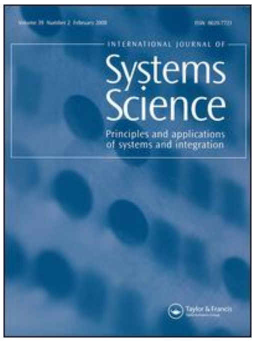

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High-order fully actuated system approaches: Part I. Models and basic procedure

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ABSTRACT
The state-space approaches have stayed in an absolutely dominant position in the field of systems and control for over a half century. Although a state-space representation is more suitable for deriving the state-response solution and observation (estimation), it does not provide as much convenience as desired for the control problem. In this paper, the concept of high-order fully actuated (HOFA) systems is firstly revisited, and it is pointed out that HOFA systems serve really as models for control systems rather than representing a small portion of physical control systems. Based on the HOFA model, a basic procedure is proposed for control of nonlinear systems satisfying certain conditions, whose first step converts the nonlinear systems into a pseudo strict-feedback system, and the second step establishes the HOFA model of the system. Once an HOFA model is derived, a controller can be immediately designed to make the closed-loop system a constant linear one with a desired eigenstructure. All the design degrees of freedom existing in the closed-loop system are also provided, which can be further utilised to achieve additional system performance. An example demonstrates the design procedure and shows the effect of the proposed HOFA approach.

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Nonlinear control; full-actuation; high-order systems; strict-feedback systems; closed-loop eigenstructure

1. First-order systems approaches

Most physical systems in the world are really governed by a series of physical laws, such as, Newton's Law, Lagrangian Equation, Theorem of Linear and Angular Momentum, Kirchhoff's Laws of Current and Voltage, etc. Modelled by these physical laws, the models of these physical systems are mostly described by differential equations of second- or high order. Therefore, second- and high-order systems are natural. Yet in the long developing period of control systems theory, these second- or high-order systems were mostly turned into first-order systems and treated in the first-order system framework. This produces the so-called first-order system approaches for the analysis and design of control systems. In contrast, high-order system approaches are those dealing with analysis and control of high-order systems using features of the high-order system representation (Duan, 2020a, 2020b, 2020c).

1.1. A brief history

First-order system approaches for control systems can be traced back to the year of 1750, when Euler proposed the order-reduction method for solving a high-order nonhomogeneous ordinary differential equation (Cui, 2010; Morris, 1990). Such an early contribution is in fact somehow equivalent to today's problem of state-response analysis of a control system represented by a state-space model. This preconceived fact has then led the way of the development of first-order system approaches for control systems.

In 1892, Aleksandr M. Lyapunov completed his Ph.D. thesis (Lyapunov, 1992), in which he proposed a complete stability theory for systems described by first-order ordinary differential equations and laid a firm theoretical base for stability analysis and design of control systems. Lyapunov stability theory was proposed for first-order systems and eventually and basically provides applications within the first-order system framework.

The middle of the last century was a very prosperous period for systems and control. Several very notable and significant contributions emerged.

Firstly, the well-known Bellman dynamic programming was proposed during the period of 1952–1957, which converts the problem of optimal control of discrete-time systems described by first-order

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difference equations into a problem of solving the well-known Hamilton–Jacobi–Bellman (HJB) equations (Bellman, 1952, 1954, 1957).

Secondly, the well-known Pontryagin's Maximum Principle was proposed in the period of 1956–1958 (Boltyanskiĭ et al., 1956, 1960; Pontryagin, 1959, 1960). Parallel to Bellman's dynamic programming, Pontryagin's Maximum Principle solves the problem of optimal control of a continuous-time nonlinear system, and the solution is given through a continuous HJB equation (Pontryagin et al., 1962; Rozonoer, 1959).

Thirdly, several great contributions were conducted by Rudolf E. Kalman. In 1959, he published the epoch-making paper Kalman (1960a) (see, also Kalman, 1960b) and proposed the celebrated state-space approaches for analysis and design of control systems described by the so-called state-space model represented by a first-order ordinary differential equation. One year later, he published the famous result, known as the Kalman filter, which solves the problem of state estimation in a linear discrete-time stochastic system described also by a first-order state-space model (Kalman, 1960c). This work was later generalised to linear continuous-time stochastic systems by himself and his coauthor, establishing the so-called Kalman-Bucy filtering theory (Kalman & Bucy, 1961). These results have been also extended to nonlinear systems (Julier & Uhlmann, 1997; Sunahara, 1970). Rudolf E. Kalman was a pioneer and advocator in bringing mathematics into control systems theory (Kalman, 2008; Kalman et al., 1969).

Guided by the above-mentioned significant developments in control systems theory, most later endeavours were also laid on the first-order state-space approaches, and consequently a great deal of contributions were made within the first-order system framework (Duan & Yu, 2013; Isidori, 1995; Khalil, 2002; Krstic et al., 1995). State-space approaches have remained dominant for over a half century. To an extent, today's world of control systems is the world of first-order system approaches. Although there are some results which adopt high-order system models, e.g. the polynomial approach for linear systems, they only occupy a rather small portion in the literature as compared with those within the state-space framework.

1.2. Deficiencies

A state-space model of a dynamical system emphasises on the state vector by integrating the state variables together and is probably the best choice for solving the problem of response analysis. However, different from the problem of seeking the state solution, controller design emphasises on the control vector. Towards this goal, a state-space representation does not provide very much convenience. For over a half century of research, results with the first-order state-space approaches have been numerous, but today we are still facing too many unsolved nonlinear control problems (Blongdel et al., 1995).

1.2.1. Control design

The basic step involved in Lyapunov stability analysis and design is to find a proper Lyapunov function. Since the condition is only sufficient for general nonlinear systems, in certain cases, it might not be able to give a definite conclusion. Furthermore, even if a Lyapunov function can be found, often a local stability result is obtained. It is generally difficult to realise global stabilisation of a general nonlinear system. The cases with Bellman dynamic programming and Pontryagin's maximum principle are similar, noting that the HJB equations which determine the solutions may be not solvable for complicated nonlinear systems.

1.2.2. Controllability and observability

Like Lyapunov stability, the concepts of controllability and observability are restricted to systems in state-space representations only (Duan, 2020b, 2020c). Unlike Lyapunov stability, controllability and observability for systems in non-state-space representations can not be defined by those of the converted state-space models, since it is well-known that a system in an input–output representation may possess, at the same time, a controllable (observable) realisation, and also an uncontrollable (unobservable) realisation (Duan, 2020b). Therefore, sticking to the state-space approaches, the concepts of controllability and observability for non-state-space systems are vacant. However, a dynamical system in a non-state-space model representation obviously also deserves the concepts of controllability and observability. Such a phenomenon clearly reveals a serious defect of the first-order state-space framework.

1.2.3. Physical background

All practical systems have physical backgrounds. When modelling a system using some physical laws, such as Lagrangian Equation or Theorem of Momentum, a second- or higher-order model is obtained, and this original model also reflects in some ways the physical backgrounds of the system. In certain situations, the background information may help in the analysis and design. However, with the first-order system approaches, the system model is reduced to a first-order state-space model, and all the background information is then removed from the original model.

1.2.4. Full-actuation

Physically, there are many fully actuated systems in the world, and their original system models established by certain physical laws are of second- or higher-order. For such systems, the control can be realised in an extremely simple way, since the full-actuation property allows us to eliminate all the nonlinear dynamics. As a consequence, a desired constant linear closed-loop system can be obtained, and the theories and techniques for analysis and design of linear systems can then be applicable. Furthermore, with the help of the full-actuation feature, many design problems, such as system decoupling, can also be solved very easily. Nevertheless, when converted into first-order state-space models, such systems no longer possess the full-actuation property, and the nonlinear dynamics needs then to be tackled with Lyapunov stability design approaches. In certain complicated cases even local stabilisation of the system can not be realised, to say nothing of giving a global stabilisation result or obtaining a constant linear closed-loop system.

1.3. A better solution

What makes the problem of nonlinear control so difficult? The answer surely is the nonlinearities. Complex nonlinearities make the analysis of the Lyapunov function very complicated, and also may create insurmountable difficulties in deriving the solutions to the concerned HJB equations. As long as nonlinearities stand, the general problem of nonlinear control can hardly be completely solved with the present state-space approaches, say, in the sense that closed-loop global stability is achieved.

Fortunately, besides the state-space models, there exists another type of system models, namely, the HOFA model (Duan, 2020a), with which the nonlinear term, no matter how complicated, can be easily eliminated using the full-actuation structure as long as the nonlinear term is measurable. Eventually, a constant linear closed-loop system is obtained.

Fully actuated systems physically exist, but only as a minor part of the set of control systems. However, when this concept is mathematically generalised, an HOFA system becomes a general model for control systems. As a matter of fact, most nonlinear systems can be either physically modelled as or converted into HOFA systems. This fact has been demonstrated in Duan (2020a, 2020b) and now is further demonstrated in this paper.

The main contribution of this paper is an HOFA system approach for control of a nonlinear system subject to certain mild conditions. The system considered is relatively general in the sense that the well-known methods of backstepping and feedback linearisation are not directly applicable. The procedure of this HOFA system approach is composed of four steps. The key steps of the approach are concerned with derivation of an HOFA model for the considered system under different circumstances. Once the set of HOFA models are obtained, the controllers of the subsystems are then immediately written out. As a consequence, the closed-loop system resulted in by the designed controller is composed of several independent constant linear subsystems with desired eigenstructures. Furthermore, all the design degrees of freedom in each of the closed-loop linear subsystems are also provided, which can be further utilised to achieve additional system requirements.

For $x \in \mathbb{R}^m$, and $A_i \in \mathbb{R}^{m\times m}$, $i = 1, 2, \ldots, n$, the following symbols are frequently used in the paper:

$$x^{(0\sim n)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n)} \end{bmatrix},$$

$$x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_i^{(0\sim n)} \\ x_{i+1}^{(0\sim n)} \\ \vdots \\ x_j^{(0\sim n)} \end{bmatrix}, \quad j \geq i,$$

$$A_{0\sim n} = \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix},$$

$$\Phi\left(A_{0\sim n}\right)=\begin{bmatrix}0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_n\end{bmatrix}.$$

2. HOFA systems

2.1. Definition

"Full-actuation" is originally a physical concept. Physically, a fully actuated system is of the following form

$$M\left(x,\dot{x},t\right)\ddot{x}+D\left(x,\dot{x},t\right)\dot{x}+K\left(x,\dot{x},t\right)x=u, \quad (1)$$

where $x,u\in\mathbb{R}^r$ are the elementary state vector and the control vector, respectively, $M(\cdot)$, $D(\cdot)$ and $K(\cdot)\in\mathbb{R}^{r\times r}$ are, respectively, the generalised mass matrix, the generalised damping matrix, and the generalised stiffness matrix. This type of systems describe a variety of practical systems in the field of robot control, spacecraft control (Duan, 2020e, 2020f), etc.

Theoretically, we may define a general second-order fully actuated system as follows:

$$E\ddot{x}=f\left(x,\dot{x},t\right)+B\left(x,\dot{x},t\right)u, \quad (2)$$

where $E\in\mathbb{R}^{r\times r}$ is a constant matrix, which is usually the identity matrix, but may be sometimes singular (Duan, 2010), $f(\cdot)\in\mathbb{R}^r$ and $B(\cdot)\in\mathbb{R}^{r\times r}$ are some sufficiently differentiable vector and matrix functions, respectively, and $B(\cdot)$ satisfies the following full-actuation condition:

$$\det B\left(x,\dot{x},t\right)\neq 0, \quad \forall x,\dot{x}\in\mathbb{R}^r, \quad t\geq 0.$$

Parallelly, we can also define, in spite of its physical meaning, an HOFA system. Consider a nonlinear system in the following form

$$Ex^{(m)}=f(x^{(0\sim m-1)},\zeta)+B(x^{(0\sim m-1)},\zeta)u, \quad (3)$$

where $m\geq 1$ is an integer, $x\in\mathbb{R}^n$ is the state vector, $u\in\mathbb{R}^r$ is the system input vector, $\zeta\in\mathbb{R}^p$ is an external vector, $E\in\mathbb{R}^{n\times n}$ is a constant matrix, which is usually the identity matrix, but may be sometimes singular, $f(x^{(0\sim m-1)},\zeta)\in\mathbb{R}^n$ is a sufficiently differentiable vector function, and $B(x^{(0\sim m-1)},\zeta)\in\mathbb{R}^{n\times r}$ is a matrix function.

Let $\Omega_i\subset\mathbb{R}^n$, $i=0,1,\ldots,m-1$, be a series of open sets and denote

$$\Omega=\Omega_0\times\Omega_1\times\cdots\times\Omega_{m-1}.$$

Then the formal definition is given as follows.

Definition 2.1: Given system (3) and the above sets $\Omega_i\subset\mathbb{R}^n$, $i=0,1,\ldots,m-1$ and Ω, if

$$\text{rank}B\left(x^{(0\sim m-1)},\zeta\right)=r=n, \quad \forall t\geq 0, \quad (4)$$

- holds for all $x^{(i)}\in\Omega_i$, $i=0,1,\ldots,m-1$, then the system (3) is called fully actuated on Ω;
- does not hold only on a hyperplane in Ω, then the system (3) is called sub-fully actuated on Ω;
- does not hold only on a set of isolated points in Ω, then the system (3) is called almost fully actuated on Ω;
- does not hold only at a finite number of points in Ω, then the system (3) is called basically fully actuated on Ω;
- holds for all $x^{(i)}\in\mathbb{R}^r$, $i=0,1,\ldots,m-1$, then the system (3) is called (globally) fully actuated. In particular, the system (3) is called a standard fully actuated system when $B(\cdot)\equiv I_r$.

Regarding the verification of the condition (4) in large dimensional cases, symbolic computation techniques may be called into use.

Remark 2.1: In the case that the condition (4) is replaced by

$$\text{rank}B\left(x^{(0\sim m-1)},\zeta\right)=n<r, \quad \forall t>0, \quad (5)$$

the system (3) is called over-actuated. In the high-order system representation, over-actuated systems may be occasionally encountered. Such systems can be similarly treated as fully actuated ones in terms of control.

The importance of fully actuated systems (where $r=n$) lies in the following simple fact (Duan, 2020a).

Proposition 2.2: *Let $A_i\in\mathbb{R}^{r\times r}$, $i=0,1,\ldots,m-1$, be a set of given matrices, then the following controller*

$$\begin{cases}u=-B^{-1}\left(x^{(0\sim m-1)},\zeta\right)\left(A_{0\sim m-1}x^{(0\sim m-1)}+u^*\right)\\ u^*=f\left(x^{(0\sim m-1)},\zeta\right)-v,\end{cases} \quad (6)$$

for the fully actuated system (3) *produces the following constant linear closed-loop system*

$$Ex^{(m)}+A_{0\sim m-1}x^{(0\sim m-1)}=v. \quad (7)$$

When expanded, the above linear system (7) can be also written as

$$Ex^{(m)} + A_{m-1}x^{(m-1)} + \cdots + A_1\dot{x} + A_0x = v.$$

Due to the arbitrariness of the series of matrices A_i, $i = 0, 1, \ldots, m-1$, closed-loop systems with desired performance can be obtained by properly selecting these matrices. A parametric approach for solving these A_i, $i = 0, 1, \ldots, m-1$ has been given in Duan (2020a), which also provides all the design degrees of freedom (see also Section 3.5).

A generalised form of system (3) appears as follows (Duan, 2020b):

$$Ex^{(m)} = f\left(x^{(0\sim m-1)}, \zeta\right) + g\left(x^{(0\sim m-1)}, \zeta, u\right), \quad (8)$$

where $g(\cdot) \in \mathbb{R}^n$ is also a sufficiently differentiable vector function. In the case of $m = 1, E = I_r$ and $p = 0$, the system turns into the following well-known form:

$$\dot{x} = f(x) + g(x, u). \quad (9)$$

The above more general system (8) is called fully actuated on Ω, if $n = r$ and for $t \geq 0$, the mapping $w = g(x^{(0\sim m-1)}, \zeta, u)$ forms a differential homeomorphism from u to w for all $\zeta \in \mathbb{R}^p$ and $x^{(i)} \in \Omega_i$, $i = 0, 1, \ldots, m-1$; it is called (globally) fully actuated if $w = g(x^{(0\sim m-1)}, \zeta, u) \in \mathbb{R}^r$ forms a differential homeomorphism from u to w for all $\zeta \in \mathbb{R}^p$ and $x^{(i)} \in \mathbb{R}^n$, $i = 0, 1, \ldots, m-1$. Sub-full-actuation on Ω can also be similarly defined.

For the more general fully actuated system (8), a result similar to Proposition 2.2 still holds. In this case, the controller corresponding to (6) becomes

$$\begin{cases} w = -\sum_{i=0}^{m-1} A_i x^{(i)} - f\left(x^{(0\sim m-1)}, \zeta\right) + v \\ u = g^{-1}\left(w, x^{(0\sim m-1)}, \zeta\right), \end{cases} \quad (10)$$

and the same closed-loop system as in (7) is achieved.

2.2. Model for control

It is observed from the above that full-actuation provides a great deal of convenience in the control of an HOFA system. Unfortunately, this great advantage was not given enough attention in the past century. The reason behind this fact might be that, affected by the physical concept, fully actuated systems are considered to be a very minor part of control systems, hence do not deserve more extensive investigation. While the fact is, when the physical concept is generalised as above, HOFA system representation (3) or (8) is able to describe most nonlinear control systems, and can thus be taken as a model for control. This point of view is briefly evidenced by the following two aspects.

2.2.1. Physical modelling

Due to the existence of the many well-known physical laws, such as, Newton's Law, Lagrangian Equation, Theorem of Linear and Angular Momentum, Kirchhoff's Current and Voltage Laws, etc., many practical systems are really originally modelled as second-order fully actuated systems (Duan, 2020f). However, almost all of such fully actuated systems are converted into first-order state-space models in order to apply existing theories and methods for state-space systems, and yet, as pointed out in the introduction Section 1.2, the results are far more from satisfactory.

In a system modelling process, once we get a series of subsystems in second-order system form using certain physical laws, on the one hand, we can further obtain a system in the first-order state-space form by variable extension, or equivalently, by defining a state vector; on the other hand, we can often finally derive an HOFA system through ways of variable elimination.

2.2.2. Model conversion

Objectively speaking, there are indeed many under-actuated systems in the world (Fantoni & Lozano, 2002), but most of them can be converted into higher-order fully actuated systems as long as they obey a certain kind of controllability property (Duan, 2020b). These include, but are certainly not limited to, the following (Duan, 2020a):

- all controllable linear systems (Duan, 2020b);
- nonlinear systems in a kind of controllable canonical forms (Duan, 2020b);
- strict-feedback nonlinear systems (Duan, 2020a); and
- all linearisable systems by state feedback (Duan, 2020a).

Such facts have been unlikely thought of by those who are extensively absorbed in the first-order state-space approaches.

Very seldom a state-space system is originally (at the modelling stage) in the first-order system form, it is often converted from second- or high-order systems through variable extension. For about a century time of dominance, state-space models have been regarded as universal. Most control scientists and practitioners have been accustomed to convert any system encountered into state-space representation. Anyhow, due to the physical laws mentioned above, first-order state-space systems which are originally (at the modelling stage) of first-order are very few, at least much fewer than HOFA systems which are originally of second- or high-order.

Since most systems can be modelled as, or converted into HOFA systems, in spite of the physical meaning, like the state-space models for dynamical control systems, the HOFA system representation (3) or (8) also deserves to be a model for control systems. The former is more suitable for deriving the state solution and observation, while the latter, as shown above, is extremely convenient for dealing with the control, hence is truly a model for control.

In the next section, we further propose a general procedure for the control of a nonlinear system through deriving the HOFA model, the process indicates the power of the HOFA approach.

3. HOFA systems approach

Consider the normal case of nonlinear system (3), that is,

$$x^{(m)} = f(x^{(0\sim m-1)}, \zeta) + B(x^{(0\sim m-1)}, \zeta)u, \tag{11}$$

where the matrix $B(x^{(0\sim m-1)}, \zeta) \in \mathbb{R}^{n\times r}$ satisfies the following full-actuation assumption:

Assumption 3.1: *For arbitrary $\zeta \in \mathbb{R}^p, x^{(0\sim m-1)} \in \mathbb{R}^{mn}$, there holds*

$$\text{rank}B(x^{(0\sim m-1)}, \zeta) = r < n. \tag{12}$$

In the case of $r = n$, the system (11) is already an HOFA system, and the controller can be immediately written out.

In the case of $m = 1$, and ζ does not exist, the above system (11) reduces to the following well-known affine nonlinear system

$$\dot{x} = f(x) + B(x)u, \tag{13}$$

In this section, we aim to provide a basic procedure of the so-called HOFA systems approach for the control of the above system (11). The procedure contains the following steps.

3.1. The pseudo strict-feedback system

Under Assumption 3.1, there exists an unimodular matrix $Q(x^{(0\sim m-1)}, \zeta)$, that is, a matrix with a nonzero constant determinant, such that

$$Q(x^{(0\sim m-1)}, \zeta)B(x^{(0\sim m-1)}, \zeta) = \begin{bmatrix} 0 \\ G(x^{(0\sim m-1)}, \zeta) \end{bmatrix}, \tag{14}$$

where $G(x^{(0\sim m-1)}, \zeta) \in \mathbb{R}^{r\times r}$ satisfies

$$\det G(x^{(0\sim m-1)}, \zeta) \neq 0, \quad \forall \zeta \in \mathbb{R}^p, \quad \text{and} \quad x^{(i)} \in \mathbb{R}^n, \quad i = 0, 1, \ldots, m-1. \tag{15}$$

For simplicity, in this paper let us consider a simple case and impose the following assumption:

Assumption 3.2: *There exists a constant nonsingular matrix $Q(x^{(0\sim m-1)}, \zeta) = Q$ satisfying* (14).

The general case will be addressed elsewhere, while the idea is illustrated in Example 2.

Under Assumption 3.2, we can define the following transformation

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Qx, \; z_1 \in \mathbb{R}^{n-r}. \tag{16}$$

Partitioning the matrix Q as

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad Q_2 \in \mathbb{R}^{r\times n},$$

and using (11) and (14), we have

$$\begin{bmatrix} z_1^{(m)} \\ z_2^{(m)} \end{bmatrix} = \begin{bmatrix} Q_1 f(x^{(0\sim m-1)}, \zeta) \\ Q_2 f(x^{(0\sim m-1)}, \zeta) \end{bmatrix} + \begin{bmatrix} 0 \\ G(x^{(0\sim m-1)}, \zeta) \end{bmatrix} u, \tag{17}$$

which can be equivalently decomposed, in view of the invertibility of the transformation (16), into two subsystems of the following forms:

$$z_1^{(m)} = g(z^{(0\sim m-1)}, \zeta), \tag{18}$$

$$z_2^{(m)} = f_2(z^{(0\sim m-1)}, \zeta) + G(z^{(0\sim m-1)}, \zeta)u. \tag{19}$$

Further, assume that the above system (18) is in the following form

$$z_1^{(m)} = f_1(z_1^{(0\sim m-1)}, \zeta) + B_1(z_1^{(0\sim m-1)}, \zeta)z_2, \tag{20}$$

then the above system (20) is again in the form of (11) when $z_2 \in \mathbb{R}^r$ is looked upon as the control input. If

$n-r>r$, by repeating the above process (if repeatable), the system (20) can also be divided into the following two subsystems:

$$z_{11}^{(m)} = g_1(z_1^{(0\sim m-1)},\zeta), \tag{21}$$

$$z_{12}^{(m)} = g_2(z_1^{(0\sim m-1)},\zeta) + G_2(z_1^{(0\sim m-1)},\zeta)z_2. \tag{22}$$

Continue the above process, under certain conditions, we obtain a pseudo strict-feedback system in the following form:

$$\begin{cases} z_0^{(m)} = g(z_{0\sim 1}^{(0\sim m-1)},\zeta) \\ z_1^{(m)} = f_1(z_{0\sim 1}^{(0\sim m-1)},\zeta) + B_1(z_{0\sim 1}^{(0\sim m-1)},\zeta)z_2 \\ z_2^{(m)} = f_2(z_{0\sim 2}^{(0\sim m-1)},\zeta) + B_2(z_{0\sim 2}^{(0\sim m-1)},\zeta)z_3 \\ \quad\vdots \\ z_{q-1}^{(m)} = f_{q-1}(z_{0\sim q-1}^{(0\sim m-1)},\zeta) + B_{q-1}(z_{0\sim q-1}^{(0\sim m-1)},\zeta)z_q \\ z_q^{(m)} = f_q(z_{0\sim q}^{(0\sim m-1)},\zeta) + B_q(z_{0\sim q}^{(0\sim m-1)},\zeta)u, \end{cases} \tag{23}$$

where $z_i \in \mathbb{R}^r, i=1,2,\ldots,q, z_0 \in \mathbb{R}^{n_0}, (n_0<r)$, with q and n_0 being integers satisfying

$$n = qr + n_0,$$

and the coefficient matrices $B_i(z_{0\sim i}^{(0\sim m-1)},\zeta) \in \mathbb{R}^{r\times r}$, $i=1,2,\ldots,q$, satisfy the following assumption:

Assumption 3.3: *For all $\zeta \in \mathbb{R}^p$, and $z_k^{(0\sim m-1)} \in \mathbb{R}^{mr}$, there hold*

$$\det B_i(z_{0\sim i}^{(0\sim m-1)},\zeta) \neq 0, \quad k=0,1,\ldots,q.$$

Remark 3.1: It is clearly observed that the linear part in each subsystem of the pseudo strict-feedback system (23) has relation with z_i but not its derivatives. If this is not the case in an application, we can convert certain related subsystems into lower order ones in order to eliminate this phenomenon. It is easily noted that in the extreme case that the related subsystems are converted into first-order systems, such derivative terms do not appear. On the other hand, it should be noted that this operation naturally changes the pseudo strict-feedback system with a fixed order into the one with mixed orders (Duan, 2020d), but this change does not affect our treatment.

3.2. The HOFA model, case of $n_0 = 0$

The last q equations in (23) form a strict-feedback system of order m. As shown in Part II of the paper Duan (2020d), this strict-feedback system is equivalent to a high-order system of the following form:

$$z_1^{(m_c)} = h(z_{0\sim 1}^{(0\sim m_c-1)},\zeta) + L(z_{0\sim 1}^{(0\sim m_c-1)},\zeta)u, \tag{24}$$

where $m_c = qm$, $h(z_{0\sim 1}^{(0\sim m_c-1)},\zeta) \in \mathbb{R}^r$ is some vector function, $L(z_{0\sim 1}^{(0\sim m_c-1)},\zeta) \in \mathbb{R}^{r\times r}$ is a matrix function satisfying

$$\det L(z_{0\sim 1}^{(0\sim m_c-1)},\zeta) \neq 0, \tag{25}$$

for all $z_0^{(0\sim m_c-1)} \in \mathbb{R}^{m_c n_0}, z_1^{(0\sim m_c-1)} \in \mathbb{R}^{m_c r}$ and $\zeta \in \mathbb{R}^p$. Therefore, the above pseudo strict-feedback system is equivalent to

$$\begin{cases} z_0^{(m)} = g(z_0^{(0\sim m-1)}, z_1^{(0\sim m-1)},\zeta) \\ z_1^{(m_c)} = h(z_{0\sim 1}^{(0\sim m_c-1)},\zeta) + L(z_{0\sim 1}^{(0\sim m_c-1)},\zeta)u. \end{cases} \tag{26}$$

When n is a multiple of r, which is always true in the case of $r=1$, we have $n_0=0$, thus the first subsystem in the above (26) vanishes. In this case the whole problem reduces to the control of the HOFA system (24), and the controller can be designed as

$$\begin{cases} u = -L^{-1}(z_{0\sim 1}^{(0\sim m_c-1)},\zeta)\left(A_{0\sim m_c-1}z_1^{(0\sim m_c-1)} + u^*\right) \\ u^* = h(z_{0\sim 1}^{(0\sim m_c-1)},\zeta) - v, \end{cases} \tag{27}$$

where v is an external signal. The closed-loop system is obtained as

$$z_1^{(m_c)} + A_{0\sim m_c-1}z_1^{(0\sim m_c-1)} = v. \tag{28}$$

3.3. The HOFA subsystem, case of $n_0 \neq 0$

Let us now continue to consider the case of $n_0 \neq 0$. For simplicity, in this paper we impose the following assumption:

Assumption 3.4: *The first subsystem in* (26) *is in the following form:*

$$z_0^{(m)} = g(z_0^{(0\sim m-1)},\zeta) + B_0 z_1, \tag{29}$$

where $B_0 \in \mathbb{R}^{n_0\times r}$ is a full-row rank matrix.

This assumption restricts the subsystem to be linear with respect to z_1, but is independent on the derivatives of z_1. The general nonlinear case will be addressed elsewhere.

Taking the derivative of order m_c on both sides of (29), and using (24), yield

$$\begin{aligned} z_0^{(m+m_c)} &= g^{(m_c)}(z_0^{(0\sim m-1)}, \zeta) + B_0 z_1^{(m_c)} \\ &= g^{(m_c)}(z_0^{(0\sim m-1)}, \zeta) + B_0\left[h(\cdot) + L(\cdot)u\right] \end{aligned} \tag{30}$$

which gives the following higher-order system

$$\begin{aligned} z_0^{(m+m_c)} &= \psi(z_0^{(0\sim m+m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta^{(0\sim m_c)}) \\ &\quad + B_0 L(\cdot)u, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &\psi(z_0^{(0\sim m+m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta^{(0\sim m_c)}) \\ &\quad = g^{(m_c)}(z_0^{(0\sim m-1)}, \zeta) \\ &\quad + B_0 h(z_0^{(0\sim m-1)}, z_1^{(0\sim m_c-1)}, \zeta). \end{aligned}$$

If we divide the control vector as

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, u_1 \in \mathbb{R}^{n_0}, \tag{32}$$

and partition $L(z_0^{(0\sim m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta)$ as

$$L(z_0^{(0\sim m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta) = [L_1 \quad L_2], L_1 \in \mathbb{R}^{r\times n_0}, \tag{33}$$

then the system (31) can be written as

$$\begin{aligned} z_0^{(m+m_c)} &= \psi(z_0^{(0\sim m+m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta^{(0\sim m_c)}) \\ &\quad + B_0 L_1 u_1 + B_0 L_2 u_2, \end{aligned} \tag{34}$$

Without loss of generality, let us assume

$$\det\left(B_0 L_1(z_0^{(0\sim m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta)\right) \neq 0, \tag{35}$$

for all $z_0^{(0\sim m_c-1)}, z_1^{(0\sim m_c-1)}$ and ζ, since otherwise we can apply a transformation to the control u, which equals to applying a series of elementary column transformations to the matrix L. Under the condition (35), system (34) is clearly a fully actuated system with respect to u_1, while u_2 and z_1 can be treated as external vectors which will be given later. Thus we can design the controller for the above HOFA system (34) as

$$\begin{cases} u_1 = -(B_0 L_1)^{-1}\left(A^z_{0\sim m+m_c-1} z^{(0\sim m+m_c-1)} + u_1^*\right) \\ u_1^* = \psi(z_0^{(0\sim m+m_c-1)}, z_1^{(0\sim m_c-1)}, \zeta^{(0\sim m_c)}) \\ \qquad + B_0 L_2 u_2 - v_z, \end{cases} \tag{36}$$

which produces the following constant linear closed-loop system

$$z_0^{(m+m_c)} + A^z_{0\sim m+m_c-1} z_0^{(0\sim m+m_c-1)} = v_z, \tag{37}$$

where v_z is an external signal of dimension n_0.

3.4. The leftover system

Note that the dimension of system (29) is n_0, while that of system (24) is $r > n_0$. Thus in the above process, not the whole system (24) contributes to the higher-order system (31), but only a part of it does. We now need to separate out the leftover part in system (24) which has not contributed to (31).

Let P be a nonsingular matrix such that

$$B_0 P = [M \quad 0], \tag{38}$$

where $M \in \mathbb{R}^{n_0\times n_0}$ is nonsingular. We then can define the transformation

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^{-1} z_1, \quad y_1 \in \mathbb{R}^{n_0}. \tag{39}$$

Correspondingly, partition the matrix P^{-1} as

$$P^{-1} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \quad N_1 \in \mathbb{R}^{n_0\times r}. \tag{40}$$

Then, the system (24) is equivalently transformed into

$$\begin{aligned} y_1^{(m_c)} &= N_1 h(z_0^{(0\sim m-1)}, y^{(0\sim m_c-1)}, \zeta) \\ &\quad + N_1 L(z_0^{(0\sim m-1)}, y^{(0\sim m_c-1)}, \zeta)u, \end{aligned} \tag{41}$$

$$\begin{aligned} y_2^{(m_c)} &= N_2 h(z_0^{(0\sim m-1)}, y^{(0\sim m_c-1)}, \zeta) \\ &\quad + N_2 L(z_0^{(0\sim m-1)}, y^{(0\sim m_c-1)}, \zeta)u. \end{aligned} \tag{42}$$

Note that

$$\begin{aligned} B_0 z_1^{(m)} &= B_0 P P^{-1} z_1^{(m_c)} \\ &= [M \qquad 0]\, y^{(m_c)} \\ &= M y_1^{(m_c)}, \end{aligned}$$

it is clearly seen from (30) that only the subsystem (41) has contributions to the higher-order system (31), while the subsystem (42) is leftover, and thus should be taken into consideration for control.

To summarise, in the case of $n_0 \neq 0$, the pseudo strict-feedback system (23) with the first equation replaced by (29) is equivalently converted into the two subsystems (42) and (31) under certain conditions. The control of system (31) has been realised in the above subsection, now let us consider the control of system (42).

In view of (32) and (33), the system (42) can be written as

$$y_2^{(m_c)} = N_2 h(z_0^{(0\sim m_c-1)}, y^{(0\sim m_c-1)}, \zeta) + N_2 L_1 u_1 + N_2 L_2 u_2. \quad (43)$$

If

$$\det\left(N_2 L_2(z_0^{(0\sim m_c-1)}, y^{(0\sim m_c-1)}, \zeta)\right) \neq 0, \quad (44)$$

for all $z_0^{(0\sim m_c-1)}, y^{(0\sim m_c-1)}$ and ζ, then system (43) can be viewed as fully actuated with respect to u_2, while with u_1 and z_0 treated as external vectors. Therefore, we can design the controller for the above HOFA system (43) as follows:

$$\begin{cases} u_2 = -(N_2L_2)^{-1}\left(A^y_{0\sim m_c-1} y_2^{(0\sim m_c-1)} + u_2^*\right) \\ u_2^* = N_2 h(z_0^{(0\sim m_c-1)}, y^{(0\sim m_c-1)}, \zeta) \\ \quad + N_2 L_1 u_1 - v_y, \end{cases}, \quad (45)$$

which produces the following linear closed-loop system

$$y_2^{(m_c)} + A^y_{0\sim m_c-1} y_2^{(0\sim m_c-1)} = v_y, \quad (46)$$

where v_y is an external signal of dimension $r - n_0$.

If the above condition (44) is not met, we can reorganise system (43) into the form of (11), with certain elements of u_2 being the control, while with u_1 being taken as an external vector, and then design the controller u_2 by starting over all of the above steps. It should be noted that by now we are handling a system in the form of (11) with very small values of n and r.

3.5. The closed-loop system

Different from many other nonlinear control approaches, the above procedure of the HOFA system approach produces a constant linear closed-loop system, given by (28), or by (37) and (46). This huge advantage eventually allows most of the theories and techniques for linear control systems design to be applicable.

The above linear systems (28), (37) and (46) can all be easily made stable by properly choosing the matrices $A_{0\sim m_c-1}, A^z_{0\sim m+m_c-1}$ and $A^y_{0\sim m_c-1}$. Here we present a simple complete parametric approach for designing these matrices.

These closed-loop systems are all of the following form, when the external signal is removed:

$$x^{(m)} + A_{0\sim m-1} x^{(0\sim m-1)} = 0,$$

which can be equivalently written in the following state-space form:

$$\dot{x}^{(0\sim m-1)} = \Phi(A_{0\sim m-1}) x^{(0\sim m-1)}.$$

The problem is to seek a matrix $A_{0\sim m-1}$ such that $\Phi(A_{0\sim m-1})$ is stable. For a solution to this problem, we have the following result, which is the Corollary 1 in Duan (2020a) (see, also Duan, 2005).

Proposition 3.5: *For an arbitrarily chosen $F \in \mathbb{R}^{mn\times mn}$, all the matrix $A_{0\sim m-1}$ and the nonsingular matrix $V \in \mathbb{R}^{mn\times mn}$ satisfying*

$$\Phi(A_{0\sim m-1}) = VFV^{-1}$$

are given by

$$A_{0\sim m-1} = -ZF^n V^{-1}(Z, F),$$

$$V = V(Z, F) = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix},$$

where $Z \in \mathbb{R}^{m\times mn}$ is an arbitrary parameter matrix satisfying

$$\det V(Z, F) \neq 0.$$

Remark 3.2: It is seen from the above result that as long as the matrix F is chosen stable, the closed-loop system is stable. Clearly, there are quite some freedom in the selection of F, while on the other hand we also have the parameter matrix Z, which provides m^2n degrees of freedom. All these degrees of freedom can be further utilised to achieve additional performance of the system (see, e.g. Duan, 1992, 1993; Duan et al., 2002, 2000; Duan & Zhao, 2020.

To end this section, we give the following summing-up points:

- In the case of $n = qr$, where q is an integer, the procedure finishes as soon as we get the higher-order fully actuated system (24). The controller is given as in (36), and the closed-loop system is given by (28). It is obvious that the single-input case always belongs to this case.
- In the case of $n_0 \neq 0$, and condition (44) is met, the procedure leads to two HOFA models of different orders. Consequently, control of these HOFA models produces two decoupled constant linear closed-loop systems, one is of order n_0, which is given by (37), the other is of order $r - n_0$, which is given by (46). The trick discovered for solving the problem is to divide the control into two parts, with each one controlling an fully actuated subsystem.
- In the case of $n_0 \neq 0$, and condition (44) is not met, we need to re-organise system (43) into the form of (11), with certain elements of u_2 being the control, while with u_1 being taken as an external vector, and start over a new round with this much smaller scale system.

4. Example

4.1. Example 1

Consider the following system (Isidori, 1995, p. 358)

$$\begin{cases} \dot{x}_1 = x_1 + x_1x_4 + x_3u_1 + u_2 \\ \dot{x}_2 = x_2 e^{x_3} + u_1 \\ \dot{x}_3 = x_2 + x_3^2 \\ \dot{x}_4 = x_1 + x_2 - x_4 + x_1x_4 \\ \qquad + (1 + x_3)\, u_1 + u_2, \end{cases} \tag{47}$$

which can be written in the form of (13), with

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$f(x) = \begin{bmatrix} x_1 + x_1x_4 \\ x_2 e^{x_3} \\ x_2 + x_3^2 \\ x_1 + x_2 - x_4 + x_1x_4 \end{bmatrix},$$

and

$$B(x) = \begin{bmatrix} x_3 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 + x_3 & 1 \end{bmatrix}.$$

Let

$$Q = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{48}$$

then it can be easily verified that

$$QB(x) = \begin{bmatrix} 0_{2\times 2} \\ G(x) \end{bmatrix},$$

with

$$G(x) = \begin{bmatrix} 1 & 0 \\ x_3 & 1 \end{bmatrix}.$$

Therefore, under the transformation

$$z = Qx, \tag{49}$$

the example system (47) is equivalently converted into the following two subsystems:

$$\dot{z}_I = g(z), \tag{50}$$

$$\dot{z}_{II} = f(z) + G(z)\, u, \tag{51}$$

where

$$z_I = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_{II} = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix},$$

$$g(z) = \begin{bmatrix} -z_1 - z_4 - z_3 e^{z_2} \\ z_2^2 + z_3 \end{bmatrix},$$

and

$$f(z) = \begin{bmatrix} z_3 e^{z_2} \\ z_4 + z_4(z_1 + z_3 + z_4) \end{bmatrix}.$$

Taking differentials on system (50), gives

$$\begin{aligned} \ddot{z}_I &= \dot{g}(z) \\ &= \begin{bmatrix} -\dot{z}_1 - \dot{z}_4 - e^{z_2}\dot{z}_3 - z_3 e^{z_2}\dot{z}_2 \\ 2z_2\dot{z}_2 + \dot{z}_3 \end{bmatrix}. \end{aligned} \tag{52}$$

Further noting (51), we have

$$\begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} z_3 e^{z_2} + u_1 \\ z_4 + z_4^2 + z_1z_4 + z_3z_4 + z_2u_1 + u_2 \end{bmatrix}. \tag{53}$$

Combining (52) and (53), yields the following HOFA model for the system:

$$\ddot{z}_I = h\left(z_{1\sim 2}^{(0\sim 1)}, z_3, z_4\right) + L\left(z_{1\sim 2}^{(0\sim 1)}, z_3, z_4\right) u,$$

where

$$L\left(z_{1\sim 2}^{(0\sim 1)}, z_3, z_4\right) = \begin{bmatrix} -z_2 - e^{z_2} & -1 \\ 1 & 0 \end{bmatrix}.$$

$$h\left(z_{1\sim 2}^{(0\sim 1)}, z_3, z_4\right) = \begin{bmatrix} -\dot{z}_1 - \eta z_4 - z_3\left(e^{2z_2} + e^{z_2}\dot{z}_2\right) \\ 2z_2\dot{z}_2 + z_3 e^{z_2} \end{bmatrix},$$

with

$$\eta = 1 + z_1 + z_3 + z_4.$$

Therefore, the controller for the system is given by

$$\begin{cases} u = -L^{-1}\left(z_{1\sim 2}^{(0\sim 1)}, z_3, z_4\right)\left(A_{0\sim 1} z_I^{(0\sim 1)} + u^*\right) \\ u^* = h\left(z_{1\sim 2}^{(0\sim 1)}, z_3, z_4\right) - v, \end{cases} \tag{54}$$

which results in the following closed-loop system

$$\ddot{z}_I + A_{0\sim 1} z_I^{(0\sim 1)} = v,$$

where v is an external vector, and $A_{0\sim 1} \in \mathbb{R}^{2\times 4}$ is a matrix given, in view of Proposition 3.5, by

$$A_{0\sim 1} = ZF^2V^{-1},$$

where

$$V = \begin{bmatrix} Z \\ ZF \end{bmatrix},$$

$F \in \mathbb{R}^{4\times 4}$ is an arbitrary stable (Hurwitz) matrix, and $Z \in \mathbb{R}^{2\times 4}$ is an arbitrary parameter matrix making the matrix V nonsingular. For this example system, a typical choice of the matrix F may be

$$F = \begin{bmatrix} -a & -b & 0 & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -d \end{bmatrix},$$

where a, b, c and d are some positive scalars.

In view of (49), it is obvious that all the variables used in the controller (54) are all available as long as the states of the original system (47) are all measurable.

Remark 4.1: According to Isidori (1995, p. 358), this example system has a zero dynamics defined on the sub-manifold

$$Z^* = \left\{x \in \mathbb{R}^4 : x_1 = x_2 = 0\right\},$$

with the representation, expressed in the (x_3, x_4) coordinates of Z^*, as

$$\begin{cases} \dot{x}_3 = x_3^2 \\ \dot{x}_4 = -x_4. \end{cases}$$

Therefore, the approach of non-interacting control fails. Even with a modified design, a stable zero dynamics still exists. While in our above design, a constant linear closed-loop system is obtained with all its four poles arbitrarily assignable.

4.2. Example 2

Consider the following pseudo strict-feedback system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) + x_3, \\ \dot{x}_3 = u, \end{cases} \tag{55}$$

where $x_i \in \mathbb{R}$, $i = 1, 2, 3$, and $u \in \mathbb{R}$ are the state variables and control input, respectively, and $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are two differentiable functions, and $f_1(x_1, x_2)$ satisfies the condition

$$\frac{\partial f_1}{\partial x_2} \neq 0, \quad \forall x_1, x_2 \in \mathbb{R}. \tag{56}$$

Following our treatment, by taking differentials of the second equation in (55), and substituting the third one into the result, we obtain the following equivalent system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \ddot{x}_2 = \dot{f}_2(x_1, x_2) + u. \end{cases} \tag{57}$$

Taking differentials of the first equation in (57), yields

$$\ddot{x}_1 = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\dot{x}_2, \tag{58}$$

which further gives

$$\begin{aligned} \dddot{x}_1 &= \left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right]' + \left[\frac{\partial f_1}{\partial x_2}\dot{x}_2\right]' \\ &= \left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right]' + \left[\frac{\partial f_1}{\partial x_2}\right]'\dot{x}_2 + \frac{\partial f_1}{\partial x_2}\ddot{x}_2 \\ &= \left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right]' + \left[\frac{\partial f_1}{\partial x_2}\right]'\dot{x}_2 + \frac{\partial f_1}{\partial x_2}\dot{f}_2(x_1, x_2) + \frac{\partial f_1}{\partial x_2}u \end{aligned}$$

$$= g(x_1^{(0\sim 2)}, x_2, \dot{x}_2) + \frac{\partial f_1}{\partial x_2} u, \tag{59}$$

where

$$g(x_1^{(0\sim 2)}, x_2, \dot{x}_2) = \left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right]' + \left[\frac{\partial f_1}{\partial x_2}\right]' \dot{x}_2 + \frac{\partial f_1}{\partial x_2}\dot{f}_2(x_1, x_2). \tag{60}$$

In view of (56), by the well-known Theorem of Inverse Functions, the state variable x_2 can be explicitly and uniquely solved from the first equation in (55) as follows

$$x_2 = g_1(x_1, \dot{x}_1). \tag{61}$$

Meanwhile, from (58) we obtain, in view of condition (56),

$$\dot{x}_2 = \left[\frac{\partial f_1}{\partial x_2}\right]^{-1} \left(\ddot{x}_1 - \frac{\partial f_1}{\partial x_1}\dot{x}_1\right). \tag{62}$$

With the help of the above relations (61) and (62), we can convert $g(x_1^{(0\sim 2)}, x_2, \dot{x}_2)$ and $\frac{\partial f_1}{\partial x_2}$ into functions of $x_1^{(0\sim 2)}$ only, that is, we have

$$\begin{cases} f(x_1^{(0\sim 2)}) \triangleq g(x_1^{(0\sim 2)}, x_2, \dot{x}_2) \\ B(x_1^{(0\sim 1)}) \triangleq \frac{\partial f_1(x_1, x_2)}{\partial x_2}. \end{cases}$$

Therefore, it follows from (59) that the original system (55) is equivalently transformed into the following third-order fully actuated system model

$$\dddot{x}_1 = f(x_1^{(0\sim 2)}) + B(x_1^{(0\sim 1)})u. \tag{63}$$

Thus, for this system, a controller can be designed as

$$u = -B^{-1}(x_1^{(0\sim 1)}) \left(a_{0\sim 3}x_1^{(0\sim 3)} + f(x_1^{(0\sim 2)}) - v\right), \tag{64}$$

which gives the closed-loop system

$$\dddot{x}_1 + a_{0\sim 3}x_1^{(0\sim 3)} = v,$$

where v is a scalar external signal, while the parameter vector

$$a_{0\sim 3} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}$$

can be easily chosen to make the closed-loop system stable.

Finally, it is pointed out that as long as the state of the original system is measurable, the above controller is realisable since the needed variables can be obtained through

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \ddot{x}_1 = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\left(f_2(x_1, x_2) + x_3\right). \end{cases} \tag{65}$$

Remark 4.2: This example system does not obey the triangular structure of a strict-feedback system, hence the well-know method of backstepping is generally not applicable. Furthermore, with only the assumption (56), the conditions for the well-known method of feedback linearisation are also not met. Yet with the proposed HOFA system approach the problem is well solved in the sense that a constant linear closed-loop system is obtained with a desired eigenstructure.

5. Conclusion

Nonlinear control has remained an unsolved problem for nearly a century. Based on the universal state-space representation, stabilisation of a nonlinear system generally leads to the problem of seeking a proper Lyapunov function. As a result, global results are seldom achieved, and in certain cases even local results are not available, to say nothing of deriving a constant linear closed-loop system.

This paper proposes an HOFA system approach for control of nonlinear systems. Generally speaking, state-space representations are more suitable for seeking the state solution (response) and observation, while for control, the proposed HOFA system approach is more suitable due to the following advantages:

(1) the controller of a nonlinear system can be immediately written out as soon as a single HOFA model or a set of HOFA models of the system are derived;
(2) a constant linear closed-loop system can always be obtained, and the analysis and design approaches for linear systems can then be applicable.
(3) a desired closed-loop eigenstructure can be assigned, and all the design degrees of freedom are provided, which can be further utilised to achieve additional system design requirements;
(4) the approach solves many nonlinear control problems that the Lyapunov approach does not solve, since Lyapunov approach heavily depends on the

complexity of the nonlinear functions in the system, while the HOFA system approach utilises only the full-actuation structure, regardless of the complexity of the nonlinear terms.

The proposed HOFA system approach can be generalised in many directions. Firstly, the relevant full-actuation Assumptions 3.1 and 3.3 can be relaxed and control of sub-fully actuated systems can be considered. Secondly, for mainly demonstrating the idea of HOFA approach, Conditions 3.2 and 3.4 are also imposed for simplicity in the paper, while these conditions can also be relaxed. Finally, the idea can be extended to discrete-time systems, stochastic systems, and systems with uncertainties, etc.

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