,m_2, … , and m_k are the cardinality of k partite sets.

A bipartite graph $G = (X,Y,E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, i.e., the vertices of X can be linearly ordered, say $x_1, x_2, \ldots, x_p$ such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$. If $G = (X,Y,E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain [9].

Cographs or *complement reducible graphs* were introduced by Lerchs [11]. Cographs are defined recursively as follows: (i) K_1 is a cograph, (ii) If G is a cograph, then G^c is a cograph, and (iii) If G_1 and G_2 are cographs, then $G_1 \cup G_2$ is a cograph. A cograph G can be represented as a rooted tree T_G, called the *cotree* of G [4]. In the cotree representation, pendant vertices of T_G are the vertices of G and each internal vertex of T_G is labelled with $\cup$ or $+$. A vertex u is said to be a *child* of a vertex v in T_G if u is adjacent to v but does not lie in the path from the root of T_G to v. Each internal vertex of T_G has at least two children. We denote the set containing the children of a vertex v in T_G as $C(T_G, v)$. A vertex of T_G with label $\cup$ (resp. $+$) corresponds to a cograph obtained from the disjoint union (resp. join) of the cographs associated with its children. A cograph G and its corresponding cotree T_G are illustrated in Figure 1.

3. Secure total domination in bounded clique-width graphs

In this section, we show that the secure total domination problem can be solved in linear time on graphs of bounded clique-width.

Let $\{E\}$ be a finite set containing one binary relation symbol E. We denote the *vocabulary* $\{E\}$ by τ_1. The presentation of any graph G as a logical structure $< V, E >$ is denoted by $G(\tau_1)$, where V is the domain of logical structure containing the vertices of G and E is the binary relation corresponding to the adjacency matrix of G.

A graph property π is expressible in τ_1-monadic second-order logic, MSOL(τ_1) for short, if π can be defined using vertices and sets of vertices of G, the logical quantifiers $\exists$ and $\forall$ over vertices and sets of vertices of G, the logical operators OR ($\vee$), AND ($\wedge$), NOT ($\neg$), the binary adjacency relation *adj*, where *adj*(u, v) holds if and only if u and v are adjacent in G, and the membership relation $\in$ to check the existence of any vertex in a set, and the equality operator $=$ for vertices of G.

Definition 3.1 ([6]). An optimization problem is a LinEMSOL(τ_1) optimization problem if it can be defined as follows: Given a graph G presented as $G(\tau_1)$ and m evaluating functions $f_1, f_2, \ldots, f_m$, find an assignment z to the free set variables in θ such that

$$\sum_{\substack{1 \le i \le l \\ l \le j \le m}} a_{ij} |z(X_i)|_j$$

$$= \min \left\{ \sum_{\substack{1 \le i \le l \\ l \le j \le m}} a_{ij} |z'(X_i)|_j : \theta(X_1, \ldots, X_l) \text{ is true for } G \text{ and } z' \right\},$$

where θ is an MSOL(τ_1) formula having free set variables $X_1, \ldots, X_l$ and for $1 \le i \le l, 1 \le j \le m, a_{ij}$ are integers, $|z(X_i)|_j = \sum_{a \in z(X_i)} f_j(a)$.

Theorem 3.2 ([1, 6]). *Let C be a class of graphs of clique-width at most k, where k is a positive integer. Then every LinEMSOL(τ_1) optimization problem on C can be solved in linear time.*

We now show that the secure total domination problem can be expressed as a LinEMSOL(τ_1) optimization problem. Given a graph G, the *total domination problem* is to find a total dominating set of G with minimum cardinality.

- Given a graph G, the following MSOL(τ_1) formula shows that G has a total dominating set.

$$\mathsf{Tdom}(X) = \forall v(\exists u(u \in X \wedge adj(u, v))).$$

- Given a graph G, the following $\mathsf{MSOL}(\tau_1)$ formula shows that G has a secure total dominating set.

$$\mathsf{STdom}(X) = \forall v(v \in X \vee (\exists u(u \in X \wedge adj(u, v)) \wedge \mathsf{Tdom}((X\backslash\{u\}) \cup \{v\})))).$$

The secure total domination problem is a $\mathsf{LinEMSOL}(\tau_1)$ optimization problem since it can be expressed as follows: given a graph G presented as $G(\tau_1)$ structure, one evaluating function f that assigns positive integer '1' to each vertex of $G(\tau_1)$, find an assignment z to the free set variable X_1 in θ such that $|z(X_1)|_1 = \min\{|z'(X_1)|_1 : \theta(X_1)$ is true for G and $z'\}$, where $\theta(X_1)$ is defined as $\theta(X_1) = \mathsf{STdom}(X_1)$. So by Theorem 3.2, we have the following theorem.

Theorem 3.3. *The secure total domination problem can be solved in linear time on graphs with clique-width bounded by a constant.*

By Theorem 3.3, it is clear that the secure total domination problem is solvable in linear time in chain graphs and cographs as clique-width of chain graphs and cographs are bounded. Courcelle et al. [6] does not give any explicit idea to obtain solution of the problem using the structure of a graph. We use the structure of chain graphs and cographs to propose linear time algorithms for computing the secure total domination number of a given chain graph and cograph. We conclude this section with the definition of a replacing vertex in a secure total dominating set of a graph G.

Definition 3.4. *Let D be a secure total dominating set of a graph G. A vertex v of D is said to be a replacing vertex for $u \in N_G(v) \cap (V\backslash D)$ if $(D\backslash\{v\}) \cup \{u\}$ is a total dominating set of G.*

4. Secure total domination in chain graphs

In this section, we present a linear time algorithm to compute the secure total domination number of a given chain graph $G = (X, Y, E)$. An ordering $\sigma = (x_1, x_2, ..., x_p, y_1, y_2, ..., y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$, and $N_G(y_q) \subseteq N_G(y_{q-1}) \subseteq \cdots \subseteq N_G(y_1)$. It is known that every chain graph admits a chain ordering [9, 15]. A chain ordering of a chain graph $G = (X, Y, E)$ can be computed in linear time [13]. The following observation is true.

Observation 4.1. ([2]). *Any secure total dominating set of a graph G contains every support vertex and pendant vertex of G.*

Theorem 4.2. *Suppose that G = (X,Y,E) is a chain graph with a chain ordering $\sigma = (x_1, x_2, ..., x_p, y_1, y_2, ..., y_q)$ and P is the set of pendant vertices of G. Let $X_P = P \cap X$ and $Y_P = P \cap Y$. Then the following are true.*

(i) *If $|X\backslash X_P| \geq 3$ and $|Y\backslash Y_P| \geq 3$, then $\gamma_{st}(G) = |P| + 4$.*
(ii) *If $|X\backslash X_P| = 2$ or $|Y\backslash Y_P| = 2$, then $\gamma_{st}(G) = |P| + 3$.*
(iii) *If $|X\backslash X_P| = 1$ and $|Y\backslash Y_P| = 1$, then $\gamma_{st}(G) = |P| + 2$.*
(iv) *If $X\backslash X_P = \emptyset$ and $Y\backslash Y_P = \emptyset$, then $\gamma_{st}(G) = 2$.*

Proof. Since σ is a chain ordering, $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$, and $N_G(y_q) \subseteq N_G(y_{q-1}) \subseteq \cdots \subseteq N_G(y_1)$. This implies that y_1 is adjacent to all vertices of X and y_2 is adjacent to all vertices of $X\backslash X_P$. Similarly, x_p is adjacent to all vertices of Y and x_{p-1} is adjacent to all vertices of $Y\backslash Y_P$. Notice that $N_G(X_P) = \{y_1\}$ and $N_G(Y_P) = \{x_p\}$. We first prove the following claim.

Claim 1. Let Q be a secure total dominating set of G. If $X\backslash X_P \not\subseteq Q$ (resp. $Y\backslash Y_P \not\subseteq Q$), then $|Q \cap (Y\backslash Y_P)| \geq 2$ (resp. $|Q \cap (X\backslash X_P)| \geq 2$).

Proof of Claim 1. Without loss of generality, assume that $X\backslash X_P \not\subseteq Q$. Let $w \in (X\backslash X_P)\backslash Q$ be an arbitrary vertex. Since Q is a secure total dominating set of G and $w \notin Q$, there exits a replacing vertex for w in Q, say w'. Clearly $w' \in Q \cap Y$ as G is a chain graph. Now $(Q\backslash\{w'\}) \cup \{w\}$ is a total dominating set of G. Since w is not adjacent to any vertex of X, $|N_G(w) \cap (Q \cap Y)| \geq 2$. This implies that $|Q \cap Y| \geq 2$. We now show that $|Q \cap (Y\backslash Y_P)| \geq 2$. First suppose that w $= x_p$. Since $x_p \notin Q, Y_P = \emptyset$; otherwise Q is a not a secure total dominating set of G as $N_G(Y_P) = \{x_p\}$. This implies that $Y\backslash Y_P = Y$. So $|Q \cap (Y\backslash Y_P)| \geq 2$ as $|Q \cap Y| \geq 2$. Now suppose that $w \neq x_p$. Since $N_G(Y_P) = \{x_p\}$ and $x_p \neq w$, w is not adjacent to any vertex of Y_P. So $|N_G(w) \cap (Q \cap (Y\backslash Y_P))| \geq 2$ as $|N_G(w) \cap (Q \cap Y)| \geq 2$. Hence $|Q \cap (Y\backslash Y_P)| \geq 2$. This completes the proof of the claim. □

(i) Let $|X\backslash X_P| \geq 3$ and $|Y\backslash Y_P| \geq 3$. We first show that $\gamma_{st}(G) \leq |P| + 4$. Let $Q = P \cup \{x_p, x_{p-1}, y_1, y_2\}$. Notice that Q is a total dominating set of G as $x_p, y_1 \in Q$. Let $u \in V(G)\backslash Q$ be an arbitrary vertex. If $u \in X\backslash X_P$, then $(Q\backslash\{y_2\}) \cup \{u\}$ is a total dominating set of G as $x_p, y_1 \in Q$. This implies that y_2 is a replacing vertex for u. Similarly, if $u \in Y\backslash Y_P$, then x_{p-1} is a replacing vertex for u. So Q is a secure total dominating set of G and hence $\gamma_{st}(G) \leq |P| + 4$.
We now show that $|P| + 4 \leq \gamma_{st}(G)$. Let Q' be a minimum secure total dominating set of G. Since Q' is total dominating set of G, $Q' \cap X \neq \emptyset$ and $Q' \cap Y \neq \emptyset$. Let $u \in Q' \cap X$ and $v \in Q' \cap Y$. By Observation 4.1, $P \subset Q'$. First assume that $X\backslash X_P \subseteq Q'$. Since $|X\backslash X_P| \geq 3$ and $\{v\} \cup P \subset Q', |P| + 4 \leq Q'$. Hence we are done. Now assume that $X\backslash X_P \not\subseteq Q'$. By Claim 1, $|Q' \cap (Y\backslash Y_P)| \geq 2$. If $|Q' \cap (Y\backslash Y_P)| \geq 3$, then $|P| + 4 \leq \gamma_{st}(G)$ as $\{u\} \cup P \subseteq Q'$ and $|Q' \cap (Y\backslash Y_P)| \geq 3$. Hence we are done. Now assume $|Q' \cap (Y\backslash Y_P)| \leq 2$. This implies that $|Q' \cap (Y\backslash Y_P)| = 2$ as $|Q' \cap (Y\backslash Y_P)| \geq 2$. So $Y\backslash Y_P \not\subseteq Q'$ as $|Y\backslash Y_P| \geq 3$. By Claim 1, $|Q' \cap (X\backslash X_P)| \geq 2$. Since $|Q' \cap (X\backslash X_P)| \geq 2$, $|Q' \cap (Y\backslash Y_P)| = 2$, and $P \subseteq Q', |P| + 4 \leq \gamma_{st}(G)$. Therefore, $\gamma_{st}(G) = |P| + 4$.

(ii) Without loss of generality, assume that $|X\backslash X_P| = 2$. This implies that $|Y\backslash Y_P| \geq 2$. We first show that $\gamma_{st}(G) \leq |P| + 3$. Let $Q = P \cup \{x_p, x_{p-1}, y_1\}$. Notice that Q is a total dominating set of G as $x_p, y_1 \in Q$. Let $u \in V(G)\backslash Q$ be an arbitrary vertex. Clearly $u \notin Y_P$ as $Y_P \subset Q$. Since $u \notin Y_P, ux_{p-1} \in E(G)$. Now $(Q\backslash\{x_{p-1}\}) \cup \{u\}$ is a total dominating set of G as $x_p, y_1 \in Q$. Hence x_{p-1} is a replacing vertex for u. So Q is a secure total dominating set of G. Therefore, $\gamma_{st}(G) \leq |P| + 3$.

We now show that $|P| + 3 \leq \gamma_{st}(G)$. Let Q' be a minimum secure total dominating set of G. Since Q' is total dominating set of G, $Q' \cap X \neq \emptyset$ and $Q' \cap Y \neq \emptyset$. Let $u \in Q' \cap X$ and $v \in Q' \cap Y$. By Observation 4.1, $P \subset Q'$. If $X\backslash X_P \subseteq Q'$, then $|P| + 3 \leq \gamma_{st}(G)$ as $|X\backslash X_P| = 2$ and $\{v\} \cup P \subset Q'$. Hence we are done. So assume that $X\backslash X_P \nsubseteq Q'$. By Claim 1, $|Q' \cap (Y\backslash Y_P)| \geq 2$. Since $\{u\} \cup P \subset Q'$ and $|Q' \cap (Y\backslash Y_P)| \geq 2, |P| + 3 \leq \gamma_{st}(G)$. Therefore, $\gamma_{st}(G) = |P| + 3$.

(iii) Let $|X\backslash X_P| = 1$ and $|Y\backslash Y_P| = 1$. This implies that $V(G) = P \cup \{x_p, y_1\}$. By Observation 4.1, $\gamma_{st}(G) = |P| + 2$.

(iv) Let $X\backslash X_P = \emptyset$ and $Y\backslash Y_P = \emptyset$. This implies that $G = K_2$. Therefore, $\gamma_{st}(G) = 2$. □

A chain ordering of a chain graph $G = (X, Y, E)$ can be computed in linear time [13]. The set P of pendant vertices of G can be computed in $O(n + m)$ time. Then by Theorem 4.2, a minimum secure total dominating set of a chain graph G can be computed in $O(n + m)$ time.

5. Secure total domination in cographs

In this section, we propose a linear time algorithm to compute the secure total domination number of a given cograph G. The following observations are easy to verify.

Observation 5.1. If G is the graph obtained from the join of the graphs $G_1, G_2, \ldots, G_k; k \geq 2$, then {u, v}, where $u \in V(G_i), v \in V(G_j)$ for $1 \leq i, j \leq k$ and $i \neq j$, is a minimum total dominating set of G.

Observation 5.2. If G is the graph obtained from the disjoint union of the graphs $G_1, G_2, \ldots, G_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i)$ and $\gamma_{st}(G) = \sum_{i=1}^{k} \gamma_{st}(G_i)$.

If G is a disconnected graph with components $G_1, \ldots, G_k$, then by Observation 5.2, $\gamma_{st}(G) = \sum_{i=1}^{k} \gamma_{st}(G_i)$. So from now onwards, we only consider connected cographs. We first prove some lemmas that will help in designing our algorithm.

If G is the cograph obtained from the join of $G_1, G_2, \ldots, G_k; k \geq 2$, where each G_i is either K_1 or a disconnected graph, then in Lemma 5.3, we characterize when the secure total domination number of G is 2. Also in Lemma 5.4, we give a sufficient condition when the secure total domination number of G is 3.

Lemma 5.3. *Suppose that G is the cograph obtained from the join of the graphs $G_1, G_2, \ldots, G_k; k \geq 2$, where each G_i is either K_1 or a disconnected graph. Then $\gamma_{st}(G) = 2$ if and only if there exist p and q, where $1 \leq p, q \leq k$ and $p \neq q$, such that G_p and G_q are K_1.*

Proof. First assume that $\gamma_{st}(G) = 2$ and $Q = \{u, v\}$ is a minimum secure total dominating set of G. Since Q is a total dominating set of G, $uv \in E(G)$ and $N_G(u) \cup N_G(v) = V(G)$. We now show that $N_G[u] = N_G[v] = V(G)$. If possible, let $w \in V(G)\backslash Q$ be a vertex such that $uw \notin E(G)$. Since $w \notin Q$ and Q is a minimum secure total dominating set of G, v is a replacing vertex for w. So $(Q\backslash\{v\}) \cup \{w\}$ is a total dominating set of G. This implies that $uw \in E(G)$. This is a contradiction to the fact that $uw \notin E(G)$. So $N_G[u] = V(G)$. Clearly $u \in V(G_a)$ for some $1 \leq a \leq k$. Notice that G_a is either K_1 or a disconnected graph. Since $N_G[u] = V(G)$, u is adjacent to each vertex of $G_a\backslash\{u\}$. This implies that G_a is connected and hence G_a is K_1. Similarly, $N_G[v] = V(G)$ and G_b is K_1, where $y \in V(G_b)$ and $1 \leq b \leq k$.

Next assume that G_p and G_q are K_1, where $1 \leq p, q \leq k$ and $p \neq q$. Let $V(G_p) = \{u'\}$ and $V(G_q) = \{v'\}$. Notice that $N_G[u'] = N_G[v'] = V(G)$. Let $Q' = \{u', v'\}$. By Observation 5.1, Q' is a minimum total dominating set of G. Let $w \in V(G)\backslash Q'$ be an arbitrary vertex. Clearly $w \in V(G_l)$ for some $1 \leq l \leq k$. Notice that $l \neq p$ and $l \neq q$ as G_p and G_q are K_1. Clearly $u'w \in E(G)$ as $N_G[u'] = V(G)$. By Observation 5.1, $(Q\backslash\{u'\}) \cup \{w\}$ is a minimum total dominating set of G. This implies that u' is a replacing vertex for w. So Q' is a minimum secure total dominating set of G and hence $\gamma_{st}(G) = 2$. □

Lemma 5.4. *Suppose that G is the cograph obtained from the join of the graphs $G_1, G_2, \ldots, G_k$, where each G_i is either K_1 or a disconnected graph. If $k \geq 3$ and there exists at most one i, where $1 \leq i \leq k$, such that G_i is K_1, then $\gamma_{st}(G) = 3$.*

Proof. By Lemma 5.3, $\gamma_{st}(G) > 2$ as there exists at most one i, where $1 \leq i \leq k$, such that G_i is K_1. Let $Q = \{u_1, u_2, u_3\}$, where $u_1 \in V(G_1), u_2 \in V(G_2)$, and $u_3 \in V(G_3)$. By Observation 5.1, $Q\backslash\{u\}$, where $u \in \{u_1, u_2, u_3\}$, is a minimum total dominating set of G. This implies that Q is a total dominating set of G. We now show that Q is a secure total dominating set of G. Let $w \in V(G)\backslash Q$ be an arbitrary vertex. Then there exists a vertex $v \in Q$ such that $vw \in E(G)$. Since $Q\backslash\{v\}$ is a minimum total dominating set of G, $(Q\backslash\{v\}) \cup \{w\}$ is a total dominating set of G. This implies that v is a replacing vertex for w. So Q is a secure total dominating set of G and hence $\gamma_{st}(G) = 3$. □

If G is the cograph obtained from the join of K_1 and a disconnected graph, then in Lemma 5.6, we find the secure total domination number of G. For this, we first prove the following lemma.

Lemma 5.5. *Suppose that G is the cograph obtained from the join of two graphs G_1 and G_2, where G_1 is a disconnected graph and $G_2 = \{u\}$. Then there exists a minimum secure total dominating set of G containing u.*

Proof. Let Q' be a minimum secure total dominating set of G. If $u \in Q'$, then we are done. So assume that $u \notin Q'$. Since Q' is a minimum secure total dominating set of G and $u \notin Q'$, there exists a replacing vertex for u in Q', say v. Let $Q = (Q' \backslash \{v\}) \cup \{u\}$. Since $u \in Q$, Q is a dominating set of G. Also $vw \in E(G)$ for some $w \in Q'$ as Q' is a total dominating set of G. Clearly $w \in Q$ as $w \neq v$. Notice that $uw \in E(G)$. This implies that Q is a total dominating set of G. We now show that Q is a secure total dominating set of G. Let $x \in V(G) \backslash Q$ be an arbitrary vertex. Clearly $x \in V(G_1)$ as $u \in Q$. If $x = v$, then u is a replacing vertex for x as Q' is a total dominating set of G. If $x \neq v$, then $x \in V(G) \backslash Q'$. This implies that there exists a replacing vertex for x in Q', say y. If $y \neq v$, then $y \in Q$. Now $(Q \backslash \{y\}) \cup \{x\}$ is a total dominating set of G as $u \in Q$ and u is adjacent to each vertex of G_1. So y is a replacing vertex for x in Q. Again if $y = v$, then $(Q \backslash \{u\}) \cup \{x\}$ is a total dominating set of G as $(Q' \backslash \{v\}) \cup \{x\}$ is a total dominating set of G. So u is a replacing vertex for x in Q. Hence Q is a secure total dominating set of G containing u. Since $|Q'| = |Q|$, Q is a minimum secure total dominating set of G containing u. □

Lemma 5.6. *Suppose that G is the cograph obtained from the join of two graphs G_1 and G_2, where G_1 is a disconnected graph and $G_2 = \{u\}$. Then $\gamma_{st}(G) = 1 + \gamma(G_1)$.*

Proof. We first show that $1 + \gamma(G_1) \leq \gamma_{st}(G)$. By Lemma 5.5, there is a minimum secure total dominating set Q of G containing u. Let $U_1, U_2, \ldots,$ and U_k be the components of G_1. Assume that $|Q \cap V(U_i)| > \gamma(U_i)$ for some $1 \leq i \leq k$. Let $Q' = (Q \backslash (Q \cap V(U_i))) \cup D(U_i)$, where $D(U_i)$ is a minimum dominating set of U_i. Notice that Q' is a dominating set of G as $u \in Q'$. Since u is adjacent to each vertex of $Q' \backslash \{u\}, Q'$ is a total dominating set of G. We now show that Q' is a secure total dominating set of G. Let $x \in V(G) \backslash Q'$ be an arbitrary vertex. Clearly $x \in V(G_1) \backslash Q'$ as $u \in Q'$. This implies that $x \in V(U_j)$ for some $1 \leq j \leq k$. If $j \neq i$, then $x \in V(G) \backslash Q$. This implies that there exists a replacing vertex for x in Q, say y. Notice that $y \in Q'$ as $y \notin V(U_i)$. Now $(Q' \backslash \{y\}) \cup \{x\}$ is a total dominating set of G as $u \in Q'$ and u is adjacent to each vertex of G_1. Again if $j = i$, then $xz \in E(G)$ for some $z \in D(U_i)$. Such a vertex z exists as $D(U_i)$ is a minimum dominating set of U_i. Now $(Q' \backslash \{z\}) \cup \{x\}$ is a total dominating set of G as $u \in Q'$ and u is adjacent to each vertex of G_1. This implies that z is a replacing vertex for x in Q'. Hence Q' is a secure total dominating set of G such that $|Q'| < |Q|$. This is a contradiction to the fact that Q is a minimum secure total dominating set of G. So $|Q \cap V(U_i)| \leq \gamma(U_i)$. By Observation 5.2, $\gamma(G_1) = \sum_{i=1}^{k} \gamma(U_k)$. So $1 + \gamma(G_1) \leq \gamma_{st}(G)$.

We now show that there exists a secure total dominating set Q of G such that $|Q| = 1 + \gamma(G_1)$. For each $1 \leq i \leq k$, let $D(U_i)$ be a minimum dominating set of U_i. Again let $Q = \{u\} \cup_{i=1}^{k} D(U_i)$. Since $u \in Q$, Q is a dominating set of G. Also u is adjacent to each vertex of $Q \backslash \{u\}$. So Q is a total dominating set of G. We now show that Q is a secure total dominating set of G. Let $x \in V(G) \backslash Q$ be an arbitrary vertex. Clearly $x \in V(G_1) \backslash Q$ as $u \in Q$. This implies that $x \in V(U_i)$ for some $1 \leq i \leq k$. Let $y \in D(U_i)$ such that $xy \in E(G)$. Such a vertex y exists as $D(U_i)$ is a minimum dominating set of U_i. Now $(Q \backslash \{y\}) \cup \{x\}$ is a total dominating set of G as $u \in Q$ and u is adjacent to each vertex of G_1. So y is a replacing vertex for x. Hence Q is a secure total dominating set of G. By Observation 5.2, $\gamma(G_1) = \cup_{i=1}^{k} D(U_i)$. So $|Q| = 1 + \gamma(G_1)$ and hence $\gamma_{st}(G) \leq 1 + \gamma(G_1)$. □

If G is the cograph obtained from the join of two graphs G_1 and G_2 such that $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$, then in Lemma 5.7, we give an upper bound to the secure total domination number of G.

Lemma 5.7. *If G is the cograph obtained from the join of two graphs G_1 and G_2 such that $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$, then $\gamma_{st}(G) \leq 4$.*

Proof. Let $Q = \{u_1, u_2, v_1, v_2\}$, where $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$. By Observation 5.1, $Q \backslash \{u_i, v_j\}$ is a minimum total dominating set of G, where $i, j \in \{1, 2\}$. This implies that Q and $Q \backslash \{w\}$, where $w \in \{u_1, u_2, v_1, v_2\}$, is a total dominating set of G. Let $x \in V(G) \backslash Q$ be an arbitrary vertex. If $x \in V(G_1)$, then $v_1 x \in E(G)$. Now $(Q \backslash \{v_1\}) \cup \{x\}$ is a total dominating set of G as $Q \backslash \{v_1\}$ is a total dominating set of G. This implies that v_1 is a replacing vertex for x. Similarly, if $x \in V(G_2)$, then u_1 is a replacing vertex for x. So Q is a secure total dominating set of G and hence $\gamma_{st}(G) \leq 4$. □

If G is the cograph obtained from the join of two disconnected graphs, then in Lemma 5.8, we give a necessary and sufficient condition under which the secure total domination number of G is 3.

Lemma 5.8. *Suppose that G is the cograph obtained from the join of two disconnected graphs G_1 and G_2. Then $\gamma_{st}(G) = 3$ if and only if $\gamma(G_i) = 2$ for some $i \in \{1, 2\}$.*

Proof. Assume that $\gamma_{st}(G) = 3$ and Q be a minimum secure total dominating set of G. We first show that $V(G_i) \cap Q \neq \emptyset$ for each $i \in \{1, 2\}$. On the contrary, assume that $V(G_1) \cap Q = \emptyset$. This implies that $|V(G_2) \cap Q| = 3$. Notice that G_2 is a disconnected graph. So $\gamma_t(G_2) \geq 4$. This implies that Q is not a total dominating set of G_2. This is a contradiction to the fact that Q is a minimum secure total dominating set of G. So $V(G_i) \cap Q \neq \emptyset$ for each $i \in \{1, 2\}$. Since $\gamma_{st}(G) = 3, |V(G_i) \cap Q| = 2$ for some $i \in \{1, 2\}$. Without loss of generality, assume that $|V(G_1) \cap Q| = 2$. Let $V(G_1) \cap Q = \{u, v\}$ and $V(G_2) \cap Q = \{w\}$. We now show that $\{u, v\}$ is a dominating set of G_1. If possible, let $x \in V(G_1) \backslash (N_{G_1}[u] \cup N_{G_1}[v])$ be an arbitrary vertex. Notice that $x \notin Q$. Since $x \in V(G_1) \backslash Q$ and Q is a minimum secure total dominating set of G, w is a replacing vertex for x. So $(Q \backslash \{w\}) \cup \{x\}$ is a total dominating set of G. This implies that $xu \in E(G)$ or $xv \in E(G)$. This contradicts the fact that $x \in V(G_1) \backslash (N_{G_1}[u] \cup N_{G_1}[v])$. So $\{u, v\}$ is a dominating set of G_1. Since G_1 is a disconnected graph, $\gamma(G_1) \geq 2$. This implies that $\{u, v\}$ is a minimum dominating set of G_1 and hence $\gamma(G_1) = 2$.

We now prove the sufficient part of this lemma. Assume that $\gamma(G_i) = 2$ for some $i \in \{1, 2\}$. Without loss of generality, assume that $\gamma(G_1) = 2$. Since $\gamma(G_1) = 2$ and G_1 is disconnected, G_1 has exactly two components, say U_1 and U_2. By Observation 5.2, $\gamma(U_1) = \gamma(U_2) = 1$. Let $\{u\}$ and $\{v\}$ be minimum dominating set of U_1 and U_2, respectively. Again let $Q = \{u, v, w\}$, where $w \in V(G_2)$. By Observation 5.1, $Q \backslash \{x\}$, where $x \in \{u, v\}$, is a minimum total dominating set of G. This implies that Q is a total dominating set of G. We now show that Q is a secure total dominating set of G. Let $y \in V(G) \backslash Q$ be an arbitrary vertex. If $y \in V(U_1) \cup V(G_2)$, then $yu \in E(G)$. Now $(Q \backslash \{u\}) \cup \{y\}$ is a total dominating set of G as $Q \backslash \{u\}$ is a minimum total dominating set of G. So u is a replacing vertex for y. If $y \in V(U_2)$, then $yv \in E(G)$. Now $(Q \backslash \{v\}) \cup \{y\}$ is a total dominating set of G as $Q \backslash \{v\}$ is a minimum total dominating set of G. So v is a replacing vertex for y. Hence Q is a secure total dominating set of G. Since G_1 and G_2 are disconnected, by Lemma 5.3, $\gamma_{st}(G) > 2$. This implies that Q is a minimum secure total dominating set of G. Hence $\gamma_{st}(G) = 3$. □

We now present our algorithm, namely MIN-STDN-CG(G) that computes the secure total domination number of a connected cograph G. We use a vertex ordering $\lambda = (c_1, c_2, \ldots, c_l)$ of cotree T_G, where $\lambda^{-1} = (c_l, \ldots, c_2, c_1)$ is an ordering of internal vertices of T_G obtained by applying breadth-first search starting at the root c_l of T_G. We call such an ordering λ of T_G as *Cotree-RBFS* of T_G. For any vertex v of T_G, we denote $T_G(v)$ as the subgraph of G induced by the pendant vertices of the subtree of T_G rooted at the vertex v. We now discuss in detail the approach of our algorithm MIN-STDN-CG(G).

[S1] The algorithm constructs a cotree T_G and computes Cotree-RBFS ordering $\lambda = (c_1, c_2, \ldots, c_l)$ of internal vertices of T_G.

[S2] Let $C(T_G, c_l) = \{x_1, \ldots, x_k\}$. Then $T_G(c_l)$ is the cograph obtained from the join of the graphs $T_G(x_1), \ldots,$ and $T_G(x_k)$ as c_l has label $+$. Now one of the following case arise.

Case 1: At least two children of c_l in T_G are pendant vertices.

In this case, $\gamma_{st}(T_G(c_l)) = 2$. This follows from Lemma 5.3.

Case 2: At most one child of c_l in T_G is a pendant vertex.

In this case, one of the following is true.

- If c_l has at least three children, then $\gamma_{st}(T_G(c_l)) = 3$. This follows from Lemma 5.4.
- If $C(T_G, c_l) = \{x_1, x_2\}$ and one of them, say x_2 is the pendant vertex, then $\gamma_{st}(T_G(c_l)) = 1 + \gamma(T_G(x_1))$. This follows from Lemma 5.6.
- If $C(T_G, c_l) = \{x_1, x_2\}$ and x_1 and x_2 are not pendant vertices, then $\gamma_{st}(T_G(c_l)) \leq 4$. This follows from Lemma 5.7. In particular, if $\gamma(T_G(x_1)) = 2$ or $\gamma(T_G(x_2)) = 2$, then $\gamma_{st}(T_G(c_l)) = 3$. This follows from Lemma 5.8.

We now present our algorithm Min-STDN-CG(G) that computes the secure total domination number of a given connected cograph G.

Algorithm 1: Min-STDN-CG(G)

Input: A connected cograph $G = (V, E)$;
Output: $\gamma_{st}(G)$;

Compute a Cotree-RBFS ordering $\lambda = (c_1, c_2, \ldots, c_l)$ of cotree T_G;
Let $C(T_G, c_l) = \{x_1, x_2, \ldots, x_k\}$;
if (c_l is adjacent to at least two pendant vertices) **then**
    $\gamma_{st}(T_G(c_l)) = 2$;
else if ($k \geq 3$) **then**
    $\gamma_{st}(T_G(c_l)) = 3$;
else if (c_l is adjacent to a pendant vertex) **then**
    $\gamma_{st}(T_G(c_l)) = 1 + \gamma(T_G(x_1))$, where x_1 is the non pendant vertex.
else if (c_l is not adjacent to a pendant vertex) **then**

$$\gamma_{st}(T_G(c_l)) = \begin{cases} 3 & \text{if } \gamma(T_G(x_1)) = 2 \text{ or } \gamma(T_G(x_2)) = 2; \\ 4 & \text{otherwise.} \end{cases}$$

$\gamma_{st}(G) = \gamma_{st}(T_G(c_l))$;

The proof of the correctness of the algorithm follows from Lemma 5.3, Lemma 5.4, Lemma 5.6, Lemma 5.7, and Lemma 5.8 as discussed in [S1] and [S2]. So the algorithm MIN-STDN-CG(G) correctly computes $\gamma_{st}(G)$ of a given connected cograph G. In the cograph G shown in Figure 1, the root c_{10} is adjacent to no pendant vertices and it has only two children. Notice that $\gamma(T_G(c_8)) = 3$ and $\gamma(T_G(c_9)) = 2$. Since $\gamma(T_G(c_9)) = 2$, by Lemma 5.8, $\gamma_{st}(T_G(c_{10})) = 3$. Hence $\gamma_{st}(G) = 3$.

We now discuss the running time of the algorithm MIN-STDN-CG(G). Assume that G is a connected cograph with n vertices and m edges. A cotree T_G of G can be constructed in $O(n + m)$ time [5]. Let $\{c_i : 1 \leq i \leq l\}$ be the set of internal vertices of T_G. Cotree-RBFS ordering λ of T_G can be computed in $O(|V(T_G)| + |E(T_G)|)$ time by using breadth first search. At the root c_l of T_G, our algorithm computes $\gamma_{st}(T_G(c_l))$. In some cases, to compute $\gamma_{st}(T_G(c_l))$, the algorithm uses $\gamma(T_G(x))$, where $x \in C(T_G, c_l)$. For this we store $\gamma(T_G(c_i))$ at each vertex c_i of T_G. Notice that for any connected cograph G, $\gamma(G) \leq 2$.

Computation of $\gamma(T_G(c_i))$:

If c_i is a pendant vertex, then $\gamma(T_G(c_i)) = 1$. If c_i is a vertex with label $\cup$, then $\gamma(T_G(c_i)) = \sum_{x \in C(T_G, c_i)} \gamma(T_G(x))$. Assume that c_i is a vertex with label $+$. If c_i has a child that is a pendant vertex, then $\gamma(T_G(c_i)) = 1$; otherwise $\gamma(T_G(c_i)) = 2$. So at any vertex c_i of T_G, the computation of $\gamma(T_G(c_i))$ can be done in $O(|N_{T_G(c_i)}|)$ time.

Computation of $\gamma_{st}(T_G(c_l))$:

The algorithm first checks whether at least two children of c_l in T_G are pendant vertices. This can be done in $O(|N_{T_G}(c_l)|)$ time. If at most one child of c_l in T_G is a pendant vertex, then the algorithm checks whether c_l has at least three children in T_G. This can be done in $O(|N_{T_G}(c_l)|)$ time. If c_l has exactly two children in T_G, then let $C(T_G, c_l) = \{x_1, x_2\}$. If x_1 and x_2 are not pendant vertices, then the algorithm checks whether $\gamma(T_G(x_1)) = 2$ or $\gamma(T_G(x_2)) = 2$. This can be done in $O(|N_{T_G}(c_l)|)$ time. If only one of them is pendant vertex, say x_2, then $\gamma_{st}(T_G, c_l) = 1 + \gamma(T_G(x_1))$. This can be done in $O(|N_{T_G}(c_l)|)$ time.

Let $C(T_G, c_l) = \{x_1, x_2, ..., x_k\}$. If $\gamma(T_G(x_i))$ is known for each $1 \leq i \leq k$, then computation of $\gamma_{st}(T_G(c_l))$ can be done in $O(|N_{T_G}(c_l)|)$ time. So in total the algorithm MIN-STDN-CG(G) takes $O(|V(T_G)| + |E(T_G))$ to compute $\gamma_{st}(T_G(c_l)) = \gamma_{st}(G)$ and hence the algorithm MIN-STDN-CG(G) takes $O(n+m)$ time. Therefore, we have the following theorem.

Theorem 5.9. *The secure total domination number of a connected cograph can be computed in linear time.*

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The author declares no conflict of interest.

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