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Some results on the existence of Hamiltonian cycles in $\mathcal{P}$-compositions of bipartite digraphs

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ABSTRACT

Let D be a digraph on n vertices $s_1, \ldots, s_n$ and let $D_1, \ldots, D_n$ be a family of vertex-disjoint bipartite digraphs. We think of $D_1, \ldots, D_n$ as 2-colored digraphs with the same color set. The $\mathcal{P}$-composition $D[D_1, \ldots, D_n]^{\mathcal{P}}$ is the digraph obtained from D by replacing each vertex s_i of D by D_i and adding an arc from each vertex of D_i to each vertex of D_j if and only if those vertices have different color and $s_i \rightarrow s_j$ is an arc of D (with $i, j \in [n]$ and $i \neq j$). Notice that this is a generalization of the usual composition D, whenever each of the digraphs $D_1, \ldots, D_n$ are k-partite, we obtain a k-partite digraph as a $\mathcal{P}$-composition and this digraph is a subdigraph of the usual composition $D[D_1, \ldots, D_n]$. In our case, we obtain a bipartite digraph. Particularly, the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, when $\mathcal{C}_n$ is the 2- or 3-cycle and each D_i is a semicomplete bipartite digraph, is also a semicomplete bipartite digraph. Gutin, Häggkvist and Manoussakis proved that a semicomplete bipartite digraph has a Hamiltonian cycle if and only if it is strong and has a cycle-factor. In this article, we prove that a digraph $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where $\mathcal{C}_n$ is the n-cycle and each D_i is a strong semicomplete bipartite digraph, has a Hamiltonian cycle if and only if it is strong and has a cycle-factor.

KEYWORDS
Generalization of tournaments; bipartite tournaments; Hamiltonian cycles; graph products; cycle factors

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The study of the generalizations of tournaments, as they were posed by Bang-Jensen in [1], had involved several classes of digraphs, like locally semicomplete digraphs, quasi-transitive digraphs, arc-locally semicomplete digraphs, path mergeable digraphs, etc. (see [2]). Those classes of digraphs are mainly couched in terms of properties on the adjacency, their neighborhoods or the structure of their paths. From those studies have arisen characterizations in terms of operation of digraphs and, particularly, in terms of compositions of digraphs. In this work, we are interested in an analogous operation which preserves the property of being multipartite.

The compositions $\mathcal{C}_n[S_1, \ldots, S_n]$, where $\mathcal{C}_n$ is the n-cycle and each S_i is a semicomplete digraph, are locally semicomplete digraphs, this is, for each vertex v of $\mathcal{C}_n[S_1, \ldots, S_n]$, the subdigraphs spanned by the out-neighborhood and in-neighborhood of v are semicomplete. In [1], Bang-Jensen proved that a locally semicomplete digraph has a Hamiltonian cycle if and only if it is strong.

Gutin, Häggkist and Manoussakis characterized the existence of Hamiltonian cycles in semicomplete bipartite digraphs in [5, 6]:

Theorem 1. *A semicomplete bipartite digraph D has a Hamiltonian cycle if and only if D is strong and contains a cycle-factor (that is, a collection of vertex-disjoint cycles covering all the vertices of D).*

Galeana-Sánchez and Goldfeder introduced in [3] an operation in digraphs which preserves multipartiteness, the $\mathcal{P}$-composition. Let $\mathcal{C}_n$ be the n-cycle and let $D_1, \ldots, D_n$ be **strong semicomplete bipartite** digraphs, $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$ is a bipartite subdigraph of $\mathcal{C}_n[S_1, \ldots, S_n]$. In fact, it is a maximal bipartite subdigraph.

In this work, we prove that this characterization is also true for digraphs of the kind $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where $\mathcal{C}_n$ es the n-cycle and $D_1, \ldots, D_n$ are strong semicomplete bipartite digraphs.

Theorem 2. *Let H be a digraph with vertex set $V(H) = \{v_1, \ldots, v_n\}$ and let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with ordered-partitions given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, H is Hamiltonian if and only H has a cycle-factor.*

1. Definitions

For general concepts we refer the reader to [2]. In this article, $D = (V(D), A(D))$ denotes a loopless *directed graph*

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(digraph) with at most one arc from u to v for every pair of vertices u and v of $V(D)$.

We denote an arc (u,v) in $A(D)$ by $u \rightarrow v$ or uv. Two distinct vertices u and v are *adjacent* if $u \rightarrow v$ or $v \rightarrow u$. An *independent set* is a set of vertices of pairwise nonadjacent vertices of D. If A and B are sets of vertices or subdigraphs of a given digraph, $A \rightarrow B$ means that for every vertex $a \in A$ and every vertex $b \in B$, we have $a \rightarrow b$. If A and B are sets of vertices or subdigraphs of a given digraph, $(A,B)_D$ denotes the set of arcs of the digraph D with tail in A and head in B.

Our paths and cycles are always directed. A *cycle-factor* of a digraph D is a collection $\mathcal{F}$ of pairwise vertex-disjoint cycles in D such that all vertices of D are in $\mathcal{F}$. A cycle-factor with k elements is a *k-cycle-factor*. We denote $\mathcal{F}$ by $\mathcal{F} = C_1 \cup \cdots \cup C_k$. An *almost cycle-factor* of a digraph D is a collection of vertex-disjoint subdigraphs of D covering $V(D)$ and such that one of the subdigraphs is a path and the others are cycles. Given a path or a cycle $P = v_0 v_1 \cdots v_p$ and $i, j \in \{1, \ldots, p\}$ such that $i < j$, $P[x_i, x_j]$ denotes the subpath $v_i v_{i+1} \cdots v_j$ of P.

A digraph is said to be *strong* if for each pair of vertices u, v there exists a path from u to v.

A digraph D is *multipartite* or *k-partite* if there exists a k-partition $V_1, \ldots, V_k$ of the vertex set of D (*i.e.*, $\cup_{i=1}^{k} V_i = V(D), V_i \neq \emptyset$ for all i and, $V_i \cap V_j = \emptyset$ whenever $i \neq j$) such that each V_i is an independent set. When $k = 2$, we say that D is *bipartite*.

A *tournament* is a digraph such that for any distinct vertices u, v, exactly one of $u \rightarrow v$ and $v \rightarrow u$ is an arc of D. A digraph D is *semicomplete* if for each pair of distinct vertices u, v, there exists at least one arc $u \rightarrow v$ or $v \rightarrow u$ in $A(D)$.

A spanning cycle C of D is a *Hamiltonian* cycle and D is *Hamiltonian* if it has a Hamiltonian cycle.

For an positive integer n, $[n]$ will denote the set $\{1, 2, \ldots, n\}$ and $[n]_0$ will denote the set $\{0, 1, \ldots, n-1\}$.

1.1. P-composition

Definition 3. Let D be a multipartite digraph. An ordered-partition $\mathcal{P}(D) = (V_k, \ldots, V_k)$ of D is a fixed ordering of the partite sets of D.

Definition 4. Let D be digraph with vertex set $V(D) = \{v_1, \ldots, v_n\}$ and let $D_1, \ldots, D_\mathrm{n}$ be k-partite digraphs with ordered-partitions $\mathcal{P}(D_i) = (V_1^i, \ldots, V_k^i)$, the $\mathcal{P}$-composition according to the ordered-partition $\mathcal{P} = ((\cup_{i=1}^{n} V_1^i) = V_1, \ldots, (\cup_{i=1}^{n} V_k^i) = V_k)$, denoted by $D[D_1, \ldots, D_n]^{\mathcal{P}}$, is the digraph H with vertex set $V(H) = \cup_{i=1}^{k} V_i$ and, for w, $z \in V(H)$, the arc $w \rightarrow z$ is in $A(H)$ if and only if

- w and z are both in D_i and $w \rightarrow z$ is in D_i or
- w is in V_k^i, z is in V_g^j with $k \neq g, i \neq j$ and $s_i \rightarrow s_j$ is in D.

We call D_i a summand of H, for all $i \in [n]$. See Figure 1.

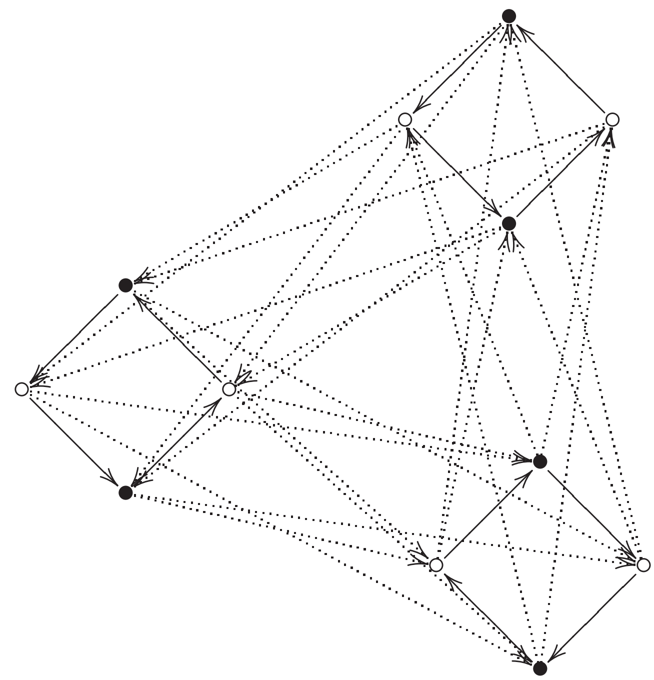

Figure 1. $C_3[C_4, C_4, C_4]^{\mathcal{P}}$.

Definition 5. If A and B are sets of vertices or subdigraphs of a multipartite digraph D with ordered-partition $\mathcal{P}(D)$, $A \rightarrow^{\mathcal{P}} B$ means that for every vertex $a \in A$ and every vertex $b \in B$ in different partite sets, we have $a \rightarrow b$.

In what follows, we only consider compositions of the type $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where $\mathcal{C}_n$ is the n-cycle and each D_i is a strong semicomplete bipartite digraph, let us call it H. Notice that H is always strong because $\mathcal{C}_n$ is strong. Furthermore, if C is a cycle in H, then either there exists $i \in [n]$ such that C is contained in the summand D_i or C passes through every summand of H. Since we only work on bipartite digraphs, we assume that each digraph D_i has an ordered-partition given by $\mathcal{P}(D_i) = (\mathcal{P}_1(D_i), \mathcal{P}_2(D_i))$. Consequently, for H we have that $\mathcal{P}_1(H) = \cup_{i=1}^{n} \mathcal{P}_1(D_i)$ and $\mathcal{P}_2(H) = \cup_{i=1}^{n} \mathcal{P}_2(D_i))$. Notice that the set $\{(\mathcal{P}_1(H), \mathcal{P}_2(H))_H, (\mathcal{P}_2(H), \mathcal{P}_1(H))_H\}$ is a partition of the arc set of H. If we take $g \in \{1,2\}$, then $\mathcal{P}_g(H)$ is either $\mathcal{P}_1(H)$ or $\mathcal{P}_2(H)$, and $\mathcal{P}_{3-g}(H)$ is either $\mathcal{P}_2(H)$ or $\mathcal{P}_1(H)$, respectively.

Henceforth, C_1 and C_2 wil be the cycles $u_0 \cdots u_{p-1} u_0$ and $v_0 \cdots v_{q-1} v_0$, respectively.

1.2. Concordant cycles

Definition 6. Let D be a digraph and let $C_1 = u_0 \cdots u_{p-1} u_0$ and $C_2 = v_0 \cdots v_{q-1} v_0$ be two cycles in D. We say that C_1 and C_2 have a good pair of arcs if there exist $i \in [p]_0$ and $j \in [q]_0$ such that both arcs $u_i \rightarrow v_j$ and $v_{j-1} \rightarrow u_{i+1}$ are in A(D).

Proposition 7 (Galeana-Sánchez and Goldfeder [4]). *Let C_1 and C_2 be two vertex-disjoint cycles in a digraph D. If there* *is a good pair of arcs between them, then there exists a cycle on the vertex set* $V(C_1) \cup V(C_2)$.

Definition 8. Let D be a digraph on n vertices and let $D_1, \ldots, D_n$ be bipartite digraphs with ordered-partitions given by $\mathcal{P}$. Consider the $\mathcal{P}$-composition $H = D[D_1, \ldots, D_n]^{\mathcal{P}}$ and take C_1 and C_2 two vertex-disjoint cycles in H such that each one passes through all the summands. Recall that for each $i \in [n]$, $\{(D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H, (D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H\}$ is a partition of $(D_i, D_{i+1})_H$, where $i+1$ is taken modulo n. We say that C_1 and C_2 are concordant if for some $i \in [n]$, one element in $\{(D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H, (D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H\}$ has arcs from both cycles C_1 and C_2, See Figure 2.

Proposition 9. *Let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where $\mathcal{C}_n$ is the n-cycle, $D_1, \ldots, D_n$ are bipartite digraphs with ordered-partitions given by $\mathcal{P}$ and C_1 and C_2 are two vertex-disjoint and non-concordant cycles in H such that each one passes through all the summands. For each $i \in [n]$, there exists $g \in \{1, 2\}$ such that $(D_i, D_{i+1})_H \cap A(C_1) = (D_i, D_{i+1})_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))$ and $(D_i, D_{i+1})_H \cap A(C_2) = (D_i, D_{i+1})_H \cap (\mathcal{P}_{3-g}(H), \mathcal{P}_g(H))$.*

Proof. Take $i \in [n]$. Recall that both $\{(D_i, D_{i+1})_H \cap A(C_1), (D_i, D_{i+1})_H \cap A(C_2)\}$ and $\{(D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H, (D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H$ are partitions of $(D_i, D_{i+1})_H$, where the subscripts are taken modulo n. Since C_1 and C_2 are non-concordant, no element in $\{(D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H, (D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H\}$ has arcs from both cycles, C_1 and C_2. Suppose without loss of generality that $(D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H$ has one arc from C_1. Since C_1 and C_2 are non-concordant, there is no arc of C_2 in $(D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H$. Since C_2 passes through all the summands of H, all the arcs of C_2 in $(D_i, D_{i+1})_H$ are in $(\mathcal{P}_2(H), \mathcal{P}_1(H))_H$. This is, $(D_i, D_{i+1})_H \cap A(C_2) \subseteq (D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H$. Since C_1 and C_2 are non-concordant, there is no arc of C_1 in $(D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H$. This is, $(D_i, D_{i+1})_H \cap A(C_2) = (D_i, D_{i+1})_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H$. Furthermore, $(D_i, D_{i+1})_H \cap A(C_1) = (D_i, D_{i+1})_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H$. □

Corollary 10. *Let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where $\mathcal{C}_n$ is the n-cycle and $D_1, \ldots, D_n$ are ordered-bipartite digraphs. If $\mathcal{F}$ is a minimum cycle-factor of H, then there is at most two cycles in $\mathcal{F}$ which pass through all the summands.*

Proposition 11. *Let H be the $\mathcal{P}$-composition $D[D_1, \ldots, D_n]^{\mathcal{P}}$, where D is a digraph with vertex set $V(D) = \{v_1, \ldots, v_n\}$ and $D_1, \ldots, D_n$ are ordered-bipartite digraphs. Take $C_1 = u_0 \cdots u_{p-1} u_0$ and $C_2 = v_0 \cdots v_{q-1} v_0$ two vertex-disjoint cycles in H such that each one passes through all the summands. If C_1 and C_2 are concordant, then they have a good pair of arcs.*

Proof. Since C_1 and C_2 are concordant cycles, there exist $t \in [n]$ and $g \in \{1, 2\}$, such that both $((D_t, D_{t+1})_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H) \cap A(C_1)$ and $((D_t, D_{+1})_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H) \cap A(C_2)$ are non-empty. This is, there exist $i \in [p]_0$ and $j \in [q]$ such that $u_i \to u_{i+1}$ is in $(D_t, D_{t+1})_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H$, where $i+1$ is taken modulo p, and $v_j \to v_{j+1}$ is in $(D_t, D_{t+1})_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H$, where $j+1$ is taken modulo q. From the definition of $\mathcal{P}$-composition, we have that $u_i \to v_{j+1}$ and $v_j \to u_{i+1}$ are in H, which are a good pair of arcs. □

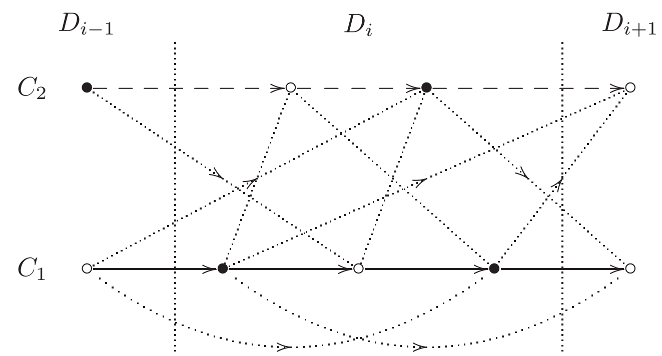

Figure 2. Example of two concordant cycles.

2. Main result

Lemma 12. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semi-complete ordered bipartite digraphs and let D be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. If H has a cycle-factor such that exactly one of its elements passes through all the summands, then H is Hamiltonian.*

Proof. If n is equal to two or three, then H is a semicomplete bipartite digraph. Since H has a cycle-factor, it follows that H is Hamiltonian by Theorem 1. Hence, we can suppose that n is greater than or equal to 4.

We proceed by induction on k, where k is the cardinality of the cycle-factor. If $k = 1$, then H is Hamiltonian. Thus, assume that the lemma holds for $k \geq 1$ and suppose that H has a $(k+1)$-cycle-factor, such that exactly one of the cycles passes through for all the summands. Notice that each one of the other cycles is contained in one summand.

Let C_1 and C_2 be vertex-disjoint two cycles in the cycle-factor, such that C_1 passes through all the summands and C_2 is contained in the summand D_j, for some $j \in [n]$. There exists a subpath $T = u_{t_1} \cdots u_{t_r}$ of C_1 such that u_{t_1} is in $V(D_{j-1}), u_{t_r}$ is in $V(D_{j+1})$ and the other vertices of T are in $V(D_j)$. Since C_2 is contained in the summand D_j and by the definition of D, we have that $\{u_{t_1}\} \to^{\mathcal{P}} C_2 \to^{\mathcal{P}} \{u_{t_r}\}$.

Let u_s be the first vertex in T such that $(C_2, \{v_s\})_H$ is non-empty. Notice that u_s is different from u_{t_1}. Take v in C_2 such that $v \to u_s$. Since v and u_s are in different partite sets, we have that u_{s-1} and v^+, the succesor of v in C, are in same summand of H but in different partite sets, hence they are adjacent. By the definition of u_s, we have that $u_{s-1} \to v^+$. See Figure 3.

The arcs $v \to u_s$ and $u_{s-1} \to v^+$ are a good pair of arcs. Hence, we can merge both C_1 and C_2 into one single cycle by Proposition 7. It follows that H has a k-cycle-factor such that one of the cycles passes for all the summands and each of the others is contained in only one summand. Therefore H is Hamiltonian. □

Now, let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where n is a positive odd number and $\mathcal{C}_n$ is the n-cycle. Assume that C_1 is a cycle in H which passes through all the summands of H. It could happen that C_1 winds several times through H (each time, C_1 passes through all summands). Let α_i^j be the number of vertices of C_1 in D_i at the j-th turn. If k is the total number of winds of C_1 around H, we have that the length of C_1 is equal to $\sum_{j=1}^{k}\sum_{i=1}^{n}\alpha_i^j$.

Recall that C_1 has even length, *i.e.*, $\sum_{j=1}^{k}\sum_{i=1}^{n}\alpha_i^j$ is even. If C_1 passes through an odd number of vertices for each summand in each turn, this is, α_i^j is an odd number for all i and j, then kn, the total number of summands in $\sum_{j=1}^{k}\sum_{i=1}^{n}\alpha_i^j$, has to be even. Since we assumed that n is odd, it follows that k is even.

Lemma 13. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. Assume that $\{C_1, C_2\}$ is a cycle-factor in H such that both cycles pass through all the summands. Let α_i^j (respectively β_i^j) be the number of vertices of C_1 (resp. C_2) in D_i at the j-th turn. If C_1 and C_2 are non-concordant cycles, then α_i^j and β_i^j have the same parity.*

Proof. Let D_i be any summand in H. Let $\rho(i,r)$ (respectively $\sigma(i,r)$) be the unique integer in $\{0, \ldots, p-1\}$ (respect. $\{0, \ldots, q-1\}$) such that $u_{\rho(i,r)}$ (resp. $v_{\sigma(i,r)}$) is the first vertex of C_1 (resp. C_2) in D_i at the r-th turn. Since C_1 and C_2 are non-concordant cycles, Proposition 9 implies that there exist $g_1, g_2 \in \{1,2\}$ such that $(D_{i-1}, D_i)_H \cap (\mathcal{P}_{g_1}(H), \mathcal{P}_{3-g_1}(H))_H = (D_{i-1}, D_i)_H \cap A(C_1)$ and $(D_i, D_{i+1})_H \cap (\mathcal{P}_{g_2}(H), \mathcal{P}_{3-g_2}(H))_H = (D_i, D_{i+1})_H \cap A(C_1)$. Hence, $u_{\rho(i,j)}$ is in $\mathcal{P}_{3-g_1}(H)$ and $u_{\rho(i,j)+\alpha_i^j}$ is in $\mathcal{P}_{g_2}(H)$. Moreover, $v_{\sigma(i,j)}$ is in $\mathcal{P}_{g_1}(H)$ and $v_{\sigma(i,j)+\beta_i^j}$ is in $\mathcal{P}_{3-g_2}(H)$. We have two cases.

Case 1. $\mathcal{P}_{3-g_1}(H)$ and $\mathcal{P}_{g_2}(H)$ are equal. This implies that α_i^j is odd. Furthermore, $\mathcal{P}_{g_1}(H)$ and $\mathcal{P}_{3-g_2}(H)$ are equal and β_i^j is also odd.

Case 2. $\mathcal{P}_{3-g_1}(H)$ and $\mathcal{P}_{g_2}(H)$ are different. This implies that α_i^j is even. Furthermore, $\mathcal{P}_{g_1}(H)$ and $\mathcal{P}_{3-g_2}(H)$ are different and β_i^j is also even. □

Lemma 14. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. If H has a cycle-factor $\{C_1, C_2\}$ such that both cycles pass through all the summands, they are non-concordant and n is odd, then there exists $j \in [n]$ such that the summand D_j has arcs of both cycles, this is, the sets $A(D_j) \cap A(C_1)$ and $A(D_j) \cap A(C_2)$ are non-empty.*

Proof. Let t be the total number of winds of C_1 around D and let α_i^j be the number of vertices of C_1 in D_i at the j-th turn. Recall that $C_1 = u_0 \cdots u_{p-1}u_0$.

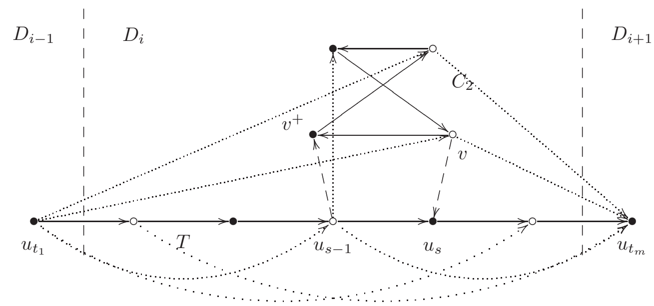

Figure 3. Proof of Lemma 12.

First, we show that there exist $j \in [t]$ and $i \in [n]$ such that in the j-th turn, the cycle C_1 passes through an even number of vertices in the summand D_i, this is, that α_i^j is even. Suppose that, on the contrary, α_i^j is odd for all $i \in [n]$ and for all $j \in [t]$. Since the length of C_1 is even, it follows that t has to be even, so that C_1 passes for at least twice in each summand.

Let $\rho(i,r)$ be the unique integer in $\{0, \ldots, p-1\}$ such that $u_{\rho(i,r)}$ is the first vertex of C_1 in D_i at the r-th turn. Hence, $u_{\rho(i,r)+(\alpha_i^r-1)}$ is the last vertex in which C_1 passes through the summand D_i in the r-th turn. We know that $u_{\rho(i,r)}$ is in $\mathcal{P}_g(H)$, for some $g \in \{1,2\}$. It follows that $u_{\rho(i,r)+(\alpha_i^r-1)}$ has to be in $\mathcal{P}_g(H)$ because α_i^r is odd. We are interested in analyzing how the cycle C_1 jumps from one summand to the next one in each turn, this is, if that arc is either in $(\mathcal{P}_1(H), \mathcal{P}_2(H))_H$ or in $(\mathcal{P}_2(H), \mathcal{P}_1(H))_H$. For this, we can assume without loss of generality that each time C_1 passes through each summand, just passes through exactly one vertex in that summand, *i.e.*, that α_i^r is one. Moreover, let us assume without loss of generality that u_0 is in $\mathcal{P}_1(H) \cap V(D_1)$. Since n is odd and we are assuming that C_1 passes through exactly one vertex by each summand in each turn, it follows that u_{n-1} is in $\mathcal{P}_1(H) \cap V(D_n)$. Furthermore, u_n is in $\mathcal{P}_2(H) \cap V(D_1)$. The arc $u_0 \to u_1$ is in $(D_1, D_2)_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H$ and the arc $u_n \to u_{n+1}$ is in $(D_1, D_2)_H \cap (\mathcal{P}_2(H), \mathcal{P}_1(H))_H$. Recall that Proposition 9 implies that it exists $g \in \{1,2\}$ such that $(D_1, D_2)_H \cap A(C_1) = (D_1, D_2)_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H$, which is a contradiction.

Hence, there exist $i_0 \in [t]$ and $j_0 \in [n]$ such that in the j_0-th turn, the cycle C_1 passes through an even number of vertices in the summand D_{i_0}, this is, that $\alpha_{i_0}^{j_0}$ is even. Therefore $\beta_{i_0}^{j_0}$ is also even by Lemma 13. Thus $A(D_{i_0}) \cap A(C_2)$ is non-empty.

Therefore, we have that both $A(D_{i_0}) \cap A(C_1)$ and $A(D_{i_0}) \cap A(C_2)$ are non-empty. □

Lemma 15. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. Assume that H has a cycle-factor $\{C_1, C_2\}$ such that both cycles pass through all the summands and they are not concordant. If there exists $j \in [n]$ such that the summand D_j has arcs from both cycles, then they have a good pair of arcs.*

Proof. Recall that $C_1 = u_0 \cdots u_{p-1}u_0$ and $C_2 = v_0 \cdots v_{q-1}v_0$. Take $i \in [k]_0$ and $j \in [m]_0$ such that $u_i \to u_{i+1}$ and $v_j \to v_{j+1}$

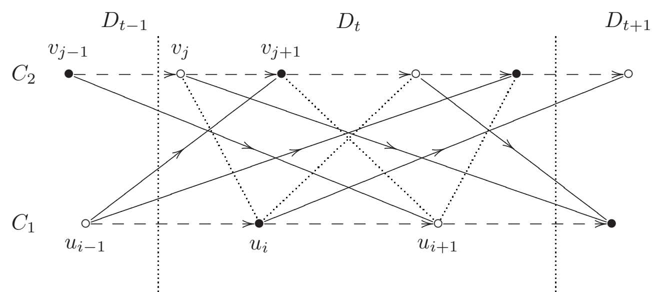

Figure 4. No matter which arc is in H, either $u_i \to v_j$ or $v_j \to v_i$, it does appear a good pair of arcs.

are the first arcs of C_1 and C_2, respectively, in D_t, for some $t \in [n]$. Hence, u_{i-1} and v_{j-1} are in D_{t-1}. Without loss of generality, suppose that $u_{i-1} \to u_i$ is in $(D_{t-1}, D_t)_H \cap (\mathcal{P}_2(D), \mathcal{P}_1(D))_H$. It follows that $(D_{t-1}, D_t)_H \cap (\mathcal{P}_2(D), \mathcal{P}_1(D))_H = (D_{t-1}, D_t)_H \cap A(C_1)$ and $(D_{t-1}, D_t)_H \cap (\mathcal{P}_1(H), \mathcal{P}_2(H))_H = (D_{t-1}, D_t)_H \cap A(C_2)$, since C_1 and C_2 are non-concordant cycles. Notice that u_i and v_j are in the same summand but in different partite sets, hence they are adjacent, as D_t is a semicomplete bipartite digraph.

Case 1. The arc $v_j \to u_i$ is in H. Notice that v_{j+1} is in $\mathcal{P}_1(H) \cap V(D_t)$ and u_{i-1} is in $\mathcal{P}_2(H) \cap V(D_{t-1})$, by the definition of the $\mathcal{P}$-composition, we have that the arc $u_{i-1} \to v_{j+1}$ is in H. Hence, the arcs $v_j \to u_i$ and $u_{i-1} \to v_{j+1}$ are a good pair of arcs. See Figure 4.

Case 2. The arc $u_i \to v_j$ is in H. The proof is analogous to the previous case.

Therefore, the cycles C_1 and C_2 have a good pair of arcs. □

Lemma 16. *Let $D_1, \dots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \dots, D_n]^{\mathcal{P}}$. Assume that:*

i. $\{C_1, C_2\}$, *with* $C_1 = u_0 \cdots u_{p-1}u_0$ and $C_2 = v_0 \cdots v_{q-1}v_0$, *is a cycle-factor in H such that both cycles pass through all the summands,*
ii. *C_1 and C_2 have no good pairs of arcs and*
iii. *no summand of H has arcs from both cycles.*

Let α_i^j (respectively β_i^j) denote the number of vertices of C_1 (resp. C_2) in D_i at the j-th turn. Let λ and μ be the number of times the cycles C_1 and C_2 wind through H, resp. Let $\rho(i,r)$ (resp. $\sigma(i,r)$) be the unique integer in $[p]_0$ (resp. in $[q]_0$) such that $u_{\rho(i,r)}$ (resp. $v_{\sigma(i,r)}$) is the first vertex of C_1 (resp. C_2) in D_i at the r-th turn. Notice that $u_{\rho(i,r)+(\alpha_i^r-1)}$ is the last vertex of C_1 in D_i at the r-th turn and $v_{\sigma(i,r)+(\beta_i^r-1)}$ is the last vertex of C_2 in D_i at the r-th turn.

1. *If $\alpha_i^j > 1$ for some $i \in [n]$ and $j \in [\lambda]$, then α_i^j is odd and $\beta_i^r = 1$ for all $r \in [\mu]$.*
2. *If $\beta_i^j > 1$ for some $i \in [n]$ and $j \in [\mu]$, then β_i^j is odd and $\alpha_i^r = 1$ for all $r \in [\lambda]$.*
3. *Take $g \in \{1,2\}$ and suppose that $u_{\rho(1,1)}$ is in $\mathcal{P}_g(H)$. For every integer odd $i \in [n]$, $u_{\rho(i,r)}$ and $u_{\rho(i,r)+(\alpha_i^r-1)}$ are in $\mathcal{P}_g(H)$ and $v_{\sigma(i,r)}$ and $v_{\sigma(i,r)+(\beta_i^r-1)}$ are in $\mathcal{P}_{3-g}(H)$. And for every integer even $i \in [n]$, $u_{\rho(i,r)}$ and $u_{\rho(i,r)+(\alpha_i^r-1)}$ are in $\mathcal{P}_{3-g}(H)$ and $v_{\sigma(i,r)}$ and $v_{\sigma(i,r)+(\beta_i^r-1)}$ are in $\mathcal{P}_g(H)$.*
4. *If $\alpha_i^j > 1$, for some $i \in [n]$, then $u_{\rho(i,j)+(\alpha_i^j-1)} \to v_{\sigma(i,r)} \to u_{\rho(i,j)}$ for every $r \in [\lambda]$.*
5. *If $\beta_i^j > 1$, for some $i \in [n]$, then $v_{\sigma(i,j)+(\beta_i^j-1)} \to u_{\rho(i,r)} \to v_{\sigma(i,j)}$ for every $r \in [\mu]$.*
6. *For all $i \in [n]$ and all $r, s \in [\lambda]$, $u_{\rho(i,r)-1} \to u_{\rho(i,s)}$.*
7. *For all $i \in [n]$ and all $r, s \in [\mu]$, $v_{\sigma(i,r)-1} \to v_{\sigma(i,s)}$.*

Proof. Since C_1 and C_2 have no good pairs of arcs, Proposition 11 implies that they are non-concordant. Furthermore, since no summand of H has arcs of both cycles, Lemma 15 implies that n is even.

If α_i^r is greater than one for some $r \in [\lambda]$, then C_1 has at least one arc entirely contained in the summand D_i. Since no summand of H has arcs from both cycles, that implies that $\beta_i^s = 1$ for every $s \in [\mu]$. Analogously, if β_i^s is greater than one for some $s \in [\mu]$, then C_2 has at least one arc entirely contained in the summand D_i. Since no summand of H has arcs from both cycles, that implies that $\alpha_i^r = 1$ for every $r \in [\lambda]$. This proves Claims 1 and 2.

Since α_i^r and β_i^s are always odd for every $i \in [n], r \in [\lambda]$ and $s \in [\mu]$ (recall Lemma 13) and H are bipartite, $u_{\rho(i,r)}$ and $u_{\rho(i,r)+(\alpha_i^r-1)}$ are in the same part and $v_{\sigma(i,s)}$ and $v_{\sigma(i,s)+(\beta_i^s-1)}$ are in the same part. This proves Claim 3.

Proposition 9 implies that there exists $g \in \{1,2\}$ such that $(D_1, D_2)_H \cap (\mathcal{P}_{3-g}(H), \mathcal{P}_{3-g}(H))_H = (D_1, D_2)_H \cap A(C_1)$ and $(D_1, D_2)_H \cap (\mathcal{P}_{3-g}(H), \mathcal{P}_g(H))_H = (D_1, D_2)_H \cap A(C_2)$. Moreover, $u_{\rho(1,r)}, u_{\rho(1,r)+(\alpha_1^r-1)} \in \mathcal{P}_g(H)$ for every $r \in [\lambda]$ and $v_{\sigma(1,s)}, v_{\sigma(1,s)+(\beta_1^s-1)} \in \mathcal{P}_{3-g}(H)$ for every $s \in [\mu]$. This implies that $u_{\rho(2,r)}, u_{\rho(2,r)+(\alpha_2^r-1)} \in \mathcal{P}_{3-g}(H)$ for every $r \in [\lambda]$ and $v_{\sigma(2,r)}, v_{\sigma(2,r)+(\beta_2^r-1)} \in \mathcal{P}_g(H)$ for every $s \in [\mu]$. Furthermore, since H is bipartite and n is even, we have that:

- $u_{\rho(i,r)}, u_{\rho(i,r)+(\alpha_i^r-1)} \in \mathcal{P}_g(H)$ for every i odd in $[n]$ and for every $r \in [\lambda]$,
- $u_{\rho(i,r)}, u_{\rho(i,r)+(\alpha_i^r-1)} \in \mathcal{P}_{3-g}(H)$ for every i even in $[n]$ and for every $r \in [\lambda]$,
- $v_{\sigma(i,s)}, v_{\sigma(i,s)+(\beta_i^s-1)} \in \mathcal{P}_{3-g}(H)$ for every i odd in $[n]$ and for every $s \in [\mu]$, and
- $v_{\sigma(i,s)}, v_{\sigma(i,s)+(\beta_i^s-1)} \in \mathcal{P}_g(H)$ for every i even in $[n]$ and for every $s \in [\mu]$.

This and the definition of $\mathcal{P}$-composition imply Claims 6 and 7.

Notice that $u_{\rho(i,r)-1} = u_{\rho(i-1,r)+(\alpha_{i-1}^r-1)}$, whenever $i > 1$, and $u_{\rho(i,r)-1} = u_{\rho_n^{r-1}+(\alpha_n^{r-1}-1)}$, if $i = 1$. Analogously, $v_{\sigma(i,r)-1} = v_{\sigma(i-1,r)+(\beta_{i-1}^r-1)}$, if $i > 1$, and $v_{\sigma(i,r)-1} = v_{\sigma_n^{r-1}+(\beta_n^{r-1}-1)}$, if $i = 1$.

Finally, since $v_{\sigma(i,r)-1}$ and $u_{\rho(i,j)+1}$ are in different parts, $v_{\sigma(i,r)-1}$ is in D_{i-1} and $u_{\rho(i,j)+1}$ is in D_i, there exists the arc $v_{\sigma(i,r)-1} \to u_{\rho(i,j)+1}$. This implies that $v_{\sigma(i,r)} \to u_{\rho(i,j)}$ (otherwise, we obtain a good pair of arcs). Similarly, since

$u_{\rho(i,j)+(\alpha_i^j-2)}$ and $v_{\sigma(i,r)+1}$ are in different parts, $u_{\rho(i,j)+(\alpha_i^j-2)}$ is in D_i and $v_{\sigma(i,r)+1}$ is in D_{i+1}, there exists the arc $u_{\rho(i,j)+(\alpha_i^j-2)} \to v_{\sigma(i,r)+1}$. This implies that $u_{\rho(i,r)} \to v_{\sigma(i,j)}$ (otherwise, we obtain a good pair of arcs). This proves Claims 4 and 5. □

Lemma 17. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. If n is odd and H has a 2-cycle-factor, then H is Hamiltonian.*

Proof. Let $C_1 = u_0 \cdots u_{p-1} u_0$ and $C_2 = v_0 \cdots v_{q-1} v_0$ be the two cycles in the cycle-factor. We consider the following cases:

Case 1. Suppose that C_1 passes through all the summands of H and C_2 is contain in one summand. Hence, by Lemma 12, it follows that H is Hamiltonian.

Case 2. Suppose that C_1 and C_2 are contained in different summands. Therefore n has to be two, which is a contradiction since n is odd.

Case 3. Suppose that each cycle C_1 and C_2 pass through all the summands.

If C_1 and C_2 are concordant cycles, then they have a good pair of arcs and we can merge both into one cycle, a Hamiltonian one. Hence, assume that C_1 and C_2 are non-concordant cycles. Lemmas 14 and 15 imply that C_1 and C_2 have a good pair of arcs, so we are done. □

Lemma 18. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. If H has a cycle-factor such that each cycle is completely contained in only one summand, then D is Hamiltonian.*

Proof. Since each one of the cycles of the cycle-factor is contained in only one summand, then each of the summands is strong and has a cycle-factor. Hence, Theorem 1 implies that each summand is Hamiltonian. Let C_i be the Hamiltonian cycle in the summand D_i, for each $i \in [n]$. Recall that $\{\mathcal{P}_1(H), \mathcal{P}_2(H)\}$ is the partition of $V(H)$ and that all our cycles are even. Let $C_i = u_0^i u_2^i \cdots u_{k_i-1}^i u_0^i$ be the Hamiltonian cycle in the summand D_i. Without loss of generality, assume that u_0^i is in $\mathcal{P}_1(H)$, for every $i \in [n]$. Hence, u_{k_i-1} is in $\mathcal{P}_2(H)$. The definition of the $\mathcal{P}$-composition implies the existence of the arcs $u_{k_{i-1}-1}^{i-1} \to u_0^i$ in D. Therefore, the cycle

$$C' = C_1\left[u_0^1, u_{k_1-1}^1\right] C_2\left[u_0^2, u_{k_2-1}^2\right] \cdots C_n\left[u_0^n, u_{k_n-1}^n\right] u_0^1$$

is Hamiltonian in H. □

Lemma 19. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. Assume that $\{C_1, C_2\}$ is a cycle-factor in H such that both cycles pass through all the summands. If there exists $i \in [n]$ such that $A(D_i) \cap A(C_1)$ and $A(D_i) \cap A(C_2)$ are empty sets, then for each u in $V(C_1) \cap V(D_i)$, there exists v in $V(C_2) \cap V(D_i)$ such that $u \to v$ and vice versa, for each w in $V(C_2) \cap V(D_i)$ there exists z in $V(C_1) \cap V(D_i)$ such that $w \to z$.*

Proof. Proposition 9 implies that for each $i \in [n]$, there exists $g \in \{1, 2\}$ such that $(D_{i-1}, D_i)_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H = (D_{i-1}, D_i)_H \cap A(C_1)$ and $(D_{i-1}, D_i)_H \cap (\mathcal{P}_{3-g}(H), \mathcal{P}_g(H))_H = (D_{i-1}, D_i)_H \cap A(C_2)$ because C_1 and C_2 are non-concordant. Since there is no arcs of C_1 and C_2 in D_i, it follows that $V(C_1) \cap V(D_i) = \mathcal{P}_{3-g}(H) \cap V(D_i)$ and $V(C_2) \cap V(D_i) = \mathcal{P}_g(H) \cap V(D_i)$ (also, we know that $(D_i, D_{i+1})_H \cap (\mathcal{P}_{3-g}(H), \mathcal{P}_g(H))_H = (D_i, D_{i+1})_H \cap A(C_1)$ and $(D_i, D_{i+1})_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H = (D_i, D_{i+1})_H \cap A(C_2)$). For each vertex $u \in V(C_1) \cap V(D_i)$, we know that u is in $\mathcal{P}_{3-g}(H) \cap V(D_i)$. Because D_i is strong, it follows that there exists v in $\mathcal{P}_g(H) \cap V(D_i)$ such that $u \to v$. Notice that v is in $V(C_2) \cap V(D_i)$. The other case is analogous. □

Lemma 20. *Let $D_1, \ldots, D_n$ be vertex-disjoint strong semicomplete bipartite digraphs with an ordered-partition given by $\mathcal{P}$ and let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$. If H has a 2-cycle-factor, then D is Hamiltonian.*

Proof. Let $\{C_1, C_2\}$, with $C_1 = u_0 \cdots u_{p-1} u_0$ and $C_2 = v_0 \cdots v_{q-1} v_0$, be the cycle-factor in H. We can assume that both cycles pass through all the summands by Lemmas 12 and 18. We can assume that C_1 and C_2 have no good pairs of arcs. Thus, Proposition 11 implies that C_1 and C_2 are non-concordant cycles. Lemma 15 implies that no summand of H has arcs from both cycles. Lemma 14 implies that n is even.

Let α_i^j (respectively β_i^j) denote the number of vertices of C_1 (resp. C_2) in D_i at the j-th turn. Let λ and μ be the number of times the cycles C_1 and C_2 wind through H, respectively. Suppose without loss of generality that $\lambda \le \mu$. Let $\rho(i,r)$ (resp. $\sigma(i,r)$) be the unique integer in $[p]_0$ (resp. $[q]_0$) such that $u_{\rho(i,r)}$ (resp. $v_{\sigma(i,r)}$) is the first vertex of C_1 (resp. C_2) in D_i at the r-th turn. Notice that $u_{\rho(i,r)+(\alpha_i^r-1)}$ is the last vertex of C_1 in D_i at the r-th turn and $v_{\sigma(i,r)+(\beta_i^r-1)}$ is the last vertex of C_2 in D_i at the r-th turn.

Now, we are going to take one turn of C_2 around H and inserting it into C_1, obtaining a new 2-cycle-factor which does have a good pair of arcs. In order to do that, we have to establish some previous results.

Take $i \in [n]$ and two different integers $r, s \in [\lambda]$. We have that $C_1[u_{\rho(i,r)}, u_{\rho(i,r)+(\alpha_i^r-1)}]$ and $C_1[u_{\rho(i,s)}, u_{\rho(i,s)+(\alpha_i^s-1)}]$ are the subpaths of C_1 in the summand D_i at the r-th and s-th turns, respectively. Moreover, $u_{\rho(i,r)-1}$ and $u_{\rho(i,s)-1}$ are the previous vertices and $u_{\rho(i,r)+\alpha_i^r}$ and $u_{\rho(i,s)+\alpha_i^s}$ are the following vertices of those subpaths, respectively. By Claim 6 of Lemma 16, we have that $u_{\rho(i,r)-1} \to u_{\rho(i,r)}, u_{\rho(i,s)-1} \to u_{\rho(i,s)}, u_{\rho(i,r)+(\alpha_i^r-1)} \to u_{\rho(i,r)+\alpha_i^r}$ and $u_{\rho(i,s)+(\alpha_i^s-1)} \to u_{\rho(i,s)+\alpha_i^s}$. By interchanging $C_1[u_{\rho(i,r)}, u_{\rho(i,r)+(\alpha_i^r-1)}]$ and $C_1[u_{\rho(i,s)}, u_{\rho(i,s)+(\alpha_i^s-1)}]$ in C_1, we obtain a new cycle with the

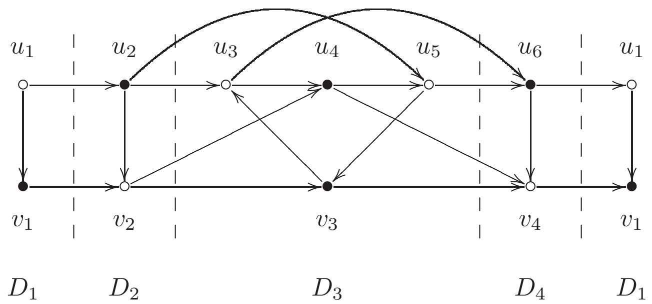

Figure 5. If $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$ is strong and has a cycle-factor, then $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$ is not necessarily Hamiltonian.

same vertex set as C_1 and which preserves the hypothesis of the lemma. We can perform similar interchanges in C_2.

The previous paragraph allows us to suppose, without loss of generality, that if C_1 or C_2 has an arc entirely contained in the summand D_i, for some $i \in [n]$, and that arc appears in the first turn, this means, α_i^1 or β_i^1 is greater than one. Recall that Claims 4 and 5 of Lemma 16 implies that $u_{\rho(i,1)+(\alpha_i^1-1)} \rightarrow v_{\sigma(i,1)} \rightarrow u_{\rho(i,1)}$, if α_i^1 is greater than one or $v_{\sigma(i,1)+(\beta_i^1-1)} \rightarrow u_{\rho(i,1)} \rightarrow v_{\sigma(i,1)}$, if β_i^1 is greater than one.

On the other hand, we have that there is no arcs of C_1 and C_2 entirely contained in the summand D_i, for some $i \in [n]$. In this case, all the subpaths of C_1 and C_2 in D_i have length zero. The former argument also allows us to interchange that subpaths. Lemma 19 implies that for every $r \in [\lambda]$, there exists $s \in [\mu]$ such that $u_{\rho(i,r)} \rightarrow v_{\sigma(i,s)}$ and, conversely, for every $s' \in [\mu]$, there exists $r' \in [\lambda]$ such that $v_{\sigma(i,s')} \rightarrow u_{\rho(i,r')}$. Hence, we can suppose, without loss of generality, that for every even integer $i \in [n]$, there exists the arc $v_{\sigma(i,1)+(\beta_i^1-1)} \rightarrow u_{\rho(i,1)}$ and that for every odd integer $i \in [n]$, there exists the arc $u_{\rho(i,1)+(\alpha_i^1-1)} \rightarrow v_{\sigma(i,1)}$.

Now, we construct the new cycle-factor. Let C_1' be the cycle

$$C_1[u_{\rho(1,1)}, u_{\rho(1,1)+(\alpha_1^1-1)}]C_2[v_{\sigma(1,1)}, v_{\sigma(2,1)+(\beta_2^1-1)}]$$
$$C_1[u_{\rho(2,1)}, u_{\rho(3,1)+(\alpha_3^1-1)}]C_2[v_{\sigma(3,1)}, v_{\sigma(4,1)+(\beta_4^1-1)}]\cdots$$
$$C_2[v_{\sigma(n-1,1)}, v_{\sigma(n,1)+(\beta_n^1-1)}]C_1[u_{\rho(n,1)}, u_{\rho(1,1)}].$$

Notice that the path $C_1[u_{\rho(n,1)}, u_{\rho(1,1)}]$ starts in the first vertex of C_1 in D_n at the first turn and passes through the rest of the turns of C_1 around H up to return to $u_{\rho(1,1)}$ and that the winding number of C_1' around H is still λ.

If the winding number of C_2, denoted by μ, is one, then C_1' is a Hamiltonian cycle. Otherwise, let C_2' be the cycle $C_2[v_{\rho(1,2)}, v_{\rho(n,\mu)+(\beta_n^\mu-1)}]v_{\rho(1,2)}$. Notice that the winding number of this new cycle is $\mu - 1$. The arcs $v_{\sigma(1,1)+(\beta_1^1-1)} \rightarrow v_{\sigma(2,2)}$ and $v_{\sigma(1,2)+(\beta_1^2-1)} \rightarrow v_{\sigma(2,2)}$ are in $A(C_2)$. Since C_1 and C_2 are non-concordant, there exists $g \in \{1,2\}$ such that both arcs are in $(D_1, D_2)_H \cap (\mathcal{P}_g(H), \mathcal{P}_{3-g}(H))_H$. In the new cycle-factor $\{C_1', C_2'\}$, those arcs are in different cycles. Hence, those cycles are concordant by definition. Proposition 11 implies that those cycles have a good pair of arcs, therefore we can merge into only one cycle, a Hamiltonian one. □

Proof of Theorem 2. Assume that H has a cycle-factor. We proceed by induction on k, where k is the cardinality of the cycle-factor. If $k = 2$, then Lemma 17 or Lemma 20 implies that H is Hamiltonian. Thus, assume that for $k \geq 2$, the theorem holds and suppose that H has a $(k+1)$-cycle-factor. If each element of the cycle-factor is completely contained in one summand, Lemma 18 implies that H is Hamiltonian. So suppose that there exists one element C_i in the cycle-factor which passes through all the summands. If there is a cycle C_j in the cycle-factor which is completely contained in one summand, we can merge both cycles into one cycle by Lemma 12, so the induction hypothesis implies that H is Hamiltonian. Hence, all the cycles pass through all the summands of H. Since the cycle-factor has at least three elements, Corollary 10 implies that two of them are concordant. Proposition 11 implies that there is a good pair of arcs, so we can merge both cycles into one cycle. Therefore, H is Hamiltonian by the induction hypothesis. □

As a last remark, let H be the $\mathcal{P}$-composition $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$, where $\mathcal{C}_n$ is the n-cycle and each D_i is a semicomplete bipartite digraph, not necessarily strong. If $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$ is strong and has a cycle-factor, then $\mathcal{C}_n[D_1, \ldots, D_n]^{\mathcal{P}}$ is not necessarily Hamiltonian. See Figure 5.

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References

[1] Bang-Jensen, J. (1990). Locally semicomplete digraphs: a generalization of tournaments. *J. Graph Theory* 14(3): 371–390.

[2] Bang-Jensen, J., Gutin, G. (2009). *Digraphs: Theory, Algorithms, and Applications*. London: Springer, 811 p.

[3] Galeana-Sánchez, H., Goldfeder, I. A. (2012). A classification of all arc-locally semicomplete digraphs. *Discrete Math.* 312(11): 1883–1891.

[4] Galeana-Sánchez, H., Goldfeder, I. A. (2014). Hamiltonian cycles in a generalization of bipartite tournament with cycle-factor. *Discrete Math.* 315-316: 135–143.

[5] Gutin, G. (1984). Criterion for complete bipartite digraphs to be Hamiltonian. *Vestsī Acad. Navuk BSSR Ser. Fīz.-Mat. Navuk* 1: 109–110.

[6] Häggkvist, R., Manoussakis, Y. (1989). Cycles and paths in bipartite tournament with spanning configurations. *Combinatorica* 9(1): 33–38.