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Gallai-Ramsey number of an 8-cycle

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ABSTRACT

Given graphs G and H and a positive integer k, the Gallai-Ramsey number $gr_k(G : H)$ is the minimum integer N such that for any integer $n \geq N$, every k-edge-coloring of K_n contains either a rainbow copy of G or a monochromatic copy of H. These numbers have recently been studied for the case when $G = K_3$, where still only a few precise numbers are known for all k. In this paper, we extend the known precise Gallai-Ramsey numbers to include $H = C_8$ for all k.

KEYWORDS
Gallai-Ramsey; rainbow triangle; 8-cycle

1. Introduction

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called *rainbow* if no two edges have the same color.

Colorings of complete graphs which contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [7] first examined this structure under the guise of transitive orientations. This result was restated in [9] in the terminology of graphs and can also be traced back to [2]. For the following statement, a *trivial partition* is a partition into only one part.

Theorem 1 ([7, 9]). *In any coloring of a complete graph with at least 2 vertices containing no rainbow triangle, there exists a non-trivial partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In honor of this result, rainbow triangle-free colorings have been called *Gallai colorings*. The partition given by Theorem 1 is called a *Gallai partition* or *G-partition* for short. Given a Gallai coloring of a complete graph and its associated G-partition, define the *reduced graph* of this partition to be the induced subgraph consisting of exactly one vertex from each part of the partition. Note that the reduced graph is a 2-colored complete graph.

When considering 2-colored complete graphs, a very natural problem to consider is the Ramsey problem of finding a monochromatic (one-colored) copy of some desired subgraph. Given a graph G, let $R_k(G)$ denote the *k-color Ramsey number* of G, namely the minimum integer M such that for any $m \geq M$, any coloring of K_m using at most k colors contains a monochromatic copy of G. We refer to the dynamic survey [11] for results about Ramsey numbers.

Combining the concepts of Ramsey numbers and rainbow triangle free colorings, we arrive at the following definition of Gallai-Ramsey numbers.

Definition 1. Given two graphs G and H, the k-colored Gallai-Ramsey number $gr_k(G : H)$ is defined to be the minimum integer n such that every coloring of the complete graph on n vertices using at most k colors contains either a rainbow copy of G or a monochromatic copy of H.

The general behavior of Gallai-Ramsey numbers when G is a triangle depends on the chromatic number of H in the following sense.

Theorem 2 ([8]). *Let H be a fixed graph with no isolated vertices. Let k be an integer with $k \geq 1$. If H is not bipartite, then $gr_k(K_3 : H)$ is exponential in k. If H is bipartite, then $gr_k(K_3 : H)$ is linear in k.*

With this result in mind, the orders of magnitude in the following general bounds for cycles should not be surprising. For the sake of notation, let C_n be the cycle of order n and let P_n be the path of order n.

Theorem 3 ([4, 10]). *Given integers $n \geq 2$ and $k \geq 1$,*

$$(n-1)k + n + 1 \leq gr_k(K_3 : C_{2n}) \leq (n-1)k + 3n.$$

Theorem 4 ([4, 10]). *Given integers $n \geq 2$ and $k \geq 1$,*

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

It is commonly believed that the lower bounds in these results are sharp. For $gr_k(K_3 : C_n)$ with $3 \leq n \leq 6$, the exact numbers are shown below.

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Theorem 5 ([1, 3, 8]).

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{otherwise.} \end{cases}$$

Theorem 6 ([4]). *For any positive integer* $k \geq 2, gr_k(K_3 : C_4) = k + 4$.

Theorem 7 ([5]). *For any positive integer* $k \geq 2$, $gr_k(K_3 : C_5) = 2^{k+1} + 1$ *and* $gr_k(K_3 : C_6) = 2k + 4$.

These and other related results in the area are collected in the dynamic survey [6]. Our main result is the following which extends the known Gallai-Ramsey numbers for even cycles to include the next open case.

Theorem 8. *For* $k \geq 1, gr_k(K_3 : C_8) = 3k + 5$.

The lower bound on the Gallai-Ramsey number in Theorem 8 follows from Theorem 3. Our proof of Theorem 8, particularly the use of Lemma 1 below, suggests that if the Gallai-Ramsey numbers were completely understood for all linear forests, then we may be able to establish the numbers for all cycles. This is somewhat complementary to the results of [10] where the bounds for even cycles were used to establish bounds for paths.

For more general notation, define $gr_k(G : H_1, H_2, \ldots, H_k)$ to be the minimum integer N such that every coloring of K_n for $n \geq N$ using at most k colors contains either a rainbow copy of G or a monochromatic copy of H_i in color i for some i. For a shorthand version of this, we will also abuse notation and let $gr_k(G : tH, (k-t)K) = gr_k(G : H, H, \ldots, H, K, K, \ldots, K)$ where H appears t times and K appears the remaining k – t times for some integer t with $0 \leq t \leq k$.

Lemma 1. *For integers k and t with* $k \geq 2$ *and* $0 \leq t \leq k$

$$gr_k(K_3 : tP_5, (k-t)P_3) \leq t + 4.$$

Proof. The proof is by induction on t. If $t = 0$, the result is trivial since we are looking for a P_3 in each color and it is easy to see that $gr_k(K_3 : P_3) = 3$ for all $k \geq 3$. So suppose $t \geq 1$.

Let G be a Gallai colored K_n where $n = t + 4$, consider a G-partition of G, and let H be a largest part of this partition. If $3 \leq |H| \leq n - 3$, then there are three vertices in $G \setminus H$ such that at least two of them have the same color on all edges to H. Since this graph contains a monochromatic $K_{2,3}$, this produces a monochromatic P_5, so we may assume that either $|H| \leq 2$ or $|H| \geq n - 2$. Our first goal is to show that $|H| \geq n - 2$.

If $|H| = 1$, then G is simply a 2-coloring of K_n for $n = t + 4$. This contains the desired monochromatic P_5 or P_3 since $R(P_3, P_5) = 5$ and $R(P_5, P_5) = 6$. So suppose $|H| = 2$. If $t = 1$, then to avoid creating a monochromatic P_5, there can be at most two vertices in $G \setminus H$ with all color 1 on edges to H but since $|G \setminus H| = 3$, there must be exactly two such vertices. The remaining vertex has both edges in another color, making the desired monochromatic P_3. Next suppose $t = 2$, so $n = t + 4 = 6$. With $|G \setminus H| = 4$, there must be precisely two pairs of vertices with each of the (first) two colors on edges to H, say red and blue. Each edge between these two pairs of vertices must be either red or blue, but any such edge would create a monochromatic P_5. Thus, we assume $t \geq 3$, so $n = t + 4 \geq 7$. Since $|H| = 2$, there are at least 5 vertices in $G \setminus H$ so at least three of these vertices have the same color on edges to H. This contains a monochromatic $K_{2,3}$, which contains the desired monochromatic P_5. Together, these observations mean that we may assume that $|H| \geq n - 2$.

Since each vertex in $G \setminus H$ has all one color on edges to H, the vertices of $G \setminus H$ must have distinct colors on edges to H to avoid a monochromatic P_5. Note that these colors must be within the first t colors, say t (and $t - 1$ if there are two such vertices), since otherwise this is already a monochromatic P_3. Also, if H contains a P_3 in one of these colors, then using the vertex of $G \setminus H$ with edges in the same color to H, we find a monochromatic P_5. By induction on t applied within H, H contains either a monochromatic P_5 in one of the first $t - 1$ colors (or $t - 2$ if $|G \setminus H| = 2$) or a monochromatic P_3 in one of the remaining $k - (t - 1)$ colors (respectively $k - (t - 2)$). This monochromatic path is either the desired path or can be used to construct the desired path as observed above, completing the proof of Lemma 1. □

In our arguments, we occasionally use classical Ramsey numbers. The following case will be helpful.

Theorem 9 ([11]). $R_2(C_8) = 11$.

For the sake of our next lemma, we need an extra definition. Given sets of graphs $\mathscr{G}$ and $\mathscr{H}$, define $R(\mathscr{G}, \mathscr{H})$ to be the minimum integer N such that any 2-coloring of K_n (say using red and blue) for $n \geq N$ contains either a copy of a graph in $\mathscr{G}$ in red or a copy of a graph in $\mathscr{H}$ in blue.

Lemma 2. $R(\{C_4, P_5\}, \{C_4, P_5\}) = 5$.

Proof. If we consider the unique 2-coloring of a K_5 with no monochromatic triangles, then there is a C_5 in each color. Thus, we also have the desired P_5 in both colors. We may therefore assume that all other 2-colorings of K_5 have a monochromatic triangle. Let $a_1, a_2, a_3 \in A$ be a monochromatic K_3, say in red, and $b_1, b_2 \in B$ be the two remaining vertices of the K_5. If all the edges from A to B are in one color, then there exists a monochromatic C_4 in that color. Without loss of generality, let e be a red edge a_1b_1. To avoid a C_4 in red, we get that the edges a_2b_1 and a_3b_1 are blue. To avoid getting a P_5 in red we get that the edges a_2b_2 and a_3b_2 are also blue. Now we can clearly see that these blue edges make a C_4 on $b_1 - a_2 - b_2 - a_3 - b_1$. □

2. Proof of Theorem 8

In order to prove Theorem 8, we actually prove the following slightly stronger result. For the precise statement, let $G_3 = C_8$, $G_2 = P_7$, $G_1 = P_5$, and $G_0 = P_3$. Note that all of these graphs are subgraphs of C_8 and represent the results of removing vertices from C_8. Theorem 8 follows from Theorem 10 by setting $i_j = 3$ for all j.

Theorem 10. *For $k \geq 1$, and for $0 \leq i_j \leq 3$ for all $1 \leq j \leq k$,*

$$gr_k(K_3 : G_{i_1}, G_{i_2}, \ldots, G_{i_k}) \leq \sum_{j=1}^{k} i_j + 5.$$

Proof. Let $\Sigma = \sum i_j$. The proof is by induction on Σ. If $\Sigma = 0$, the result is trivial since in each color we are only looking for P_3 and it is easy to see that $gr_k(K_3 : P_3) = 3$. Thus, suppose $\Sigma \geq 1$ so $n \geq \Sigma + 5 \geq 6$. Let G be a k-coloring of K_n with no rainbow triangle and no monochromatic G_{i_j} for any j. Let T be a largest set of vertices in G with the properties that

- each vertex in T has one color on all its edges to $G \setminus T$, and
- $|G \setminus T| \geq 4$.

Note that $T = \emptyset$ is possible. Let $T_1, T_2, \ldots, T_k$ denote the sets of vertices in T such that each vertex in T_j has all edges in color j to the vertices in $G \setminus T$. If $|T_j| > i_j$, then $T_j \cup (G \setminus T)$ contains the desired monochromatic copy of a graph G_{i_j} in color j. Thus, $|T_j| \leq i_j$ for all j. More generally, suppose $T \neq \emptyset$. Then, by induction on Σ applied within $G \setminus T$, there exists a copy, say H, of $G_{i_j - a}$ in color j for some j where $a = |T_j|$ with $1 \leq a \leq 3$. Then the graph consisting of edges of color j induced on $H \cup T_j$ along with $a - 1$ other vertices of $G \setminus (T_j \cup H)$ contains a copy of G_{i_j} in color j, the desired subgraph. Thus, we may assume that $T = \emptyset$.

Consider a G-partition of G and let A be a largest part of this partition. Note that if $|A| \geq 4$, we can let $T = G \setminus A$ and apply induction as above so we may assume $|A| \leq 3$. By the choice of A, the following fact becomes immediate.

Fact 1. *Every part of the G-partition has order at most* 3.

By Lemma 2, if there are at least five parts of order at least 2, then there is a monochromatic C_8 since the 2-blow-up of a C_4 or a P_5, replacing each vertex by a 2-set of vertices, each containing a C_8. We now prove several helpful claims, most of which provide a monochromatic C_8 under certain restrictions.

Claim 1. *If there are two parts of order 3 and at least five more vertices, then there exists a monochromatic C_8.*

Proof. Let A and B be the two parts of order 3, say with all red edges between them. Let $C = \{v_1, v_2, v_3, v_4, v_5\}$ be a set of 5 of the remaining vertices in $G \setminus (A \cup B)$. If all edges between C and $A \cup B$ were blue, there is clearly a blue C_8, so suppose there are some red edges, say from v_1 to A. To avoid creating a red C_8, all other vertices in C must have blue edges to B. To avoid creating a blue C_8, all of C must have red edges to A and so, by symmetry, v_1 (and so all of C) must also have blue edges to B. Any two red edges within C would produce a red C_8 and any two blue edges within C would produce a blue C_8 so there can be at most one red and at most one blue edge within C. Since, by Fact 1, all parts of the G-partition have order at most 3, this is clearly a contradiction, completing the proof of Claim 1. □

Claim 2. *If there is one part of order 3, one part of order at least 2 and at least six additional vertices, then there exists a monochromatic C_8.*

Proof. Let A be the set of order 3 and let B be the set of order at least 2 and assume all edges between A and B are red. By Claim 1, we may assume that $|B| = 2$ and none of the additional vertices form a part of the G-partition of order 3. Label the additional vertices as v_i where $1 \leq i \leq 6$. If there is a vertex, say v_1, with red edges to B and two other vertices, say v_2 and v_3, with red edges to A, then we have a red C_8 using $B - v_1 - B - A - v_2 - A - v_3 - A - B$.

Suppose first that no vertex v_i has red edges to B, which means that all vertices v_i have all blue edges to B. No three vertices v_i can have blue edges to A since otherwise we could find a blue C_8, so this means that at least four vertices v_i must have red edges to A. Without loss of generality, let $C = \{v_1, \ldots, v_4\}$ be this set of four vertices. Any two red edges within C would allow for the construction of a red C_8. Also since no three of the vertices in C form a part of our G-partition, there can be at most two edges of colors other than red or blue within C and these must induce a matching. This means that at least three edges within C are blue and they must contain a blue P_4, say $v_1v_2v_3v_4$. If both v_5 and v_6 have blue edges to A, then $v_1 - v_2 - v_3 - B - v_5 - A - v_6 - B - v_1$ is the desired blue C_8. Thus, we may assume, without loss of generality, that v_5 also has red edges to A. By the same argument, the blue graph induced on $C \cup \{v_5\}$ contains a blue P_5, say P from v_1 to v_5. Then $v_1 - P - v_5 - B - v_6 - B - v_1$ produces a blue C_8.

The previous argument means that we may assume there is a vertex, say v_1, with red edges to B. As noted, this means that at most one other vertex, say v_2, can have red edges to A, so all other vertices in $\{v_3, \ldots, v_6\}$ have blue edges to A. Certainly no two of these vertices may have blue edges to B, meaning that at least three of them, say v_3, v_4, v_5 have red edges to B. By the same argument as above with v_3 in place of v_1, there can actually be at most one vertex v_i with red edges to A, meaning that there are five vertices with blue edges to A. Let C be this set of vertices and note that at least four of the vertices in C also have red edges to B. If there are two blue edges within C, we may construct a blue C_8, so suppose there is at most one. Since the vertices of C do not form parts of order 3 in our G-partition, there are at most two edges of colors other than red or blue within C and these must induce a matching. This leaves at least 7 red edges within C. Trivially C contains a red P_5, say P, starting and ending at vertices with red edges to B, say v_4 and v_5. Then $v_4 - P - v_5 - B - A - B - v_4$ is the desired red C_8, completing the proof of Claim 2. □

Claim 3. *If there is one set of order at least 3 and at least nine more vertices, then there exist a monochromatic C_8.*

Proof. Let A be the part of order 3. We define B to be the set of vertices with red edges to A, and C to be the set of vertices with blue edges to A. By the pigeon hole principle at least five edges will have the same color edges to A, say $|B| \geq 5$.

If $|B| = 5$, then $A \cup B$ induces a red $K_{3,5}$ and $|C| = 4$ so $A \cup C$ induces a blue $K_{3,4}$. To avoid a rainbow triangle, each edge between B and C must be red or blue. Within this 2-colored $K_{4,5}$, the graph induced on the edges between B and C, there must be a monochromatic P_3. Regardless of the color or placement, this easily creates a monochromatic C_8.

Now suppose $|B| \geq 6$. Within B, there is at most one red edge since otherwise we could easily construct a red C_8. Also the edges that are neither red nor blue induce a matching since there is no part of the G-partition of order at least 3 within $B \cup C$ (by Claim 1). In particular, this means that the minimum degree of the graph induced on the blue edges within B is at least $|B| - 3$, so there is a blue Hamiltonian cycle within B. If $|B| = 8$, then this is the desired blue C_8 and if $|B| = 9$, we actually have even more blue edges so the blue graph is pancyclic and we again find a blue C_8. Otherwise, each vertex in C has at most one red edge to B because otherwise we could construct a red C_8. To avoid a rainbow triangle, this means that all but one edge from each vertex in C to B must be blue, meaning that each vertex in C can be absorbed into a blue Hamiltonian cycle of B to again create a blue C_8, completing the proof of Claim 3. □

Claim 4. *If there are three sets of order at least 2 and at least five more vertices, then there exists a monochromatic C_8.*

Proof. Let A, B and C be the sets of order 2 and label the remaining vertices as v_i where $1 \leq i \leq 5$. First suppose A, B and C have all red edges between them. If two of the other vertices, say v_1 and v_2, have red edges to at least two of the sets, say A and B, we can find a red C_8, $v_1 - A - C - B - v_2 - B - C - A - v_1$. Therefore, we can have either at most one vertex v_i with red edges to the sets or at most one set with red edges to the vertices v_i. This means that at least 4 vertices outside have blue edges to at least two of the sets. This induces a blue $K_{4,4}$ which contains a blue C_8.

Thus, we may assume that the edges between A and C are blue while all edges from B to $A \cup C$ are red. If none of the vertices v_i have red edges to A or C, then this induces a blue $K_{4,5}$ which contains the desired blue C_8. Thus we may assume that at least one vertex, say v_1, has red edges to either A or C, say A. To avoid a red C_8, all other vertices v_i for $i \geq 2$ must have blue edges to C. To avoid a blue C_8, no three of these vertices can have blue edges to A, so that means at least two of them, say v_2 and v_3, have red edges to A. By symmetry, this means that v_1 also has blue edges to C.

To avoid a red C_8, there can be at most one red edge within $\{v_1, v_2, v_3\}$. Since there is no part of our G-partition of order 3 among the vertices v_i and a part of order 2 within $\{v_1, v_2, v_3\}$ would mean that the remaining vertex has two edges of the same color to the part, this means that there are at least 2 blue edges within these three vertices, say $v_1 v_2$ and $v_2 v_3$. If both v_4 and v_5 have blue edges to A, then $C - v_4 - A - v_5 - A - C - v_1 - v_2 - v_3 - C$ is a blue C_8. This means that one of v_4 or v_5, say v_4, must have red edges to A.

To avoid a red C_8, there can be at most one red edge within $\{v_1, v_2, v_3, v_4\}$ and since there is no part of order 3 and a 2-part would imply two edges of the same color, we must have a blue P_4 within these vertices, say $v_1 v_2 v_3 v_4$. Now if v_5 has blue edges to A, then $C - v_5 - A - C - v_1 - v_2 - v_3 - v_4 - C$ is a blue C_8. This means that v_5 must also have red edges to A. By the same logic as above, there are at most 3 non-blue edges within $\{v_1, ..., v_5\}$, so there is a blue P_5, say $v_1 v_2 ... v_5$. Then $C - A - C - v_1 - v_2 - v_3 - v_4 - v_5 - C$ is a blue C_8, completing the proof. □

By Theorem 9, there are at most 10 parts in our G-partition. By Fact 1, no part has order larger than 3 and by Lemma 2, there are at most 4 parts of order at least 2. By Claim 1, if there are 2 parts of order 3, then $n \leq 10$. By Claim 2, if there is one part of order 3 and at least one part of order 2, then $n \leq 10$ again. By Claim 3, if there is any part of order 3, then $n \leq 11$. Thus, we may assume that either $n \leq 11$ or all parts have order at most 2. By Claim 4, if there are 3 parts of order 2, then $n \leq 10$ so we may assume there are at most 2 parts of order 2. With at most 10 parts total, this means that $n \leq 12$.

To complete the proof of Theorem 10, we consider cases based on small values of n, and therefore small values of $\Sigma = n - 5$.

Case 1. $\Sigma = 1$.

With loss of generality, suppose $G_1 = P_5$ and $G_i = P_3$ for $i \geq 2$. Therefore, we have $G = K_6$ we want to show $gr_k(K_3 : P_5, P_3, P_3, ..., P_3) = 6$. Since red is the only color allowed to contain adjacent edges, each other color induces only a matching. In fact, to avoid a rainbow triangle, the edges induced on all colors other than red together must induce a matching. The complement of this matching contains a P_5 in red to easily complete the proof in this case.

Case 2. $\Sigma = 2$.

Subcase 2.1. $gr_k(K_3 : P_7, P_3, ..., P_3) = 7$

In this case, all colors other than red together induce a matching M. In $K_7 \setminus M$, it is easy to find a P_7.

Subcase 2.2. $gr_k(K_3 : P_5, P_5, P_3, ..., P_3) \leq 7$.

This result follows from Lemma 1.

Case 3. $\Sigma = 3$.

Subcase 3.1. $gr_k(K_3 : C_8, P_3, P_3, ..., P_3) = 8$.

In this case, all colors other than red together induce a matching M. In $K_8 \setminus M$, it is easy to find a C_8.

Subcase 3.2. $gr_k(K_3 : P_7, P_5, P_3, ..., P_3) = 8$.

Since $R_2(P_7, P_5) = 8$, we may assume that there are at most 7 parts in the partition. Thus, there must exist a part of the partition of order at least 2. Other than the first two colors red and blue, all other colors together induce a matching so if we choose our G-partition to have the most possible parts, we may assume all parts have order at most 2.

First suppose there exists exactly one part of order 2, call it A. To avoid creating a blue P_5, there can be at most 2 vertices in $G \setminus A$ with blue edges to A. Call these vertices A_{blue} and let A_{red} denote the remaining vertices of $G \setminus A$, those with all red edges to A. Note that $|A_{red}| \geq 4$. To avoid creating a red P_7, each vertex of A_{blue} has at most 2 red edges to A_{red}, so all other edges from A_{blue} to A_{red} must be blue. If $|A_{blue}| = 2$, then we have a blue P_5 immediately using these blue edges to A_{red} so suppose $|A_{blue}| \leq 1$, meaning that $|A_{red}| \geq 5$. To avoid creating a red P_7, there can be at most 1 red edge within A_{red}, so there must be the claimed blue P_5 within A_{red}.

Next suppose 2 sets have size 2, call them A and B. If blue appears between A and B then all other edges will be red to the 2 sets. This gives us a $K_{4,4}$ which contains a P_7. Therefore the edges between A and B must be red. If there are at least 2 vertices outside with red to A and one vertex to B then there is a P_7 in red. On the other hand if there are 2 vertices outside with blue to A, then we might as well have blue in between the 2 sets. Therefore we have found our desired P_7 in one color and P_5 in the other color.

Subcase 3.3. $gr_k(K_3 : P_5, P_5, P_5, P_3, ...P_3) \leq 8$.

This subcase follows from Lemma 1.

The remaining cases, when $\Sigma \in \{4, 5, 6, 7\}$, follow from similar (albeit tedious) case analysis or by straightforward computer search. □

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No potential conflict of interest was reported by the authors.

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