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# On the complexity of some quorum colorings problems of graphs

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## ABSTRACT

A partition  $\pi = \{V_1, V_2, \dots, V_k\}$  of the vertex set  $V$  of a graph  $G$  into  $k$  color classes  $V_i$ , with  $i \in \{1, \dots, k\}$  is called a quorum coloring if for every vertex  $v \in V$ , at least half of the vertices in the closed neighborhood  $N[v]$  of  $v$  have the same color as  $v$ . The maximum order of a quorum coloring of  $G$  is called the quorum coloring number of  $G$  and is denoted  $\psi_q(G)$ . In this paper, we give answers to two open problems stated in 2013 by Hedetniemi, Hedetniemi, Laskar and Mulder. In fact, we prove that the decision problem associated with  $\psi_q(G)$  is  $NP$ -complete when the input graph is a 4-regular graph. We also show that the decision problem asks whether a given graph  $G$  has a quorum coloring of order at least 2 is  $NP$ -complete too.

## KEYWORDS

Defensive alliance; quorum colorings; satisfactory partition; cost-effective partition; complexity

## 2000 MATHEMATICAL SUBJECT CLASSIFICATION

05C15; 05C69

## 1. Introduction

Let  $G = (V, E)$  be a simple graph with order  $n = |V|$ . The complement graph of  $G$  is denoted by  $\bar{G}$ . The graph induced in  $G$  by a subset  $S$  of  $V$  is denoted by  $G[S]$ . For every vertex  $v \in V$ , the open neighborhood  $N_G(v)$  is the set  $\{u \in V(G) : uv \in E(G)\}$  and the closed neighborhood of  $v$  is the set  $N_G[v] = N_G(v) \cup \{v\}$ . The degree of a vertex  $v$  in  $G$  is  $d_G(v) = |N_G(v)|$ , while the degree of  $v$  in the complement graph  $\bar{G}$  is denoted by  $\bar{d}_G(v)$ . More generally, the degree of a vertex  $v$  in  $G[S]$  is denoted by  $d_S(v)$ , while the degree of  $v$  in  $\bar{G}[S]$  is denoted by  $\bar{d}_S(v)$ .

A partition  $\pi = \{V_1, V_2, \dots, V_k\}$  of the vertex set  $V$  of a graph  $G$  into  $k$  color classes  $V_i$ , with  $i \in \{1, \dots, k\}$  is called a quorum coloring if for every vertex  $v \in V$ , at least half of the vertices in the closed neighborhood  $N_G[v]$  have the same color as  $v$ . The color classes  $V_i$  are called quorum classes. The maximum order of a quorum coloring of  $G$  is called the quorum coloring number of  $G$  and is denoted by  $\psi_q(G)$ . A quorum coloring of order  $\psi_q(G)$  is called a  $\psi_q$ -coloring. Quorum colorings were introduced by Hedetniemi, Hedetniemi, Laskar and Mulder [5]. The concept of Quorum colorings is closely related to the concept of defensive alliances in graphs introduced by Kristiansen, Hedetniemi and Hedetniemi [4]. Indeed, a defensive alliance in  $G$  is a subset  $S$  of  $V$  such that for every vertex  $v \in S$ ,  $|N_G[v] \cap S| \geq |N_G(v) \cap (V \setminus S)|$ . Hence every color class of a  $\psi_q$ -coloring is a defensive alliance. Note that Haynes and Lachniet in [3] were the first to introduce the problem of partitioning the vertex set  $V$  into defensive alliances. This problem was also studied later by Eroh and Gera [2]. However, we will adopt in this paper the definitions and notations given in [5].

A matching in a graph  $G = (V, E)$  is a set of edges  $M \subseteq E$  having the property that no two edges in  $M$  have a vertex in common. The matching number  $\beta_1(G)$  equals the maximum cardinality of a matching in  $G$ .

In [5], Hedetniemi et al. raised the following problems.

1. It is easy to see that for 1-regular graphs  $G$  of order  $n$ ,  $\psi_q(G) = n$ . It is also easy to determine the value of  $\psi_q(G)$  for any 2-regular graph  $G$ . In addition, since  $\psi_q(G) = \beta_1(G)$  for 3-regular graphs  $G$ , it is easy to determine, in polynomial time, the value of  $\psi_q(G)$  for 3-regular graphs. This leads us to the following decision problem:

### 4-REGULAR QUORUM

**Instance:** A 4-regular graph  $G = (V, E)$ , positive integer  $K \leq |V|$ .

**Question:** Does  $G$  have a quorum coloring of order at least  $K$ ?



2. What is the complexity of the following decision problem:

### QUORUM-ONE

**Instance:** Graph  $G = (V, E)$ .

**Question:** Is  $\psi_q(G) > 1$ ?

In 2018, Sahbi and Chellali [6] showed that the problem associated with  $\psi_q(G)$ , to which we refer as QUORUM-K, is  $NP$ -complete in general graphs, thus answering to an open problem stated in [5]. However, Questions 1 and 2 remain open. In this paper, we first show that the problem 4-REGULAR QUORUM is  $NP$ -complete by reducing the PARTITION INTO TRIANGLES problem in 4-regular graphs, to which we refer as 4-REGULAR PARTITION INTO TRIANGLES, to the problem 4-REGULAR QUORUM. The  $NP$ -hardness of the problem 4-REGULAR

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PARTITION INTO TRIANGLES was proven in 2011 by van Rooij et al. [7].

#### QUORUM-K

**Instance:** Graph  $G = (V, E)$ , positive integer  $K \leq |V|$ .

**Question:** Does  $G$  have a quorum coloring of order at least  $K$ ?

#### 4-REGULAR PARTITION INTO TRIANGLES

**Instance:** A 4-regular graph  $G = (V, E)$ .

**Question:** Can  $V$  be partitioned into 3-element sets

$S_1, S_2, \dots, S_{|V|/3}$  such that each  $S_i$  forms a triangle in  $G$ ?

Then, we prove that the problem QUORUM-ONE is  $NP$ -complete by firstly showing that its balanced version, to which we refer as BALANCED QUORUM, is  $NP$ -complete and secondly by reducing the problem BALANCED QUORUM to the problem QUORUM-ONE. The problem BALANCED QUORUM was inspired from the BALANCED SATISFACTORY PARTITION problem introduced by Bazgan et al. [1] and the polynomial reduction from the problem BALANCED QUORUM to the problem QUORUM-ONE is an adaptation of the proof of Proposition 1 in [1].

#### BALANCED QUORUM

**Instance:** Graph  $G = (V, E)$  of even order.

**Question:** Is there a partition  $\{V_1, V_2\}$  of  $V$  such that  $|V_1| = |V_2|$  and for every  $v \in V$ , if  $v \in V_i$  then  $d_{V_i}(v) + 1 \geq d_{V_{3-i}}(v)$ ?

For a graph  $G = (V, E)$  of even order, a bipartition  $\{V_1, V_2\}$  of the vertex set  $V$  is a *balanced quorum coloring* if  $|V_1| = |V_2|$  and each  $V_i$ , with  $i \in \{1, 2\}$  is a quorum class, that is, for every  $v \in V$ , if  $v \in V_i$  then  $d_{V_i}(v) + 1 \geq d_{V_{3-i}}(v)$ .

#### BALANCED SATISFACTORY PARTITION

**Instance:** Graph  $G = (V, E)$  of even order.

**Question:** Is there a partition  $\{V_1, V_2\}$  of  $V$  such that  $|V_1| = |V_2|$  and for every  $v \in V$ , if  $v \in V_i$  then  $d_{V_i}(v) \geq d_{V_{3-i}}(v)$ ?

In fact, we prove the  $NP$ -completeness of the problem BALANCED QUORUM by reducing it to the BALANCED CO-SATISFACTORY PARTITION problem, to which we refer in this paper as BALANCED COST EFFECTIVE BIPARTITION. The  $NP$ -completeness of the problem BALANCED COST EFFECTIVE BIPARTITION was proven in 2005 by Bazgan et al. [1].

#### BALANCED COST EFFECTIVE PARTITION

**Instance:** Graph  $G = (V, E)$  of even order.

**Question:** Is there a partition  $\{V_1, V_2\}$  of  $V$  such that  $|V_1| = |V_2|$  and for every  $v \in V$ , if  $v \in V_i$  then  $d_{V_i}(v) \leq d_{V_{3-i}}(v)$ ?

## 2. Answer to Question 1

**Theorem 1.** *Problem 4-REGULAR QUORUM is  $NP$ -complete.*

**Proof.** 4-REGULAR QUORUM is a member of  $\mathcal{NP}$ , since we can check in polynomial time that any partition of the vertices of a 4-regular graph  $G$  into at least  $K$  color classes is a quorum coloring. Let  $G$  be a 4-regular graph of order  $3q$ , instance of both problems 4-REGULAR PARTITION INTO TRIANGLES and 4-REGULAR QUORUM, and set  $K = q$ . We will show that the instance  $G$  has a solution with respect to the first problem if and only if the same instance has a solution with respect to the second problem, that is, we will show that  $G$  has a partition into triangles if and only if  $\psi_q(G) \geq K$ .

Suppose that  $G$  has a partition into triangles and let  $\pi = \{V_1, \dots, V_q\}$  be such a partition. Then, one can easily see that each  $V_i$  is a quorum class. Hence,  $\pi$  is a quorum coloring of order  $q = K$ .

Conversely, suppose that  $G$  has a quorum coloring of order at least  $K = q$  and let  $\pi' = \{V'_1, \dots, V'_q\}$  be such a coloring. Clearly,  $|V'_i| \geq \lceil \frac{4+1}{2} \rceil = 3$ , for every  $i \in \{1, \dots, q\}$ . Therefore,  $|V'_i| = 3$ , for every  $i \in \{1, \dots, q\}$  for otherwise,  $|V(G)| > 3q$ . Since each class of  $\pi'$  is a quorum class, all vertices of each  $V'_i$  are necessarily pairwise adjacent, that is, each  $V'_i$  is a triangle. Consequently,  $\pi'$  is a partition of  $G$  into triangles.  $\square$

The  $NP$ -completeness of the problem 4-REGULAR QUORUM being now established, an interesting open problem is to determine more generally the complexity of the decision problem associated with the quorum coloring number in  $r$ -regular graphs, for  $r > 4$ .

## 3. Answer to Question 2

In [1], Bazgan et al. proved that problem BALANCED SATISFACTORY PARTITION is polynomial-time reducible to problem SATISFACTORY PARTITION, where SATISFACTORY PARTITION was defined as follows:

#### SATISFACTORY PARTITION

**Instance:** Graph  $G = (V, E)$ .

**Question:** Is there a partition  $\{V_1, V_2\}$  of  $V$  such that for every  $v \in V$ , if  $v \in V_i$  then  $d_{V_i}(v) \geq d_{V_{3-i}}(v)$ ?

**Proposition 2.** [1] *BALANCED SATISFACTORY PARTITION is polynomial-time reducible to SATISFACTORY PARTITION.*

In the following proposition, we prove similarly that problem BALANCED QUORUM is polynomial time reducible to problem QUORUM-ONE by using the reduction of Proposition 2 and adapting its proof.

**Proposition 3.** *Problem BALANCED QUORUM is polynomial time reducible to problem QUORUM-ONE.*

**Proof.** Let  $G = (V, E)$  be a graph, instance of BALANCED QUORUM on  $n$  vertices. The graph  $G' = (V', E')$ , instance of QUORUM-ONE, is obtained from  $G$  by adding two cliques of size  $\frac{n}{2}$ ,  $A = \{a_1, \dots, a_{\frac{n}{2}}\}$  and  $B = \{b_1, \dots, b_{\frac{n}{2}}\}$ . In  $G'$ , in addition to the edges of  $G$ , all vertices of  $V$  are adjacent to all vertices of  $A$  and  $B$ . Also, each vertex  $a_i \in A$  is linked to all vertices of  $B$  except  $b_i$ ,  $i \in \{1, \dots, n\}$ .

Let  $\{V_1, V_2\}$  be a balanced quorum coloring of  $G$ . Then,  $\{V'_1, V'_2\} = \{V_1 \cup A, V_2 \cup B\}$  is a quorum coloring of  $G'$ . Indeed, we have

$$\begin{aligned}
 \text{for every } v \in A, \quad d_{V'_1}(v) + 1 &= d_A(v) + d_{V_1}(v) + 1 \\
 &= |A| - 1 + |V_1| + 1 \\
 &= |A| + |V_1| = |B| + \\
 &|V_2| = d_{V'_2}(v) + 1 > d_{V'_2}(v), \text{ and}
 \end{aligned}$$

$$\begin{aligned}
 \text{for every } v \in V_1, \quad d_{V'_1}(v) + 1 &= d_A(v) + d_{V_1}(v) + 1 \\
 &= |A| + d_{V_1}(v) + 1 \\
 &= |B| + d_{V_1}(v) + 1 \geq \\
 &|B| + d_{V_2}(v) = d_{V'_2}(v).
 \end{aligned}$$

By symmetry, we obtain analogously:

$$\begin{aligned}
 \text{for every } v \in B, \quad d_{V'_2}(v) + 1 &> d_{V'_1}(v), \text{ and} \\
 \text{for every } v \in V_2, \quad d_{V'_2}(v) + 1 &\geq d_{V'_1}(v).
 \end{aligned}$$

Conversely, let  $\{V'_1, V'_2\}$  be a quorum coloring of  $G'$ . Set  $V'_1 = V_1 \cup A_1 \cup B_1$  and  $V'_2 = V_2 \cup A_2 \cup B_2$  where  $V_i \subseteq V, A_i \subseteq A$  and  $B_i \subseteq B$ , with  $i \in \{1, 2\}$ . We claim that  $\{V_1, V_2\}$  is a balanced quorum coloring of  $G$ . We consider the following two cases.

**Case 1.** Either  $A_1 \cup B_1 = \emptyset$  or  $A_2 \cup B_2 = \emptyset$ .

Suppose without loss of generality that  $A_2 \cup B_2 = \emptyset$ . Therefore,  $A_1 \cup B_1 = A \cup B$  (i.e.:  $A_1 = A$  and  $B_1 = B$ ). Since  $\{V'_1, V'_2\}$  is a quorum coloring of  $G'$ , then we have by definition for every  $v \in V_2, d_{V'_2}(v) + 1 \geq d_{V'_1}(v)$ . Hence,  $d_{V_2}(v) + 1 \geq d_{V_1}(v) + n$ , or equivalently  $d_{V_2}(v) \geq d_{V_1}(v) + n - 1$ . Thus,  $G = K_n$  ( $n$  even) and any clique of even order has a balanced quorum coloring.

**Case 2.** Now assume that neither  $A_1 \cup B_1$  nor  $A_2 \cup B_2$  is empty. We consider the following two subcases.

**Subcase 2.1.**  $A_1 \cup B_1 = A$  and  $A_2 \cup B_2 = B$  (i.e.:  $B_1 = \emptyset$  and  $A_2 = \emptyset$ ).

Since  $\{V'_1, V'_2\}$  is a quorum coloring of  $G'$ , we have

$$\begin{aligned}
 \text{for every } v \in A, \quad d_{V'_1}(v) + 1 &= d_A(v) + d_{V_1}(v) + 1 \\
 &= |A| - 1 + |V_1| + 1 \\
 &= |A| + |V_1| \geq d_{V'_2}(v) \\
 &= d_B(v) + d_{V_2}(v) \\
 &= |B| - 1 + |V_2| \\
 &= |A| - 1 + |V_2| \text{ (since } |A| = |B| \text{)}.
 \end{aligned}$$

Consequently,

$$|V_1| \geq |V_2| - 1. \quad (1)$$

By symmetry, we obtain

$$\text{for every } v \in B, \quad |V_2| \geq |V_1| - 1. \quad (2)$$

Inequalities (1) and (2) imply that  $|V_1| \in \{|V_2| - 1, |V_2|, |V_2| + 1\}$ . Since  $n$  is even, we get  $|V_1| = |V_2|$ .

Then, by removing the vertices of  $A \cup B$ , we remove for each vertex of  $V_1$  and each vertex of  $V_2$ ,  $\frac{n}{2}$  neighbors in its class and  $\frac{n}{2}$  neighbors in the other class. Thus,  $\{V_1, V_2\}$  is a balanced quorum coloring of  $G$ .

**Subcase 2.2.** Either  $A$  or  $B$  is cut by the partition, with one part in  $V'_1$  and the second part in  $V'_2$ .

We now show that if  $a_i \in A_1$  for some  $i$ , then also  $b_i \in B_2$  for the same  $i$ . Assume by contradiction that  $b_i \in B_1$ . We have,

$$\begin{aligned}
 d_{V'_1}(a_i) + 1 \geq d_{V'_2}(a_i) &\iff d_{A_1}(a_i) + d_{B_1}(a_i) + d_{V_1}(a_i) + 1 \\
 &\geq d_{A_2}(a_i) + d_{B_2}(a_i) + d_{V_2}(a_i) \\
 &\iff |A_1| - 1 + |B_1| - 1 + |V_1| + 1 \\
 &\geq |A_2| + |B_2| + |V_2| \iff |V'_1| \geq |V'_2| + 1
 \end{aligned} \quad (3)$$

Now, let  $a_j \in A_2$ . Then, either  $b_j \in B_1$  or  $b_j \in B_2$ .

If  $b_j \in B_2$ , by analogy with the previous case we obtain,

$$|V'_2| \geq |V'_1| + 1 \quad (4)$$

However, inequality (4) contradicts inequality (3). Hence,  $b_j \in B_1$ .

Since  $\{V'_1, V'_2\}$  is a quorum coloring of  $G'$ , then

$$\begin{aligned}
 d_{V'_2}(a_j) + 1 \geq d_{V'_1}(a_j) &\iff d_{A_2}(a_j) + d_{B_2}(a_j) + d_{V_2}(a_j) + 1 \\
 &\geq d_{A_1}(a_j) + d_{B_1}(a_j) + d_{V_1}(a_j) \\
 &\iff |A_2| - 1 + |B_2| + |V_2| + 1 \\
 &\geq |A_1| + |B_1| - 1 + |V_1| \iff |V'_1| \leq |V'_2| + 1
 \end{aligned} \quad (5)$$

(3) and (5) imply that  $|V'_1| = |V'_2| + 1$ , which is impossible since  $|V(G')| = 2n$ .

In conclusion,  $|A_1| = |B_2|$  and  $|A_2| = |B_1|$ . So,  $|A_1 \cup B_1| = |A_2 \cup B_2| = \frac{n}{2}$ .

Moreover, we have

$$\begin{aligned}
 \text{for every } v \in A_1 : \quad d_{V'_1}(v) + 1 &\geq d_{V'_2}(v) \\
 &\iff d_{A_1}(v) + d_{B_1}(v) + d_{V_1}(v) \geq d_{A_2}(v) + d_{B_2}(v) + d_{V_2}(v) \\
 &\iff |A_1| - 1 + |B_1| + |V_1| + 1 \geq |A_2| + |B_2| - 1 \\
 &\quad + |V_2| \iff |V_1| \geq |V_2| - 1,
 \end{aligned} \quad (6)$$

and by symmetry for every  $v \in A_2$ ,

$$|V_1| \leq |V_2| + 1. \quad (7)$$

Inequalities (6) and (7) imply that  $|V_1| \in \{|V_2| - 1, |V_2|, |V_2| + 1\}$ . However,  $n = |V_1| + |V_2|$  is even and hence,  $|V_1| = |V_2|$ . Thus,  $\{V_1, V_2\}$  is a balanced quorum coloring of  $G$ .  $\square$

We state now our NP-completeness result.

**Theorem 4.** *Problem QUORUM-ONE is NP-complete.*

*Proof.* Clearly, QUORUM-ONE is in  $\mathcal{NP}$ . We reduce BALANCED COST EFFECTIVE PARTITION to BALANCED QUORUM which shows the NP-completeness of BALANCED QUORUM. Proposition 3 implies the NP-completeness of QUORUM-ONE. The reduction is as follows.

Let  $G$  be a graph of even order, instance of BALANCED COST EFFECTIVE PARTITION, and consider  $\bar{G}$  as instance of BALANCED QUORUM. We will show that the instance  $G$  has a solution if and only if the instance  $\bar{G}$  has a solution.

Suppose that the instance  $G$  of BALANCED COST EFFECTIVE PARTITION has a solution  $\{V_1, V_2\}$ . This is equivalent to,

$$\begin{aligned}
 \text{for every } i \in \{1, 2\}, \quad d_{V_i}(v) &\leq d_{V_{3-i}}(v) \\
 &\iff |V_i| - 1 - d_{V_i}(v) \geq |V_i| - 1 - d_{V_{3-i}}(v) \\
 &\iff \bar{d}_{V_i}(v) + 1 \geq |V_{3-i}| - d_{V_{3-i}}(v) \\
 &\iff \bar{d}_{V_i}(v) + 1 \geq \bar{d}_{V_{3-i}}(v)
 \end{aligned} \quad (8)$$

Inequality (8) means that  $\{V_1, V_2\}$  is a balanced quorum coloring of  $\bar{G}$ .  $\square$

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### References

- [1] Bazgan, C., Tuza, Z., Vanderpooten, D. (2005). Complexity and approximation of satisfactory partition problems. *Proceedings of the 11th International Computing and Combinatorics Conference (COCOON 2005)*, LNCS 3595: 829–838.
- [2] Eroh, L., Gera, R. (2012). Alliance partition number in graphs. *Ars Combin.* 103: 519–529.
- [3] Haynes, T. W., Lachniet, J. A. (2007). The alliance partition number of grid graphs. *AKCE Int. J. Graphs Combin.* 4(1): 51–59.
- [4] Hedetniemi, S. M., Hedetniemi, S. T., Kristiansen, P. (2004). Alliances in graphs. *J. Combin. Math. Combin. Comput.* 48: 157–177.
- [5] Hedetniemi, S. M., Hedetniemi, S. T., Laskar, R., Mulder, H. M. (2013). Quorum colorings of graphs. *AKCE Int. J. Graphs Comb.* 10(1): 97–109.
- [6] Sahbi, R., Chellali, M. (2018). On some open problems concerning quorum colorings of graphs. *Discrete Appl. Math.* 247: 294–299.
- [7] van Rooij, M. M., van Kooten Niekerk, M. E., Bodlaender, H. L. (2011). Partition into triangles on bounded degree graphs. *Proceedings of the 37th Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM 2011)*, *Lecture Notes in Computer Science* 6543: 558–569.