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Min-max and max-min graph saturation parameters

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ABSTRACT
Let P be a graph theoretic property concerning subsets of the vertex set V, which is either hereditary or super hereditary. We associate with this property P five graph theoretic parameters, namely a minimum parameter, a maximum parameter, a max-min parameter, a min-max parameter and a partition parameter. We initiate a study of these parameters for the properties of independence, domination and irredundance.

KEYWORDS
Domination; independence; irredundance

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1. Introduction

By a graph $G=(V,E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [5].

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of the fundamentals of domination in graphs is given in the book by Haynes et al. [7]. Surveys of several advanced topics in domination are given in the book edited by Haynes et al. [8].

Let $G=(V,E)$ be a graph. Let $v \in V$. The open neighbourhood $N(v)$ and the closed neighbourhood $N[v]$ are defined by $N(v)=\{u \in V : uv \in E\}$ and $N[v]=N(v)\cup\{v\}$. A subset S of V is said to be an independent set if no two vertices in S are adjacent. A set S is called a dominating set of G if every vertex in $V-S$ is adjacent to at least one vertex in S. A subset S of V is called an irredundant set if for every vertex $v \in S$ there exists a vertex w such that $N[w]\cap S=\{v\}$.

We observe that the maximality condition for an independent set is the definition of dominating set and the minimally condition for a dominating set is the definition of irredundant set.

The above concepts of independence, domination and irredundance lead to the following six parameters.

$$i(G)=min\{|S| : S \text{ is a maximal independent set in } G\}.$$
$$\beta_0(G)=max\{|S| : S \text{ is an independent set in } G\}.$$
$$\gamma(G)=min\{|S| : S \text{ is a dominating set in } G\}.$$
$$\Gamma(G)=max\{|S| : S \text{ is a minimal dominating set in } G\}.$$
$$ir(G)=min\{|S| : S \text{ is a maximal irredundant set in } G\}.$$
$$IR(G)=max\{|S| : S \text{ is an irredundant set in } G\}.$$

These parameters are respectively called the independent domination number, the independence number, the domination number, the upper domination number, the irredundance number and the upper irredundance number. These parameters satisfy the following inequality chain.

$$ir(G)\le\gamma(G)\le i(G)\le\beta_0(G)\le\Gamma(G)\le IR(G).$$

This inequality chain was first observed by Cockayne et al. [6] and has become one of the strongest focal points for research in domination theory.

With respect to each of the properties domination, independence and irredundance several authors have investigated the problem of partitioning the vertex set into dominating sets, independent sets and irredundant sets leading respectively to the concept of domatic number, chromatic number and irratic number.

Acharya [1] introduced the concept of domsaturation number $ds(G)$ of a graph, which is defined to be the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. Arumugam and Kala [2] observed that for any graph G, $ds(G)=\gamma(G)$ or $\gamma(G)+1$, and obtained several results on $ds(G)$. Motivated by this concept Arumugam and Subramanian [3] introduced the concept of independence saturation number of a graph and Arumugam et al. [4] introduced the concept of irredundance saturation number of a graph.

Definition 1.1. Let $G=(V,E)$ be a graph and let $v \in V$. Let $IS(v, G)$ denote the maximum cardinality of an independent set in G which contains v. Then $IS(G)=min\{IS(v,G) : v \in V\}$ is called the independence saturation number of G.

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Thus $IS(G)$ is the largest positive integer k such that every vertex of G lies in an independent set of cardinality k.

The independence saturation number $IS(G)$ extends the domination chain, as shown in the following theorem.

Theorem 1.2. [3] *For any graph G we have* $ir(G) \leq \gamma(G) \leq i(G) \leq IS(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$.

Definition 1.3. Let $G = (V, E)$ be a graph and let $v \in V$. Let $IRS(v, G)$ denote the maximum cardinality of an irredundant set in G which contains v. Then $IRS(G) = min\{IRS(v, G) : v \in V\}$ is called the irredundance saturation number of G.

Theorem 1.4. [4] *For any graph G we have* $ir(G) \leq \gamma(G) \leq i(G) \leq IS(G) \leq IRS(G) \leq IR(G)$.

Definition 1.5. Let $G = (V, E)$ be an arbitrary graph and let k be any positive integer. The trestled graph of index $k, T_k(G)$, is the graph obtained from G by adding k copies of K_2 to each edge uv of G and joining u and v to the end vertices of each K_2, respectively.

Corollary 1.6. [4] *For any graph G,* $IS(T_2(G)) = IRS(T_2(G))$.

Lemma 1.7. [4] *For any graph G of size* $m, IS(T_2(G)) = IS(G) + 2m$.

Theorem 1.8. [3] *The problem of determining whether* $IS(G) \geq k$ *for any graph G is NP-complete.*

In this paper we propose a general theoretical frame work for min-max and max-min graph saturation parameters.

Let P be a graph theoretic property concerning subsets of the vertex set V. A subset S of V is called a P-set if S has P, otherwise S is a $\bar{P}$-set. A subset S of V is called a maximal P-set if S is a P-set and any proper superset $S_1 \supset S$ is a $\bar{P}$-set. The set S is called a 1-maximal P-set if S is a P-set and for every $v \in V - S$, $S \cup \{v\}$ is a $\bar{P}$-set. The set S is called a minimal P-set if S is a P-set and any proper subset $S_1 \subset S$ is a $\bar{P}$-set. The set S is called a 1-minimal P-set if S is a P-set and for every $v \in S$, $S - \{v\}$ is a $\bar{P}$-set. A property P of sets of vertices is said to be hereditary if whenever a set S has property P, so does every proper subset $S' \subset S$. A property P is super hereditary if whenever a set S has property P, so does every proper superset $S' \supset S$. Minimal P-sets are always 1-minimal P-sets. Similarly maximal P-sets are always 1-maximal but the converse is not true. However in some cases theses definitions are equivalent.

Theorem 1.9. [7] *Let G be a graph and let P be a hereditary property. Then a set S is a maximal P-set if and only S is a 1-maximal P-set.*

Theorem 1.10. [7] *Let P be a super hereditary property. Then a set S is a 1-minimal P-set if and only if S is a minimal P-set.*

In this paper for any graph theoretic property P(which may be hereditary or super hereditary) we associate five parameters namely the minimum parameter, the maximum parameter, the min-max parameter, the max-min parameter and the partition parameter. There is abundant scope of extending this study for several graph theoretic properties.

2. Basic definitions

Definition 2.1. Let P be a hereditary property. We define five graph theoretic parameters of a graph G as follows.

1. *The P-number* $\lambda_p(G) = min\{|S| : S \text{ is a maximal P-set}\}$.
2. *The upper P-number* $\Lambda_p(G) = max\{|S| : S \text{ is a P-set}\}$.
3. *The min-max P-number* $\lambda_p^{M,m}(G)$ *is defined as follows. For any* $v \in V(G)$, *let* $\lambda_p^{min}(v, G) = min\{|S| : v \in S$ *and* S *is a maximal P-set*$\}$ *and let* $\lambda_p^{M,m}(G) = max\{\lambda_p^{min}(v, G) : v \in V\}$. *Thus* $\lambda_p^{M,m}(G)$ *is the smallest positive integer k, with the property that every vertex of G lies in a maximal P-set of cardinality at most k.*
4. *The max-min P-number* $\Lambda_p^{m,M}(G)$ *is defined as follows. For any* $v \in V(G)$, *let* $\Lambda_p^{max}(v, G) = max\{|S| : v \in S$ *and* S *is a maximal P-set*$\}$ *and let* $\Lambda_p^{m,M}(G) = min\{\Lambda_p^{max}(v, G) : v \in V\}$. *Thus* $\Lambda_p^{m,M}(G)$ *is the largest positive integer k, such that every vertex of G lies in a P-set of cardinality k.*
5. *The P-partition number* $\pi_p(G)$ *is the minimum order of a partition of V(G) into P-sets.*

Observation 2.2. *We observe that for any* $v \in V(G)$, $\lambda_p(G) \leq \lambda_p^{min}(v, G) \leq \Lambda_p(G)$ *and* $\lambda_p(G) \leq \Lambda_p^{max}(v, G) \leq \Lambda_p(G)$. *Hence* $\lambda_p(G) \leq \lambda_p^{M,m}(G) \leq \Lambda_p(G)$ *and* $\lambda_p(G) \leq \Lambda_p^{m,M}(G) \leq \Lambda_p(G)$. *In general, the parameters* $\lambda_p^{M,m}(G)$ *and* $\Lambda_p^{m,M}(G)$ *are not comparable.*

Definition 2.3. Let P be a super hereditary property. We define five graph theoretic parameters of a graph as follows.

1. The P-number $s_p(G) = min\{|S| : S \text{ is a P-set}\}$.
2. The upper P-number $S_p(G) = max\{|S| : S \text{ is a minimal } P\text{-set}\}$.
3. The min-max P-number $s_p^{M,m}(G)$ is defined as follows. For any $v \in V(G)$, let $s_p^{min}(v, G) = min\{|S| : v \in S \text{ and S is a minimal P-set}\}$ and let $s_p^{M,m}(G) = max\{s_p^{min}(v, G) : v \in V\}$. Thus $s_p^{M,m}(G)$ is the largest positive integer k, with the property that every vertex of G lies in a minimal P-set of cardinality at least k.
4. The max-min P-number $S_p^{m,M}(G)$ is defined as follows. For any $v \in V(G)$, let $S_p^{max}(v, G) = max\{|S| : v \in S$ and S is a minimal $P-set\}$ and let $S_p^{m,M}(G) = min\{S_p^{max}(v, G) : v \in V\}$. Thus $S_p^{m,M}(G)$ is the least positive integer k such that every vertex of G lies in a P-set of cardinality k.
5. The P-partition number $\Pi_p(G)$ is the maximum order of a partition of $V(G)$ into P-sets.

Observation 2.4. *We observe that for any* $v \in V(G)$, $s_p(G) \leq s_p^{min}(v, G) \leq S_p(G)$ *and* $s_p(G) \leq S_p^{max}(v, G) \leq S_p(G)$. *Hence* $s_p(G) \leq s_p^{M,m}(G) \leq S_p(G)$ *and* $s_p(G) \leq S_p^{m,M}(G) \leq$

$S_p(G)$. *In general, the parameters* $s_p^{M,m}(G)$ *and* $S_p^{m,M}(G)$ *are not comparable.*

Example 2.5. Let P denote the property that a subset S of V is independent. Clearly independence is a hereditary property. The parameters $\lambda_p(G)$, $\Lambda_p(G)$ and $\pi_p(G)$ are respectively the independent domination number $i(G)$, the independence number $\beta_0(G)$ and the chromatic number $\chi(G)$ and these parameters have been extensively studied. The parameter $\Lambda_p^{m,M}(G)$ is the independence saturation number $IS(G)$ which has been investigated in [3]. The parameter $\lambda_p^{M,m}(G)$ has not been studied so far.

3. Dom saturation parameters

In this section we investigate the min-max and max-min parameters corresponding to independence, domination and irredundance. The min-max and max-min parameters corresponding to independence are denoted by $i^{M,m}(G)$ and $\beta_0^{m,M}(G)$ respectively. The min-max and max-min parameters corresponding to domination are denoted by $\gamma^{M,m}(G)$ and $\Gamma^{m,M}(G)$ respectively. Similarly the min-max and max-min parameters corresponding to irredundance are denoted by $ir^{M,m}(G)$ and $IR^{m,M}(G)$. With this notation the independence saturation number $IS(G)$ and the irredundance saturation number $IRS(G)$ are respectively $\beta^{m,M}(G)$ and $IR^{m,M}(G)$.

Observation 3.1. *The following inequality chains follow from Theorem 1.2 and Observation 2.2.*

1. $ir(G) \le \gamma(G) \le i(G) \le i^{M,m}(G) \le \beta_0(G) \le \Gamma(G) \le IR(G)$.
2. $ir(G) \le \gamma(G) \le i(G) \le \beta^{m,M}(G) \le \beta_0(G) \le \Gamma(G) \le IR(G)$.
3. $ir(G) \le \gamma(G) \le \gamma^{M,m}(G) \le \Gamma(G) \le IR(G)$.
4. $ir(G) \le \gamma(G) \le \Gamma^{m,M}(G) \le \Gamma(G) \le IR(G)$.
5. $ir(G) \le ir^{M,m}(G) \le IR(G)$.
6. $ir(G) \le IR^{m,M}(G) \le IR(G)$.

Observation 3.2. *For any graph* G, $\beta_0^{m,M}(G) = \Gamma^{m,M}(G) = IR^{m,M}(G) = 1$ *if and only if* $\Delta(G) = n - 1$.

Observation 3.3. *For any graph* G, $ir^{M,m}(G) = \gamma^{M,m}(G) = i^{M,m}(G) = 1$ *if and only if* $G \cong K_p$.

Observation 3.4. *For any graph* G, $ir^{M,m}(G) = \gamma^{M,m}(G) = i^{M,m}(G) = \beta_0^{m,M}(G) = \Gamma^{m,M}(G) = IR^{m,M}(G) = n$ *if and only if* $G \cong \overline{K_n}$.

Observation 3.5. *For the cycle* C_n, *we have* $\beta_0^{m,M}(C_n) = \Gamma^{m,M}(C_n) = IR^{m,M}(C_n) = \lfloor n/2 \rfloor$ *and* $ir^{M,m}(C_n) = \gamma^{M,m}(C_n) = i^{M,m}(C_n) = \lfloor n/3 \rfloor$.

Observation 3.6. *For the path* P_n, *we have* $\beta_0^{m,M}(P_n) = \Gamma^{m,M}(P_n) = IR^{m,M}(P_n) = \lfloor n/2 \rfloor$ *and* $ir^{M,m}(P_n) = \gamma^{M,m}(P_n) = i^{M,m}(P_n) = \lfloor n/3 \rfloor$.

Observation 3.7. *For the complete bipartite graph* $K_{m,n}$ $m \le n$, *we have* $\beta_0^{m,M}(K_{m,n}) = \Gamma^{m,M}(K_{m,n}) = IR^{m,M}(K_{m,n}) = m$ *and* $ir^{M,m}(K_{m,n}) = \gamma^{M,m}(K_{m,n}) = 2$ *and* $i^{M,m}(K_{m,n}) = n$.

Observation 3.8. *For any graph* G, $\beta_0^{m,M}(G) \le \Gamma^{m,M}(G) \le IR^{m,M}(G) \le n - \Delta$.

Observation 3.9. *For any graph* G, $ir^{M,m}(G) \le \gamma^{M,m}(G) \le i^{M,m}(G) \le n - \delta$.

Observation 3.10. *The parameters* $\Gamma^{m,M}(G)$ *and* $\gamma^{M,m}(G)$, $IR^{m,M}(G)$ *and* $ir^{M,m}(G)$, $\beta^{m,M}(G)$ *and* $i^{M,m}(G)$ *are not comparable. For the star* $K_{1,n}$, *we have* $\Gamma^{m,M}(G) = IR^{m,M}(G) = \beta^{m,M}(G) = 1$ *and* $\gamma^{M,m}(G) = ir^{M,m}(G) = i^{M,m}(G) = n$. *Also for the complete bipartite graph* $K_{n,n}$, *we have* $\Gamma^{m,M}(G) = IR^{m,M}(G) = n$ *and* $\gamma^{M,m}(G) = ir^{M,m}(G) = 2$. *Also for the graph* $C_4 \circ 2K_1$, $i^{M,m}(G) = 6$ *and* $\beta_0^{m,M}(G) = 7$.

Theorem 3.11. *For any graph* G, $\beta^{m,M}(G) \le \Gamma^{m,M}(G) \le IR^{m,M}(G)$ *and* $ir^{M,m}(G) \le \gamma^{M,m}(G) \le i^{M,m}(G)$.

Proof. Since every maximal independent set is a minimal dominating set and every minimal dominating set is a maximal irredundant set, we have $\beta^{max}(v,G) \le \Gamma^{max}(v,G)$, $\gamma^{min}(v,G) \le i^{min}(v,G)$, $\Gamma^{max}(v,G) \le IR^{max}(v,G)$ and $ir^{min}(v,G) \le \gamma^{min}(v,G)$ for all $v \in V(G)$. Hence it follows that $\beta^{m,M}(G) \le \Gamma^{m,M}(G) \le IR^{m,M}(G)$ and $ir^{M,m}(G) \le \gamma^{M,m}(G) \le i^{M,m}(G)$. □

Corollary 3.12. *Given any graph G and a positive integer k, the problem of determining whether* $\Gamma^{m,M}(G) \ge k$ *is NP-complete.*

Proof. It follows from Corollary 1.6 and Lemma 1.7 that $\Gamma^{m,M}(T_2(G)) = \beta^{m,M}(G) + 2m$. Hence $\beta_0^{m,M}(G) \ge k$ if and only if $\Gamma^{m,M}(T_2(G) \ge k + 2m$ and the result follows from Theorem 1.8. □

Corollary 3.13. *For any graph* G, $ir(G) \le \gamma(G) \le i(G) \le \beta_0^{m,M}(G) \le \Gamma^{m,M}(G) \le IR^{m,M}(G) \le IR(G)$ *and* $ir(G) \le ir^{M,m}(G) \le \gamma^{M,m}(G) \le i^{M,}\ m(G) \le \beta_0(G) \le \Gamma(G) \le IR(G)$.

Proof. The result follows from Theorem 1.2 and Theorem 3.11. □

The relationship between the twelve parameters corresponding to domination, independence and irredundance are summarized in the Hasse diagram given in Figure 1.

4. Existence results

In this section we prove the existence of graphs such that specified parameters assume specified values.

Theorem 4.1. *Given any two positive integers a and b their exists a graph* G_1, *with* $ir^{M,m}(G_1) = \gamma^{M,m}(G_1) = a$ *and* $IR^{m,M}(G_1) = \Gamma^{m,M}(G_1) = b$.

Proof. **Case i.** $2 \le a \le b$.

Let $G = K_{b-a+2} \times K_2$. Let G_1 be the graph obtained from G by joining any vertex of G with a vertex of degree 2 in $P_{a-2} \circ K_1$. Clearly $ir^{min}(v,G_1) = \gamma^{min}(v,G_1) = a$ for all $v \in V(G_1)$. Also $IR^{max}(v,G_1) = \Gamma^{max}(v,G_1) = b - a + 2 + a -$

$2 = b$ for all $v \in V(G_1)$. Hence $ir^{M,m}(G_1) = \gamma^{M,m}(G_1) = a$ and $IR^{m,M}(G_1) = \Gamma^{m,M}(G_1) = b$.

Case ii. $b \leq a$.

Take a path $P = (v_1, v_2, ..., v_b)$ on b vertices. Join u_i to v_i for $i = 1, 2, ..., b-1$, and then attach $w_1, w_2, ..., w_{a-b+1}$ pendants to v_b. For this graph $ir^{min}(v_i, G_1) = \gamma^{min}(v_i, G_1) = b$ for $i = 1, 2, ..., b$, $ir^{min}(u_i, G_1) = \gamma^{min}(u_i, G_1) = b$ for $i = 1, 2, ..., b-1$, $ir^{min}(w_i, G_1) = \gamma^{min}(w_i, G_1) = a$ for $i = 1, 2, ..., a-b+1$. $IR^{max}(v_b, G_1) = \Gamma^{max}(v_b, G_1) = b$, $IR^{max}(v, G_1) = \Gamma^{max}(v, G_1) = a$ for all $v \in (V(G_1) - v_p)$. Hence $ir^{M,m}(G_1) = \gamma^{M,m}(G_1) = a$ and $IR^{m,M}(G_1) = \Gamma^{m,M}(G_1) = b$. □

Theorem 4.2. *Given any two positive integers a and b their exists a graph G_3, with $i^{M,m}(G_3) = a$ and $\beta_0^{m,M}(G_3) = b$.*

Proof. **Case i.** $3 \leq a \leq b$.

Let $G = K_{b-a+2, b-a+2} - M$, where M is a perfect matching. Let G_3 be the graph obtained from G by joining any vertex of G with a vertex of degree 2 in $P_{a-2} \circ K_1$. For this graph $i^{min}(v, G_3) = a$ for all $v \in V(G_3)$ and $\beta_0^{max}(v, G_3) = b$ for all $v \in V(G_3)$. Hence $i^{M,m}(G_3) = a$ and $\beta_0^{m,M}(G_3) = b$.

Case ii. $b \leq a$.

Consider the bistar $G_3 = B(a-1, b-1)$. Clearly $i^{M,m}(G_3) = a$ and $\beta_0^{m,M}(G_3) = b$. □

Theorem 4.3. *Given any three positive integers a, b and c with $2 \leq a \leq b \leq c$, their exists a graph G with $\gamma(G) = a, \Gamma^{m,M}(G) = b$ and $\Gamma(G) = c$.*

Proof. Let a, b and c be three positive integers with $2 \leq a \leq b \leq c$.

Case i. $a = 2$.

$$\text{Let } k = \begin{cases} 0 & \text{if } c \leq 2b-1 \\ c-2b+1 & \text{if } c > 2b-1 \end{cases}$$

$$\text{and let } \alpha = \begin{cases} 2b-1-c & \text{if } c \leq 2b-1 \\ 0 & \text{if } c > 2b-1. \end{cases}$$

Let $P_3 = (v_1, v_2, v_3)$ be a path on three vertices. Attach b – 1 pendant vertices $u_1, u_2, ..., u_{b-1}$ to v_1 and $b-1+k$ pendant vertices $w_1, w_2, ..., w_{b-1+k}$ to v_3. Add the edges $u_1w_1, u_2w_2, ..., u_\alpha w_\alpha$. For the resulting graph G, we have $\gamma(G) = 2 = a$. Now $\{u_1, u_2, ..., u_{b-1}, v_3\}$ is a minimal dominating set of maximum cardinality containing v_3 and hence $\Gamma^{max}(v_3, G) = b$. Also $\{w_1, w_2, ..., w_{b-1+k}, v_1\}$ is a minimal dominating set of maximum cardinality containing v_1 and hence $\Gamma^{max}(v_1, G) = b + k$. Also $\{v_2, u_1, ..., u_{b-1}, w_{\alpha+1} + w_{\alpha+2}, ..., w_{b-1+k}\}$ is a minimal dominating set of maximum cardinality containing v_2 and hence $\Gamma^{max}(v_2, G) = 2b-1+k-\alpha$. If $c \leq 2b-1$, then $2b-1+k-\alpha = 2b-1+0-(2b-1-c) = c$. If $c < 2b-1$, then $2b-1+k-\alpha = c$. Thus $\Gamma^{max}(v_2, G) = c$. By a similar argument we have $\Gamma^{max}(u_i, G) = \Gamma^{max}(w_i, G) = c$. Hence it follows that $\Gamma^{m,M}(G) = b$ and $\Gamma(G) = c$.

Case ii. $a \geq 3$.

$$\text{Let } k = \begin{cases} 0 & \text{if } c \leq 2b-a \\ c-2b+a & \text{if } c > 2b-a \end{cases}$$

$$\text{and let } \alpha = \begin{cases} 2b-a-c & \text{if } c \leq 2b-a \\ 0 & \text{if } c > 2b-a. \end{cases}$$

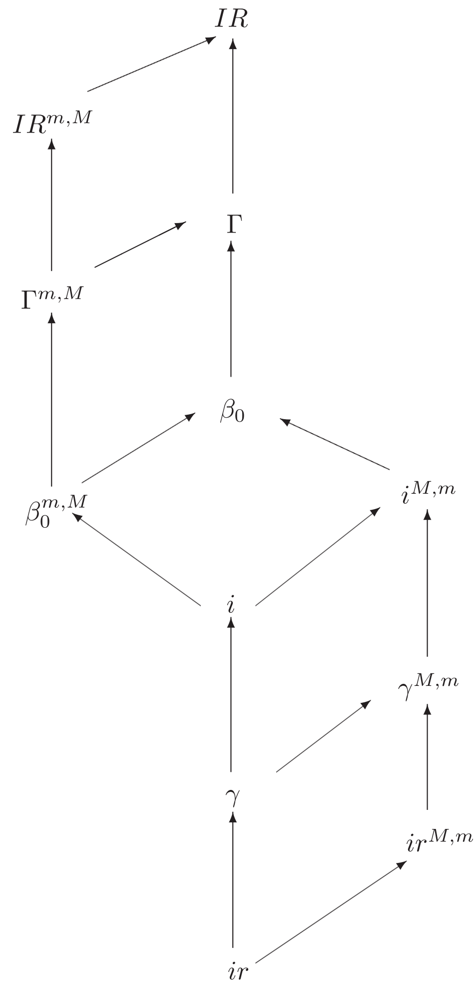

Figure 1. Relation between parameters.

Let $P = (v_1, v_2, ..., v_a)$ be a path on a vertices. Attach $b-(a-1)$ pendant vertices $u_1, u_2, ..., u_{b-(a-1)}$ to v_1, attach $b-(a-1)+k$ pendant vertices $w_1, w_2, ..., w_{b-(a-1)+k}$ to v_a and attach a pendant vertex x_i to each v_i, $2 \leq i \leq a-1$. If $c \leq 2b-a$, we add the edges $u_1w_1, u_2w_2, ..., u_\alpha w_\alpha$. For the resulting graph G, $\{v_1, v_2, ..., v_a\}$ is a minimum dominating set of G and hence $\gamma(G) = a$. Now $\{v_1, v_2, ..., v_{a-1}, w_1, w_2, ..., w_{b-(a-1)+k}\}$ is a dominating set of maximum cardinality containing $v_1$and hence

$$\Gamma^{m,M}(v_1) = b + k = \begin{cases} b & \text{if } c \leq 2b-a \\ c-2b+a & \text{if } c > 2b-a. \end{cases}$$

Also $\{u_1, u_2, ..., u_{b-a+1}, v_2, v_3, ..., v_a\}$ is a dominating set of maximum cardinality containing v_a and hence $\Gamma^{max}(v_a, G) = b$. Also $I = \{v_2, v_3, ..., v_{a-1}, u_1, u_2, ..., u_{b-a+1}, w_{\alpha+a}, ..., w_{b-(a-1)+k}\}$ is a dominating set of maximum cardinality containing v_i, $2 \leq i \leq a-1$, and hence $\Gamma^{max}(v_i, G) = c$. Hence it follows that $\gamma(G) = a$, $\Gamma^{m,M}(G) = b$ and $\Gamma(G) = c$. □

5. Graphs with $\Gamma^{m,M}(G)=2$

In this section we present a few results on graphs with $\Gamma^{m,M}(G) = 2$.

Theorem 5.1. *Let G be a connected graph. Then $\Gamma^{m,M}(G) = 2$ if and only if $\Delta < n-1$ and there exists a vertex $v \in V$ such that the following conditions hold.*

(i) *The induced subgraph $\langle V(G) - N[v] \rangle$ is complete.*
(ii) *If there exists a subset $S \subset N(v)$ such that S dominates $V(G) - N[v]$, then there exists a vertex $x \in S$ such that x dominates $V(G) - N[v]$.*

Proof. Suppose $\Gamma^{m,M}(G) = 2$. Clearly $\Delta < n-1$. Now let $v \in V$ be such that $\Gamma^{max}(v, G) = 2$.

(i) If $\langle V(G) - N[v] \rangle$ is not complete, then there exist at least two vertices u and w in $V(G) - N[v]$ which are non-adjacent and hence $\Gamma^{max}(v, G) \geq 3$, which is a contradiction. Hence $\langle V(G) - N[v] \rangle$ is complete.
(ii) Suppose there exists a subset $S \subset N(v)$ such that S dominates $V(G) - N[v]$. If no vertex $x \in S$, dominates $V(G) - N[v]$, then $\Gamma^{max}(v, G) \geq 3$, which is a contradiction. Hence there exists a vertex $x \in S$ such that x dominates $V(G) - N[v]$.

To prove the converse, let $\Delta < n-1$ and there exists a vertex v satisfying the conditions of the theorem. Then $\{v, w\}$, where $w \in V(G) - N[v]$, is a minimal dominating set so that $\Gamma^{max}(v, G) \geq 2$. Now let R be any minimal dominating set of maximum cardinality containing the vertex v. The set R contains at most one element of $V(G) - N[v]$, by (i). If $|R \cap (V(G) - N[v])| = 1$, then $|R| = 2$. Suppose now $R \subseteq N[v]$. Since $\Gamma^{max}(v, G) \geq 2, R \cap N(v) \neq \phi$ and it follows from (ii) that $|R \cap N(v)| = 1$. Hence $|R| = 2$. Thus $\Gamma^{max}(v, G) = 2$ and since $\Delta < n-1$, $\Gamma^{m,M}(G) = 2$. □

Observation 5.2. *Let G be a graph with $n \geq 3$ and $\delta = 0$. Then $\Gamma^{m,M}(G) = 2$ if and only if $G = G_1 \cup K_1$, where $\Delta(G_1) = n-2$.*

Theorem 5.3. *Let G be a connected graph of order n with $\Gamma^{m,M}(G) = 2$. Then $(n-1) \leq |E(G)| \leq \lfloor \frac{n(n-2)}{2} \rfloor$. Also $|E(G)| = n-1$ if and only if G is isomorphic to the graph G_1 obtained from the star $K_{1,n-2}$ with exactly one edge subdivided or the graph G_2 obtained from the star $K_{1,n-3}$ with one edge subdivided twice. Further $|E(G)| = \lfloor \frac{n(n-2)}{2} \rfloor$ if and only if $G \cong K_n - M$ where M is a perfect matching, when n is even and $M = M_1 \cup \{e\}$ where M_1 is a maximum matching and e is any edge incident with the unique M_1-unsaturated vertex, when n is odd.*

Proof. Let G be a connected graph of order n with $\Gamma^{m,M}(G) = 2$. The lower bound is obvious since G is connected. Since $\Gamma^{m,M}(G) = 2$, we have $\Delta < n-1$. Hence $\deg v \leq n-2$ for all $v \in V(G)$, so that $|E(G)| \leq \frac{n(n-2)}{2}$.

Let G be a graph with $\Gamma^{m,M}(G) = 2$ and $|E(G)| = n-1$. Then G is a tree. Let $v \in V(G)$ be such that $\Gamma^{max}(v, G) = 2$. Since $\langle V(G) - N[v] \rangle$ is complete it follows that $\deg v = n-2$ or $n-3$. If $\deg v = n-2$, then G is isomorphic to G_1, and if $\deg v = n-3$, then G is isomorphic to G_2. The converse is obvious.

Now, let G be a connected graph with $\Gamma^{m,M}(G) = 2$ and $|E(G)| = \lfloor \frac{n(n-2)}{2} \rfloor$. It follows that if n is even, then $\deg v = n-2$ for all $v \in V$ and if n is odd, exactly one vertex of G has degree $n-3$ and all the remaining vertices have degree $n-2$. Hence the result follows. □

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References

[1] Acharya, B. D. (1980). The strong domination number of a graph and related concepts. *J. Math. Phys. Sci.* 14(5):471–475.
[2] Arumugam, S, Kala, R. (2002). Domsaturation number of a graph. *Indian J. Pure Appl. Math.* 33(11):1671–1676.
[3] Arumugam, S, Subramanian, M. (2007). Independence saturation and extended domination chain in graphs. *AKCE J. Graphs. Combin.* 4(2):59–69.
[4] Arumugam, S., Favaron, O, Sudha, S. (2010). Irredundance saturation number of a graph. *Aust. J. Combin.* 46:37–49.
[5] Chartrand, G, Lesniak, L. (2005). *Graphs and Digraphs*, 4th ed. Boca Raton: CRC.
[6] Cockayne, E. J., Hedetniemi, S. T, Miller, D. J. (1978). Properties of hereditary hypergraphs and middle graphs. *Can. Math. Bull.* 21(4):461–468.
[7] Haynes, T. W., Hedetniemi, S. T, Slater, P. J. (1998a). *Fundamentals of Domination in Graphs*. New York: Marcel Dekker, Inc.
[8] Haynes, T. W., Hedetniemi, S. T, Slater, P. J. (1998b). *Domination in Graphs - Advanced Topics*. New York: Marcel Dekker, Inc.