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Exploring the Mathematical Knowledge for Teaching of Japanese Teachers

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ABSTRACT

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In the past two decades there has been an increased effort to understand the depth to which mathematics teachers must know their subject to teach it effectively. Researchers have termed this type of knowledge mathematical knowledge for teaching (MKT). Even though recent studies have focused on MKT, the current literature on the subject indicates that this area remains underdeveloped. In an attempt to further refine our conception of MKT this study looked at MKT in Japan. In this thesis I explored and categorized the MKT of three experienced Japanese cooperating teachers (CTs) by looking at the content of their conversations with three Japanese student teachers (STs). I separated the MKT mentioned in these conversations into three categories: knowledge about the students' mathematical knowledge, knowledge about mathematics, and knowledge about school mathematics. I also discussed various implications of this work on the field of MKT.

Keywords: mathematical knowledge for teaching, Japanese mathematics education, practice of teaching

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Dr. Doug Corey, my advisor, once told me that he owed everything he had accomplished in field of mathematics education to his doctoral advisor. I feel the same. Thank you Doug for all of the many hours we put into this thing. I never felt alone thanks to you. Also a great deal of thanks to Dan and Blake. Your help throughout this research has helped me to end up with so much more. Finally, a special thank you goes to my wife. Love you.

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Introduction

Collectively, Japanese students have consistently proven themselves to be particularly high achieving in the area of mathematics. They have consistently scored higher on the TIMSS (Gonzales et al., 2008) and PISA (OECD, 2010) standardized tests than most of their worldwide peers, including students from the United States. This consistent achievement of Japanese students has continued to interest researchers (Baker, 1993; Hiebert et al., 2003; Jacobs et al., 2003; Stevenson, Lee, & Stigler, 1986; Stigler, 1990; Westbury, 1992) and is one reason that has led them to explore various components of the Japanese education system. Stevenson, Lee, and Stigler (1986) explored the impact of the parents on student achievement. Stigler (1990) looked at Japanese students' mathematical ability and test-taking proficiency. Baker (1993) and Westbury (1992) investigated the effects of Japanese curriculum on student achievement.

Other researchers such as Jacobs and Morita (2002) and Schaub and Baker (1991) considered a teacher's role. Schaub and Baker (1991) found that Japanese teachers felt responsible for synthesizing students' diverse reasoning and unifying the class under a common integrated understanding. Jacobs and Morita (2002) found that Japanese teachers viewed themselves to have the responsibility to present math content by involving, guiding, and observing students while also creating a productive learning atmosphere in the classroom and maintaining a positive flow and pace in each lesson.

Some efforts that have focused on the role of the teacher have explored teacher knowledge. One such effort was to characterize the type of mathematical knowledge required in the work of teaching. Throughout this paper, the phrase "the work of teaching" will be used to mean supporting and facilitating an increase in students' mathematical proficiency as mentioned in the strands (National Research Council, 2001). Researchers in this field such as Ball et al.

(2008), Hill et al. (2008), Rowland et al. (2003) and others term this type of teacher mathematical knowledge mathematical knowledge for teaching (MKT). Speaking on the importance of MKT in developing students' mathematical understanding Morris, Hiebert, and Spitzer (2009) stated that

> The concept of MKT provides the most promising current answer to the longstanding question of what kind of content knowledge is needed to teach mathematics well. (p. 492)

For this reason MKT has remained a frequently researched topic since Shulman introduced the concept of a domain of knowledge specific to teaching in 1986.

Although there has been substantial work done on conceptualizing and identifying the components of MKT, there is still a lot that we do not know about it. It was the goal of this research to further the conversation about MKT by looking into the MKT valued by Japanese teachers. Many researchers have looked into MKT in the United States (Ball, Thames, & Phelps, 2008; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; Kersting, Givvin, Sotelo, & Stigler, 2010; Morris, Hiebert, & Spitzer, 2009; Schoenfeld & Kilpatrick, 2008; Wang, Spalding, Odell, Klecka, & Lin, 2010), yet few have looked at international teachers when creating a model of MKT and no researchers have looked into Japan.

There are three reasons that Japan provides a rich environment for scholars to study MKT. First Japanese mathematics teachers present rich instruction centered on developing mathematical understanding in their students (Fernandez & Yoshida, 2004). Second, Japanese teachers share a strong conception of what constitutes good instruction (Jacobs et al., 2006; Stigler & Hiebert, 1999a, 1999b; Whitman & Lai, 1990). Third, researchers have found significant differences between the teaching styles of the Japanese and other countries, including the United States (Stigler & Hiebert, 2004).

Furthermore, in *The Teaching Gap* Stigler and Hiebert (1999a) found that math instruction did not vary significantly between teachers from the same country (USA, Japan, and Germany). In that same study they also found that instruction varied greatly between teachers across countries. These two findings suggest that looking outside of the US and documenting the type of MKT that is valued in other countries, such as Japan, could prove to be beneficial in helping us refine our current conception of MKT.

Also, typical Japanese mathematics instruction closely matched that of various reform efforts here in the United States (Even & Tirosh, 1995; Senk & Thompson, 2003). This overlap increases the potential of this study. Three such examples of this overlap suggested by reform researchers are that 1) mathematics instruction in the United States be more focused on student thinking (Even & Tirosh, 1995; Schifter & Fosnot, 1993), 2) mathematics instruction should include challenging mathematics (Hiebert & Grouws, 2007), and 3) math instruction should build on the relationships between math topics (Senk & Thompson, 2003). The work of Corey, Peterson, Lewis, and Bukarau (2010) and Stigler and Hiebert (1999, 2004) showed evidence of this overlap from the Japanese perspective.

In this study I discuss the characteristics of MKT valued among Japanese teachers by attending to the conversations that cooperating teachers (CT) had with student teachers (ST) before a lesson was taught. The following question guided this study. What did the conversations imply about the type and nature of MKT that was valued by Japanese teachers?

Literature Review

This literature review will discuss two main topics. The first topic will deal with Japanese instruction and outline the work that has been done on characterizing the Japanese style of teaching mathematics. The second topic will cover MKT. There I will discuss various models developed to characterize MKT which will provide a context to the field of MKT.

Japanese instruction

The first section will focus on Japanese instruction and how it is based on a strongly shared, student-centered, and non-procedural exploration of fresh and challenging mathematical concepts. The second section will focus on a study conducted by Corey, Peterson, Lewis, and Bukarau (2010) that delved into the same data that I will be looking at for this study. Their study focused on a different research question that was based on characterizing high quality mathematics instruction.

Research has shown that mathematics instruction in Japan differs from instruction in the United States (Jacobs & Morita, 2002; Stigler & Hiebert, 1999a). The existing literature on Japanese mathematics instruction suggested that the Japanese had both a well understood and highly shared conception of what constituted good instruction (Jacobs & Morita, 2002). Jacobs and Morita (2002) studied video tapped lessons and had both teachers from the United States and Japan comment on those videos. They found that teachers from Japan made similar comments about the lessons they viewed. They also documented that Japanese lessons usually followed a common pattern. This pattern started as the teacher posed a problem. The students worked on the problem. They then presented their ideas and generated a formula or generalization to explain the answer to the problem.

The literature also showed that the Japanese conception of mathematics instruction involved student-centered lessons (Lewis & Tsuchida, 1997; Stevenson & Stigler, 1992). Lewis and Tsuchida (1997) noted that Japanese instruction had changed since World War II to include more student exploration, presentations, and discussions. Stevenson and Stigler (1992) drew similar conclusions after finding that Japanese teachers gave students open-ended tasks that the students were expected to solve and present to the class. They found that the teachers actively drew upon students' work to guide the lesson in the desired direction. Similarly, Jacobs and Morita (2002) found that as Japanese teachers reviewed films of math lessons they agreed that student presentations were an effective way of accomplishing the desired learning goal.

Researchers have also shown that Japanese lessons were focused on non-procedurally based problems (Hiebert et al., 2005; Stigler & Hiebert, 1999a, 2004; Whitman & Lai, 1990). This meant that Japanese teachers put a great deal of planning into presenting tasks that students had not previously received instruction on how to solve (Stigler & Hiebert, 1999a, 2004). Hiebert et al. (2005) found that Japanese teachers presented problems to students that required them to spend 65 percent of their work time on using non-procedural mathematics. This meant that Japanese students spent most of their work time using what they know about math in ways not previously demonstrated to them by the teacher.

The current research suggested that Japanese teachers spent more time introducing new mathematics and did not spend an excessive amount of time reviewing principles (Hiebert et al., 2005). Studying the TIMSS videos, Hiebert et al. (2005) found that within the sample, 76% of Japanese lessons introduced new content. Hiebert et al. found that teacher's reviewed old material when students had to remember pieces of older information in order to work on new tasks, but that the majority of their class time was spent on studying fresh, new topics.

Many researchers have studied the Japanese and have asked what it is that allows them to hold such a conception about teaching. Stigler and Hiebert (1997) addressed this by stating that the Japanese conceptions of instruction "undoubtedly follow from different instructional goals and are probably based on different assumptions about the nature of mathematics, the ways in which students learn, and the appropriate role of the teacher (p. 91)."

Second, a study conducted by Corey, Peterson, Lewis, and Bukarau (2010) explored the nature of mathematics and mathematics teaching that Japanese teachers value. Within this study Corey et al. used a grounded theory approach as they studied 19 conversation sessions between six Japanese student and cooperating teachers. Within this study Corey et al. attended to the features of an effective lesson as it came up in the pre-lesson conversations. They found that Japanese instruction was focused on six principles: student intellectual engagement, goals, flow, unit, adaptive instruction, and preparation principles. I will briefly outline each of these six principles of high-quality Japanese instruction outlined in this study.

The intellectual engagement principle was identified when Corey et al. (2010) found that it was important that Japanese teachers engage their students in meaningful mathematics. Meaningful mathematics meant that the mathematics presented to the students was both challenging and relevant to the students' progression through mathematics. This topic came up in all 19 of the conversations. It came up as Japanese cooperating teachers brought up the importance of having mathematically rich tasks with their student teachers. The task was considered rich if it allowed students the opportunity to struggle with meaningful mathematics.

The goal principle arose when Corey et al. (2010) noticed that it was important to the cooperating teacher that each Japanese lesson be centered on a goal that dealt with student motivation, performance, and understanding. Motivation dealt with the students' desire to engage in the mathematics. Performance dealt with the students' ability to correctly solve a problem. Understanding dealt with the students' ability to comprehend the mathematics at hand. It was seen as stress was made on students grasping the mathematical reasons to why various solution patterns would work on the same problem. The goal principle came up in 14 of the 19 conversations. Most of the goals dealt with more than one of these three areas and was always a driving factor in the type of instruction that cooperating teachers would suggest to their student teachers.

The flow principle came up when Corey et al. (2010) realized that Japanese teachers focused on helping students move from one topic to the next by presenting a task that was carefully crafted to build off of students' prior knowledge. This principle was mentioned in all 19 of the conversations. It occurred when student teachers were routinely asked to think about tasks or principles within the lesson that were successively more complicated from the previous task or principle. Often the objective was to help students arrive at the desired learning goal by first working with accessible subtasks that allowed them build on prior knowledge in a natural way.

The unit principle was identified when Japanese cooperating teachers suggested that the student teacher should consider the lesson they were assigned to teach in the context of previous and future lessons (Corey et al., 2010). They did this to ensure that lessons progressed in a natural way. It also helped student teachers be aware of what had already been taught and what was going to be taught next. The unit principle came up in 10 out of the 19 conversations. It came up when cooperating teachers found it necessary to inform their student teachers of the grade level of their students and what their students had covered in previous units or would cover in the future. The CTs wanted to ensure that the STs understood the context behind the lessons they were preparing to teach.

The adaptive instruction principle came about when Corey et al. (2010) noticed that Japanese teachers sought to engage students with appropriate tasks that challenged the students at the students' personal level of mathematical understanding. The adaptive instruction principle appeared in 10 out of the 19 conversations. It usually happened as student teachers were helped in choosing a task that was rich with multiple entry points. These tasks allowed struggling students the ability to solve the question by counting and looking for patterns, whereas more advanced students were able to explore general equations that governed the specific answer.

The preparation principle arose when Corey et al. (2010) found that Japanese teachers stressed the importance of well laid out lesson plans that were both coherent and addressed the previous five principles. It also was important that they interconnect the five previously mentioned categories in a coherent way. This preparation principle came up in 18 of the 19 conversations. The researchers noticed that it occurred when either the student teacher or cooperating teachers addressed the layout and/or the content of their lesson plan.

Current research on Japanese and American conceptions of mathematics instruction suggest that there are many differences between the two countries (Jacobs & Morita, 2002; Stigler & Hiebert, 1999a). In *The Teaching Gap* (1999a) Stigler and Hiebert looked at Japanese, American, and German mathematics lessons. They found a great contrast in the way American teachers taught as compared to their Japanese counterparts. They found that a majority of the American lessons were based mostly procedural in nature and that American teachers spent more time reviewing old subjects. Both the Stigler and Hiebert and the studies above indicated that Japanese teachers attend to task based instruction that is focused on engaging students with new and challenging mathematics whereas the focus of the American teachers were very different. It was this type of focus on effective mathematics lessons that has made Japan a rich field of study.

Mathematical knowledge for teaching

In this section I first make an argument that MKT is a worthwhile field of study. Next I delineate popular models that have helped define the field of MKT. I have also included these models in this part of the literature review. Finally after each model I explained how each them helped validate this study.

Shulman (1986) was one of the first researchers who addressed a domain of teaching knowledge that was independent of knowledge of a certain subject. This meant that not only did teachers need to know the subject they taught but they also needed to know other elements of the subject that normal professionals in that same field did not need to know. He originally proposed that this domain of teaching knowledge consisted of three categories. The first category, content knowledge, referred to the amount of knowledge of a subject as well as the way it was organized in the mind of the teacher. The second category, called pedagogical content knowledge (PCK), went beyond the first category and entered the realm of student interaction. Shulman's PCK consisted of the representations of specific content ideas and how students would view those. It also dealt with what ideas could be difficult for students to grasp. This category quickly became the center of attention for mathematics education researchers such as Deborah Ball (Ball et al., 2008) and others. The third category was curricular knowledge, which dealt with the teacher's knowledge of the existing materials and tasks designed to help the students learn the content material.

Shulman's ideas of content and pedagogical content knowledge impacted the field of education research. Shulman (1986) termed the lack of stress on content and pedagogical content knowledge as the *missing paradigm.* He stated that not only was PCK found lacking in schools but it was also missing in the research on education. Before Shulman brought up these ideas teaching decisions focused primarily on general pedagogy and practice, and not on the type of teacher actions that are were specific to a particular discipline (Ball et. al., 2008). This knowledge was later termed MKT.

Ball and other researchers have worked on proving the existence of MKT as a domain of knowledge whose composite parts did not all reside within the realm of mathematical knowledge. In 2004 Hill et al. developed tests that were designed to assess the MKT of a teacher. They gave these tests to mathematics teachers and found that the MKT scores they had assigned to teachers based on observations of their teaching correlated with the scores the teachers earned on the tests. In this same study they also found a positive correlation between teachers' scores and student achievement. This helped to substantiate the existence of a set of knowledge about mathematics that if possessed could help students achieve more in mathematics.

To further prove the distinct nature of MKT from mathematical knowledge Hill, Dean, and Goffney (2007) gave the MKT tests to both math teachers and mathematicians. They found that in a sample of 40 mathematicians, the average score was 78% with scores ranging from 48% to 94% on an elementary MKT test. Though mathematicians were able to answer content knowledge related questions with ease they showed very little flexibility when asked to interpret non-traditional solution methods. They also found that mathematicians often struggled on pedagogical content knowledge problems where they were asked to interpret student responses. The finding that mathematicians scored on average 78% on an MKT test showed that MKT contains components of knowledge that are not fully dependent on the type of mathematics which mathematicians use on a regular basis. If MKT, as a set of knowledge, resided wholly in knowledge about mathematics one would expect a higher average score from the mathematicians.

Despite the work done in the field of MKT research has shown that teachers lack MKT. Mewborn (2003) confirmed Shulman's conjecture of the existence of a missing paradigm in mathematics. She found that mathematics teachers had a strong knowledge of mathematical procedures but little to no understanding of the mathematical concepts. Furthermore, Tirosh, Even, and Robinson (1998) found that some practicing intermediate school teachers were unaware of students' common mistakes despite years of experience. Ma (1999) found a substantial lack of American elementary teachers' mathematical content knowledge compared to that of their Chinese counterparts. Instead she found that American teacher knowledge was procedurally based and not interconnected. These studies suggested that implementation of MKT principles was problematic. These studies further suggested that an approach that focused on teachers' in practice mention of MKT principles could be useful to teachers because it would give them a context into how the MKT could be used in a teaching scenario.

There has been very little work done on characterizing the MKT that teachers of a different country value. The only two studies that I have found that did something similar was Delaney and Kwon. Both Delaney (Delaney, Ball, Hill, Schilling, & Zopf, 2008) and Kwon (Kwon, Thames, & Pang, 2012) adapted the MKT tests that were developed by Ball, Hill and others in the United States so that they could fairly test and rate the MKT of Irish primary school and Korean teachers. In their individual studies they both found correlations between the MKT model developed in the United States and the one that teachers used in Ireland. However, due to the nature of their methods, they were unable to draw any concrete conclusions about how the model of MKT in their county compared with that of the US. Both authors explicitly stated that the main purpose of the study was more focused on the adaptation of US materials, such as MKT tests, to other countries rather than the analysis of the MKT used among Irish primary school and Korean teachers. Neither author was able to draw meaningful conclusions about the characteristics of MKT of the teachers they studied. Another interesting finding was that these studies showed that MKT differed slightly, but it was difficult due to the focus of the studies to classify those differences. These studies showed that using MKT tests to classify the MKT of teachers in a different country was problematic. It is then reasonable to deduce that a different method could yield a meaningful look into an international source of MKT.

Shulman's (1986) ideas were general to the field of education and were not content specific. Furthermore, the mention of the missing paradigm lead many researchers in mathematics education to attempt to conceptualize PCK in mathematics. Researchers used different approaches to accomplish this end. One of the first attempts was made by Marks (1990). He conducted a case study of eight fifth-grade teachers and focused on their completion of task based interviews that dealt with planning a lesson, critiquing a classroom videotape, and identifying and remediating students' errors. He found that among these eight teachers pedagogical content knowledge consisted of four categories. These four categories were the teachers' subject matter knowledge, students' understanding, media available for instruction, and process for instruction. It is important to note that Marks called PCK what later researchers such as Ball et al. (2008) would have called MKT. The term MKT was not used in 1990 when Marks did this study. Therefore, subject matter knowledge was considered by Marks to be a category of PCK. The model of this conception of PCK is seen below in Figure 1.

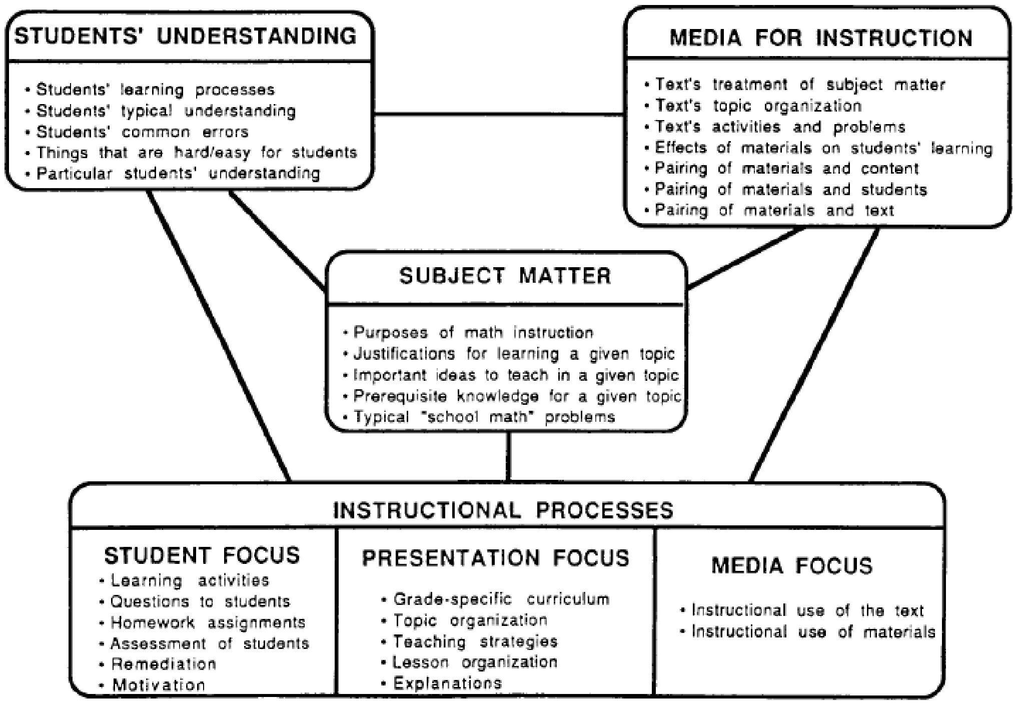

Figure 1. Model presented by Marks (1990, p. 5) of the pedagogical content knowledge of fifth grade teachers.

Within this model clear examples of each of the four categories are provided for the reader. Within the model instructional process deals with each of the other three categories and places it within the context of the realm of teacher-student interactions. This is why the instructional process category includes portions from each of the other three categories.

Marks' (1990) was the first of its kind. It helped researchers see a way to classify areas of MKT. The model that Marks presented shows that it was difficult to separate elements of subject matter knowledge and pedagogical content knowledge. Later studies were needed to classify this difference between the two categories and better document their distinct nature.

Davis and Simmt (2006) used a complexity science lens to address mathematics-for-teaching. Within their research they use the term mathematics-for-teaching to mean both the teachers' knowledge of established mathematics and the teachers' knowledge of how mathematics is established in the mind of the learner. In their research they analyzed their conversations with 24 teachers ranging from Kindergarten through high school. Their research discussed the nested or "intertwined" nature of four subcategories of mathematical teacher knowledge.

These subcategories were mathematical objects, curriculum structures, classroom collectivity, and subjective understanding. In this model Davis and Simmt also accounted for dynamic and stable natures of teacher knowledge. I will now discuss what they meant by the four subcategories and the two natures of teacher knowledge.

Mathematical objects dealt with metaphors, images, and constructs used to make sense of mathematical principles. An example of this would be viewing multiplication as a certain amount of jumps of a particular size on a number line.

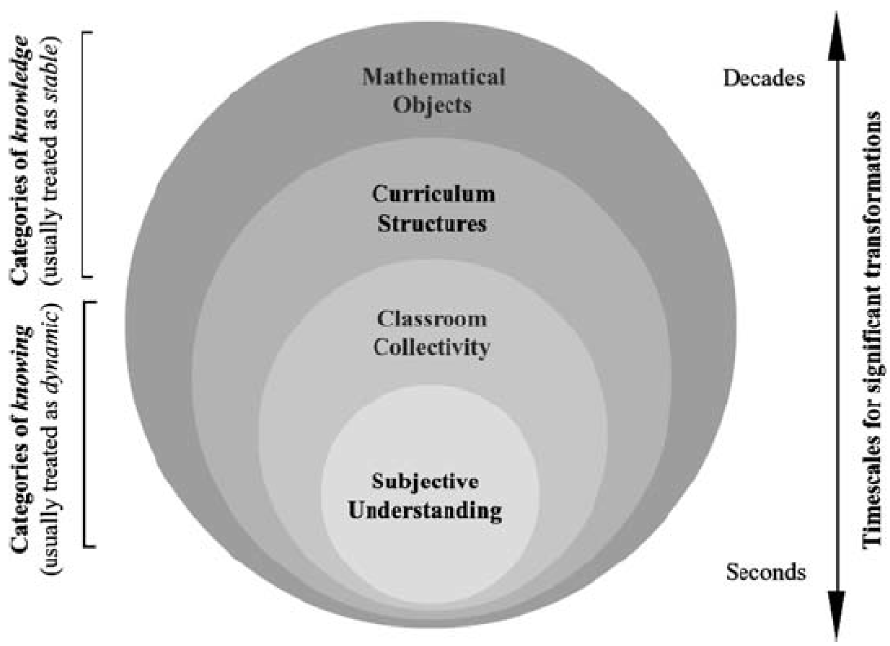

Figure 2. Model of mathematics-for-teaching presented by Davis and Simmt (2006, p.296).

Curriculum structures referenced the efforts to draw upon commonalities between topics when searching for tasks and models that were used to teach certain principles. Here emphasis was given to the tasks and models that had the widest array of applications. An example of this could be a grid-based table that could be used to calculate 25×45 as $(20 + 5) \times (40 + 5)$. The component parts would be multiplied (20×40, 20×5, 5×40, 5×5) and then added together to get the product. That same grid-based table could then be used to help the student calculate $2\,\frac{1}{2} \times 4\,\frac{1}{2}$ as $(2 + \frac{1}{2}) \times (4 + \frac{1}{2})$. Students would then be able to use the same grid-based model to calculate polynomial multiplication such as $(2 + \text{n}) \times (4 + \text{n})$.

Classroom collectivity dealt with the type of learning atmosphere that was set up by the teachers, students, and even researchers. Each group was learning because of the interactions between each of these groups. An example of this could be the learning that takes place when presenting a multiplication lesson. The students learn different representations of multiplication from the teacher and other students. Meanwhile, the teacher is learning from the students about how effective the different representations are for the students or even making sense of new models that the students presented that the teacher did not previously think about. The researchers could also learn and influence the learning of the other two previously mentioned groups by attending to the learning that took place and interjecting as they see necessary.

Subjective understanding addressed the awareness that teachers should have about how learners come to make sense of different mathematical topics. Davis and Simmt (2006) stated that students do this in unique ways. This may be caused by a number of different factors such as developmental issues, life experiences, and life circumstances. An example of this could be seen in two students in the same class who viewed multiplication differently. One student, from America, saw 3×4 as three groups of four and another student, from Europe, saw 3×4 as three things in each of four groups despite receiving the same classroom instruction.

The dynamic and stable nature of teacher knowledge attended to the different roles in which the knowledge was used or stored. As the name suggests, dynamic knowledge came into play when teachers were aware of, or updated, their knowledge based on student responses to instruction. Because classroom collectivity and learning subjectivity dealt with teacher-student interactions they fell into Davis and Simmt's (2006) dynamic nature of teacher knowledge. Stable knowledge dealt with the teachers' knowledge of established facts, principles, and tools that could be used to help students understand mathematical concepts. Because mathematical objects and curriculum structures were based on knowledge in these established areas and did not change based on the students, they pertained to the stable nature of teacher knowledge.

Davis and Simmt (2006) showed that a novel approach, such as using a completely different lens, could yield important new view of how to look at MKT. The complexity science view of the nested nature of learning and instruction highlighted the openness of the MKT field of research. No one had previously noted the static and dynamic nature of teaching knowledge. This study further showed that novel approaches, like the one I will explain in the methods section below, can lead to meaningful contributions into the field of MKT.

Finally, Ball et al. (2008) presented the conceptualization of MKT shown in Figure 3. Within this conceptualization Ball broke up mathematical knowledge for teaching into two parts, subject matter knowledge (SMK) and pedagogical content knowledge, which was different than the PCK discussed by Shulman in 1986.

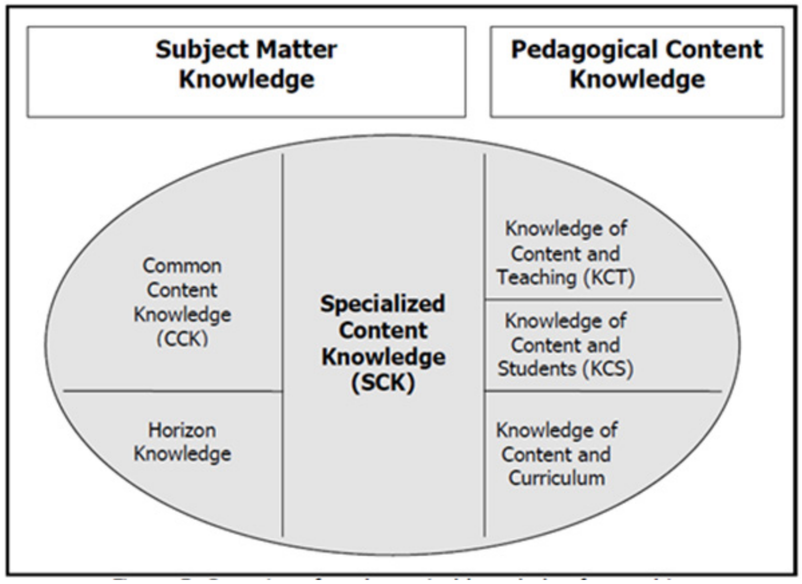

Figure 3. Model of MKT presented by Ball, Thames, and Phelps (2008, p.403).

Here it is important to understand what Ball et al. meant by these two main topics. Subject matter knowledge dealt with the teacher's knowledge of mathematics. It not only focused on the knowledge of procedures but it also focused on the deep mathematical principles of that behavior. It also addressed the connectedness of math topics. Pedagogical content knowledge could be seen when a teacher used his or her subject matter knowledge in teaching. A teacher used pedagogical content knowledge when he or she anticipated students' reactions to tasks, considered what tasks would be beneficial in order to introduce a new topic, and thought about what materials existed to help students in their study of math topics.

Ball et al. (2008) expounded on these two areas of knowledge by categorizing and defining its component parts. First, for subject matter knowledge, Ball et al. discussed the existence of common content knowledge (CCK), specialized content knowledge (SCK), and horizon knowledge. CCK refers to the mathematical knowledge and skill of a well-educated adult. For example, knowing the answer to 6×5. SCK is a teacher's knowledge and skill that well-educated adults do not generally have. For example, one could know different strategies for calculating 6×5. Finally, horizon knowledge is the knowledge of the connectedness of mathematical topics. For example, knowing how the multiplication of 6×5 is connected to an array or grouping representation that can possibly connect to future mathematical topics such as division, variable multiplication, or exponents.

Second, for pedagogical content knowledge Ball et al. (2008) expounded on the existence of three domains: knowledge of content and teaching (KCT), knowledge of content and students (KCS), and knowledge of content and curriculum. KCT refers to the knowledge of mathematics as it applies to teaching. An example of this would be knowing the advantages and disadvantages of different representations of 6×5. KCS requires an understanding of how students will respond to mathematical tasks presented to them. This includes students' common misconceptions such as a student giving an erroneous answer to 6×5 of 11 or 25.

Finally, knowledge of content and curriculum refers to the knowledge about tasks and materials that exist to help facilitate student understanding. An example of this would be knowledge of a unit on manipulatives designed to provide students with a stronger feeling for what 6×5 really means.

This model was developed using studies that had been done in the United States. It did not mention the MKT developed by teachers of a different country. The study from Stigler and Hiebert (1999a) showed that Japanese teachers valued a different type of instruction than American teachers. It is reasonable to surmise that since Japanese teachers value different instruction that the knowledge it takes them to accomplish such instruction would also be different.

Framework

Taking the research above into consideration, it is important now to establish three tools that I will take into this study that will help me to address the research questions. These three tools are a definition of MKT, a framework that will help to identify MKT, and a set of characteristics that classify Japanese instruction.

Definition of MKT

The definition of MKT used throughout this paper was the same definition of MKT given by Ball et al. (2008). In their work they defined MKT as what mathematics "teachers need to know and be able to do in order to carry out the work of teaching effectively" (p. 20). The term "work of teaching effectively" was defined by Ball et al. (2008) as what teachers do when teaching mathematics and how what they do requires mathematical understanding, reasoning, insight, and skill. This broad definition of MKT encompassed definitions of MKT used in other studies (Davis & Simmt, 2006).

I chose this definition for a two reasons. First, this definition was broad enough to cover occurrences of teaching mathematics that might show up in a novel set of data. Also this definition is based on teaching which matches the data set well. The data set covers the CTs and STs conversations as they prepared to teach. The conversations covered the tasks of teaching and the mathematics behind the tasks the STs would need to know before they taught the lessons. Ball et al. (2008) stated that the focus of their definition was on the tasks of teaching and the mathematics required to accomplish those tasks. For these two reasons the Ball et al. definition of MKT was a good fit for this study.

Framework

In this section I will delineate two main topics. First, I will explain why a framework is necessary. Second, I will present the framework I developed to identify episodes of MKT within the conversations between Japanese cooperating and student teachers.

As explained in the literature review there have been many attempts to classify the elements of MKT. The Ball et al. (2008), Davis and Simmt (2006), and Marks (1990) models are the more well-known examples of these attempts. However, no attempts to classify MKT have been developed using international sources except for two (Delaney et al., 2008, Kwon et al., 2012). As mentioned in the review of the literature Delaney et al. state the limitations of their approach in identifying the differences and conceptualizing a framework for MKT of teachers in a different country. However, they did find that in some aspects MKT was different in Ireland. Since identifying conceptions of MKT was the main goal of this study I had to approach the issue of gaining insight into Japanese teachers' MKT from a different angle.

Furthermore, as stated above, since Japanese teachers had a different conception of effective math lessons it is very likely that they have a different conception of MKT. Hiebert et al. (2005) mentioned that this could be due to the fact that Japanese teachers attended to different pedagogical elements when teaching. This was further justification to approach creating a model of Japanese MKT from a unique perspective.

For these reasons I could not adopt a current framework (Ball et al., 2008; Davis & Simmt, 2006; Marks, 1990). This was because an adopted framework would most likely miss the type of MKT that Japanese teachers were expected to learn and exercise.

Due to the differences explained above it was necessary to study MKT in Japan from a more fundamental vantage point. This was done by looking at the different components of a classroom setting and considering the instructional triangle presented below in Figure 4 adapted from *Adding It Up* (2005).

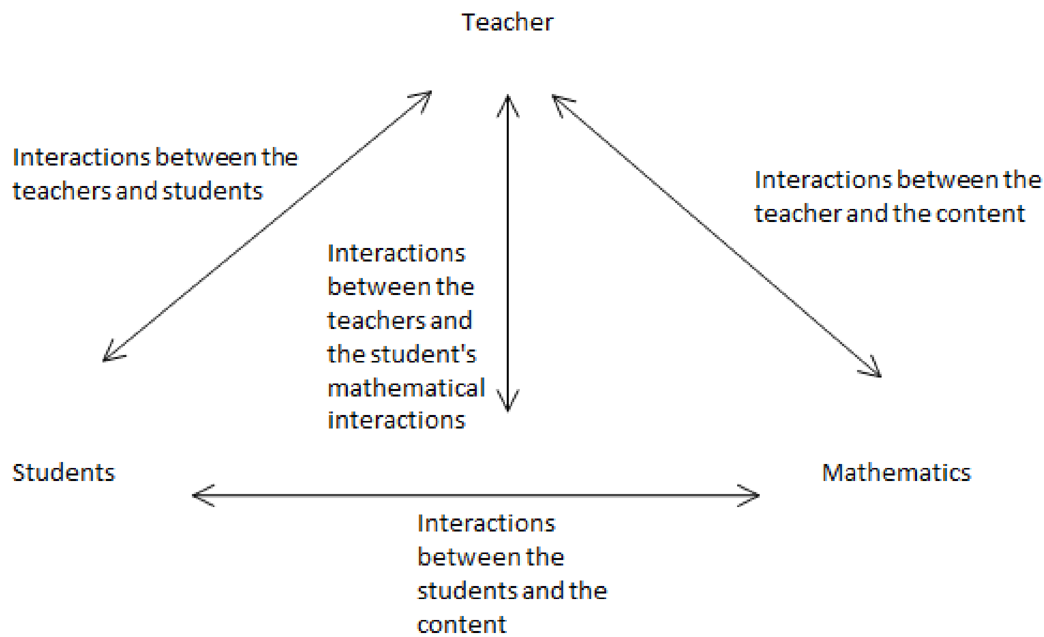

Figure 4. Framework of the instructional triangle that was used to classify interactions between Japanese CTs and STs. Adapted from *Adding it up: Helping children learn mathematic* (National Research Council, 2001, p.314).

This instructional triangle took into account the teacher, mathematics content, and students. Meaningful discussions pertaining to MKT could potentially arise within certain interactions between these three components. These interactions are indicated with an arrow between two of the three components and one between a component and a different interaction. Among the four sets of interactions above the two that held the most promise for finding occurrences of MKT were in discussions about the interactions between the teachers and student mathematical interactions and the interactions between the teacher and the content.

First, attention was placed on the interactions between teachers and the content. This category of interactions dealt with the teacher's familiarity of mathematics field. Within the context of this study these interactions dealt with teacher's ability to understand a mathematical concept accurately and correctly utilize the procedure being presented to students (National Research Council, 2001).

A preliminary examination of the data showed that when attention was given to the interactions between the teachers and content it highlighted episodes of MKT. In a conversation between the cooperating teacher (CT) Sasaki and student teacher (ST) Motori, CT Sasaki instructed ST Motori about the importance of knowing the connections between the different solution methods when solving quadratic equations.

CT Sasaki: Are you ok with the complete the square?
ST Motori: What do you mean by ok?
CT Sasaki: You understand it, don't you?
ST Motori: Yes, I do.
CT Sasaki: Can you derive a quadratic formula using the complete the square method? Do you understand the process of how $ax^2 + bx + c = 0$ can be changed to $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
ST Motori: Hold on one second, please.

This example demonstrated the type of requisite knowledge about mathematics that cooperating teachers in Japan feel their student teachers should have. This episode dealt with MKT because Motori Sensei was expected to know how to derive the quadratic formula from a general quadratic equation using the complete the square method. These types of episodes became apparent within the data when attention is given to the interactions between teacher and content.

Second, the interactions between the teacher and the students' mathematical interactions also yielded fruitful insight into MKT. This was because, as stated previously, within these interactions the teacher consciously made decisions based on how he or she felt the students would react to the mathematical concepts that were taught to them. A preliminary glance at the same conversation between CT Ueno and ST Motori showed the type of data that was highlighted when the interactions between the teacher and the students' mathematical interactions became the focus. In this part of the conversation ST Motori proposed a task to help the students work on variables. CT Ueno was concerned and offered the following instruction.

ST Motori: My intension is for the students to determine whether this is a number of something or a price.
CT Ueno: Are you going to ask them to work on this individually or as a group?
ST Motori: As a group. I will ask this question and have them think in groups.
CT Ueno: Teacher, Please imagine. You will be using Udon (Japanese style noodle), egg, and Ebiten (shrimp tempura) and have students make an expression, right? You will have them make an expression individually, right? Kids will make any expressions they want. For example, one student might say "I will eat two bowls of udon." In this case, it is not going to be a multiplication problem because it cannot be multiplied by anything. What if a student says that he will put 5 eggs in it? You have to use a specific number, right? Well, then suppose a number of udon becomes A and a number of eggs becomes B and shrimp. Suppose students make an expression individually and make a complicated expression. Individuals make expressions like this and what are students going to do about that in groups?

When emphasis was placed on examining the interactions between teacher and the students interactions with the mathematics several MKT relevant themes arise. Here CT Ueno stated that if students are left to create anything they wanted they would have a hard time making sense of a variable because among the students they may have used different variables for the same food. CT Ueno has clearly instructed ST Motori to account for student thinking before presenting this task to the students. He saw that ST Motori had not thought hard enough about the best way for students to engage with this

These two examples supported the use of the instructional triangle within the framework of this study. It was clear that when attention was given to the interactions between the teacher and the content and the teacher and the students' interactions with the content components of MKT were both identified. Due to the nature of MKT and its definition stated above MKT could only be present where an element of the teaching was present. The interactions between the teachers and the students that were devoid of mathematical content were not considered MKT relevant. Admittedly it was hard to claim that no MKT relevant conversations could have taken place in the possible interactions between the students and the mathematics, however, the nature of the data ruled this category out. This was because the conversations were pre-lesson conversations between CTs and STs and not observed student interactions with mathematics. However, even if this were not the case it could be argued that a category of possible interactions devoid of teaching would not be MKT relevant. Finally, no MKT relevant conversations could take place within any one vertex of the instructional triangle due to the fact that it would be missing an element of either teaching or mathematics.

It is also important to explicitly mention that the previous examples from the data illuminated the way that the instructional triangle will be used to guide and inform the data analysis. The instructional triangle helped guide data analysis by providing a method for sorting the interactions between the cooperating and student teachers as MKT or not. If the interactions were between the teachers and the content or the connections between the teachers and the student's interactions with the mathematics then that interaction was considered an MKT interaction and was coded. If the interaction was about anything else, such as the interactions between teachers and students or teachers and the Japanese language, then that interaction was not considered as an MKT interaction. An example of these could be found in the conversation between CT Kimura and ST Akahiko. In this particular part of the conversation the teachers were talking about the write-up of the lesson plan.

CT Kimura: You didn't use parenthesis here
ST Akahiko: Oh, I didn't. I didn't know how to. I could only find this.
CT Kimura: Aren't there other fonts in Word?
ST Akahiko: There is no font in Word. I was looking for it, but I couldn't find it.
CT Kimura: I'll type it for you.
ST Akahiko: Oh, I am sorry.

It was clear that this interaction had nothing to do with MKT. The teachers were talking about how to make a parenthesis on the computer. Similar interactions that were not MKT relevant dealt with poorly worded sentences in the teacher's lesson plan.

Another example of unit of analysis that had to do with a task of teaching but did not contain any MKT relevant interactions happened in the conversation between CT Ueno and ST Tomoko. Towards the end of the conversation ST Tomoko expressed her fears that the students would act up during the class. CT Ueno gave the following advice.

> ST Tomoko: I am worried about it.
> CT Ueno: They don't mean to disturb your lesson so you should relax more as you teach. I am with them all the time so I can tell them anything. Besides they understand my jokes; and it doesn't cause any problems in the class. However, if you do the same it may cause some problems. So please choose words carefully and don't insist too much.

Here CT Ueno gave ST Tomoko advice about teaching his students. Though the advice was helpful to ST Tomoko it did not contain any elements of mathematics. Therefore, this interaction would be placed on the teachers' interactions with the students' on the instructional triangle.

As shown above in the three excerpts MKT relevant conversations could be found in the interactions between the teachers and the students' interactions with the mathematics and the interactions between the teachers and the mathematics. A preliminary exploration of the data did show that the STs and CTs did talk about subjects that were not MKT relevant, however, the triangle was used to classify the types of interactions and identify the ones that possibly contained MKT relevant conversations.

A Japanese conception of high quality instruction

As will be explained in more depth in the methods section, I attended to the qualities of the first five principles of highly effective Japanese instruction mentioned by Corey, Petersen, Lewis, and Bukarau (2010). The planning principle was not included in this study because it was interrelated with the other five principles. Within this section I will explain the reason behind using these five principles and delineate their interpretation from an MKT standpoint. The MKT interpretation of the five principles were used to code within the sections of the conversation that were determined could contain MKT relevant conversations by using the instructional triangle (mentioned above). I covered in depth how I used the five principles in the coding process in the methods chapter.

First, I used the five principles because the data which I analyzed was the same data that was used in the study in which these principles were found (Corey et al., 2010). Therefore, it was certain I would find episodes dealing with these five principles in the conversations. Also these principles were found to typify Japanese instruction and were, therefore, consistent with the Japanese context of teaching mentioned in the literature review.

Second, the six principles found by Corey et al. (2010) dealt with aspects of instruction. I, however, focused on what the data says about MKT. Therefore, I analyzed the mathematical knowledge that will be necessary to implement those principles. This was done because the implementation of each of these principles required certain aspects of teacher knowledge. This analysis aided in the creation of codes that were used to categorize various episodes of MKT. The following are the first five principles of Corey et al. (2010) principles of highly-effective mathematics instruction adapted to an MKT perspective.

The mathematical knowledge that was required to implement the intellectual engagement principle entailed knowledge that was required to ensure that the students would be busy making sense of meaningful mathematics. This would include knowledge of tasks that addressed meaningful mathematics as well as how students would interact with the tasks. The presented task should not only be fun but should connect students with appropriate level mathematical concepts.

The mathematical knowledge needed to apply the goal principle would include knowledge about how to ensure that meaningful goals were set and kept for each lesson. These goals dealt with student motivation, performance, and understanding. This area of knowledge included statements that explained the need for appropriate goals and reasons that were given for changing the particular goals. It also included mention of the obtaining the goal when discussing the development of a lesson. It dealt with what goals would motivate students to engage in the task and what goals were mathematically meaningful based on the students' mathematical background. It also dealt with the students' disposition towards math and whether the goal helped students see the usefulness of mathematics.

The mathematical knowledge requisite to effectively apply the flow principle dealt with knowledge required to ensure that the lesson naturally progressed. One example of this category would be the mention of time issues as the lesson developed. Another example would be the teachers' knowledge about what mathematics built off of one another. This domain would also address when to ask crucial questions or present certain findings to the class.

The knowledge to appropriately utilize the unit principle entailed the teacher's knowledge of the cohesion of math certain mathematical topics across units and grade levels. This included the discussions on the placement of the lesson within the unit. It also included mention of students' current grade level and what that means about the lessons they have already experienced and had yet to experience.

Finally, the knowledge necessary to properly implement the adaptive instruction principle included the knowledge that of what mathematics would interest students at their personal level of understanding. This included knowledge of students' mathematical abilities as well as knowledge of a set of strategies that could be used to help struggling students. It also included knowledge of a set of techniques that could be used to extend a task for talented students.

As mentioned above these interpretations of the five principles of highly-effective Japanese instruction to an MKT perspective directed the creation of codes and aided in the final construction of the model presented below in the results.

Methods

This study was different than most other studies in the field of MKT in its methodology. The current literature suggested four common methodologies that have been used to add to the field of MKT. First, researchers in this field have relied on direct and task based interviews with teachers to illuminate the MKT that existed in the minds of mathematics teachers (Marks, 1990). Second, researchers have inferred a teacher's MKT based on observed lessons they taught along with follow up interviews (Davis & Simmt, 2006). Third, researchers have sought to develop exams that classify a teacher's MKT and then compare that with observed lessons to assess the usefulness of MKT in the context of the lesson (Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005). Fourth, others have attempted to delineate the domains of MKT by theorizing and taking into account prior research and experience on the subject of MKT (Ball & Bass, 2000; Ball et al., 2008). In each of these cases the framework is either being developed or, as in the case of testing MKT, was implemented as a precursory tool which was used to develop the various questions on the test.

For this study I used a method which provided insight into the MKT of Japanese teachers in a novel way. In this study student teachers were observed as they presented their lesson plans to cooperating teachers in Japan. These teachers were not asked to speak about MKT but rather engaged in feedback sessions before a mathematics lesson was taught. Therefore, these conversations were not centered on MKT but instead typified the types of conversations that occurred pre-lesson. Within these conversations the experienced cooperating teacher had the goal of instructing the student teacher on how to create highly effective mathematics lessons. One necessary component of a highly effective mathematics lesson is the amount and type of mathematics the teacher knows. Often within these conversations the cooperating teachers both implied and stated the type of knowledge that is expected of their student teachers.

There are advantages to using these conversations to study MKT. First, research (Leatham, 2006) suggested that basing teacher beliefs off of direct interviews can be problematic. He stated that when teachers were directly questioned about their practices they may tend to answer questions the way they feel they should rather than saying what they truly believed. Furthermore, Leatham stated that teachers may not be fully aware that they hold a certain belief. However, the open ended nature of these conversations provided me with a way to study elements of MKT that teachers might have otherwise not made explicit. Second, these conversations were a common part of the student teaching processes and, therefore, provided me with an opportunity to observe elements of Japanese MKT in a natural setting.

It was a concern of this study that the MKT that was addressed by the CTs and STs would only cover a small portion of the MKT really possessed. However, in the exit interviews the CTs addressed the wide range of abilities of their STs. Though some needed relatively minor adjustments others needed a lot of help when considering the type of MKT that was necessary to teach well. This meant that the teachers could address a wide range of MKT relevant issues. Also the conversation between CT Ueno and ST Motori produced over 25 pages of transcripts and with over 60 MKT relevant units. This showed that the CT felt free to address any problems with MKT the ST was having regardless of how long it took. It also showed that this data set addressed a wide view of MKT issues.

Data collection

The pre-lesson data consisted of Japanese cooperating teachers meeting with their student teachers as they planned out lessons. During these sessions the student teachers received feedback on their lesson plans.

The data was previously collected and was situated in the student teaching program run within a Japanese school system. Student teaching followed a different format in Japan than in the United Sates (Peterson, 2005). In Japan student teaching took on a more apprenticing format where the student teachers learned firsthand from their cooperating teachers' about acceptable practices. Japanese student teachers usually taught fewer lessons during their student teaching period than student teachers in the United States. Due to the fact that they spent less time teaching they had more time to prepare lessons and discuss how those lessons played out in the classroom.

In the pre-lesson conversations the student teachers discussed lesson plans which the cooperating teacher would review and then provide verbal feedback. In order to teach the proposed lessons the student teacher would have to earn a stamp on the lesson plan signifying that the lesson was ready to be taught in class. In order to obtain the stamp the cooperating teacher would discuss and suggest different aspects of the lesson that lacked forethought about mathematics and/or the abilities of the students. After receiving the requisite stamp the student teacher would then teach the class. The cooperating teacher and fellow student teachers attended this class and then held a post lesson conversation.

In this study seven student teachers were assigned to three cooperating teachers, however, only three of the seven student teachers' conversations were videotaped and transcribed. The three student teachers (ST) were Akihiko, Tomoko, and Motori. These three were not chosen for any particular reason other than it allowed the researcher to fully record all of their pre and post-lesson conversations. This was due to the fact that there was only one researcher recording these conversations and the other student teachers were in different conversations at the same time.

In the pre-lesson conversations one of the student teachers met with a cooperating teacher various times before the lesson to receive feedback and get various changes approved. Then after each lesson the student teacher met with the cooperating teacher and other student teachers to discuss how the lesson went. Each of the three student teachers worked with cooperating teachers (CT) Sasaki Sensei for the first lesson, Kimura Sensei for the second lesson, and Ueno Sensei for their third lesson. Each of the three student teachers taught three lessons to seventh, eighth, and ninth grade classes. In total this gave me eight pre-lesson conversations due to a missing conversation between CT Ueno and ST Akahiko. Because the pre-lesson conversations usually took place over a few visits they usually lasted one hour and forty-five minutes to two hours.

Data analysis

Data analysis consisted of first reading and coding the pre-lesson conversations. The primary coding and analysis was based on a set of codes developed from the first five principles of highly effective instruction mentioned in Corey, Peterson, Lewis, and Bukarau (2010). These codes will be presented at the end of this section.

However, before coding began I had to establish a unit of analysis. For this study I split each conversation into a small enough unit as to avoid dealing with too many MKT topics at once. At the same time each unit was big enough in order to find meaningful data that focused on one or two main MKT ideas. A preliminary examination of the data showed that pre-lesson conversations were led by the CT. The discussions followed the flow of the student teachers' lesson plan. Within these pre-lesson conversations the topics followed one after the other until the CT's points were made. Depending on the importance of a certain topic within the Japanese style of teaching, and the level to which the CT felt like he or she needed to address it, a topic varied anywhere from two to six exchanges with very few exceeding six exchanges. For this study I took an individual topic as the basic unit of analysis. I then used the instructional triangle to identify if the unit dealt with the interactions between the teachers and the students' interactions with the mathematics or the interactions between the teachers and the mathematics. If the unit dealt with one of these categories I then used the codes presented below to code the unit. I also spanned multiple consecutive units that to ensure that the context of the unit was preserved and to ensure that the coded section was coded with the correct code.

I first made sure that each unit of analysis had MKT relevant material. I did this by using the educational triangle in the framework and placing the unit either on the arrow representing the teacher's interactions with the mathematics or the teacher's interactions between the students and the mathematics. If the unit did not deal with either interaction it was not considered MKT relevant. Examples of units that were not MKT relevant were interactions about their use of proper Japanese language, administrative issues, and other subjects not dealing with the lesson they were teaching (small talk). I relied on the coding scheme developed by the first five principles of highly effective instruction, which I adapted to an MKT perspective in the framework. I also created emergent codes that did not fall under any the codes developed from the five principles mentioned in the framework. I created emergent codes when a unit of analysis that was MKT relevant contained parts of conversation that were not coded by the developed codes. Using the teaching triangle I determined if the un-coded section of an MKT relevant interaction was itself MKT relevant. I did this by applying the instructional triangle in the same manner used in determining if a particular unit of analysis was MKT relevant. If it was MKT relevant I created a new code or set of codes for it. The one emergent category that I found during this process was content knowledge so it was added to the coding scheme. It was the only set of new codes that arose during the coding process.

The following were the codes developed from the five principles mentioned in the framework and used in the preliminary coding process.

Intellectual engagement:

1) Knowledge about how students will react to the task.
2) Knowledge about how cognitively demanding a task could be for the students.
3) Knowledge about what makes a task fun or interesting.
4) Knowledge about what constitutes or defines a hard or easy concept.
5) Knowledge about what constitutes meaningful mathematics.
6) Task knowledge.

Goal:

1) Knowledge about what types of goals will motivate students.
2) Knowledge about what goals are appropriate for the students.
3) Knowledge about building and maintaining a positive disposition towards math.
4) What mathematics is important.

Flow

1) Knowledge about where in a task students might struggle.
2) Knowledge about how to craft a task that will build off of student's previous knowledge in order to facilitate students reaching the desired learning goal.
3) Knowledge about time allocation.
4) Knowledge about how and when to transition between various parts of the lesson.
5) Knowledge about when and/or how to ask particular mathematical questions.
6) Knowledge about how to create an intellectual need or motivate students with lively mathematics.

Unit

1) Knowledge about students' previous knowledge as it pertains to previous units students covered.
2) Knowledge about struggles that students might encounter due to the differences of one unit to another.
3) Knowledge about how the mathematical ideas of a unit connect to those in a different unit.

Adaptive Instruction:

1) Common misconceptions that students may have (could be some overlap here).
2) Knowledge of different entry points that students might take to engage with a task.
3) Knowledge about what accommodations can be made to help struggling students.
4) Knowledge about how to extend or change the task for those that aren't challenged.
5) Knowledge about what makes a certain task difficult to learn.
6) Knowledge of tasks or problems that contain multiple entry points.

Content knowledge

1) Knowledge about how to correctly apply a certain mathematical principle.
2) CT tests the content knowledge of the ST.

Figure 5. Primary codes used in the coding process.

These codes were developed with help from my advisors and were based on our knowledge of this data set from the prior study we conducted.

I began by coding five pre-lesson conversations. I chose one conversation from each of the three CTs each with different student teachers. I then chose two other pre-lesson conversations at random. Using the method outlined above I classified and categorized the MKT mentioned in the conversations then I tested those MKT categories as I coded and analyzed subsequent conversations. I also went back to previously coded and analyzed conversations to ensure the accuracy of the findings.

After defining each unit of analysis and using the instructional triangle I identified 389 MKT relevant units of analysis within the eight pre-lesson conversations. This abundance of MKT relevant conversations implied the validity of the data set to identify important MKT themes that came up in the pre-lesson conversations. These 389 units were spread out fairly evenly among the conversations with exception to the ST Matori conversation with CT Ueno Sensei. This conversation had a total of 64 MKT relevant units of analysis. The other seven conversations had on average 47 MKT relevant units of analysis. This meant that the CT Ueno/ST Motori conversation had more than 35% more units of analyses than the average.

I coded and analyzed the 389 units of analysis using the codes mentioned above. I also created new codes to account for the emergent codes that came up as I coded. I first attempted to find a correlation to MKT topics that frequently appeared in conjunction with one another by seeing if codes came up in correlation with each other with any type of frequency within the same unit. I explored various multi-level statistical models and none of them yielded any type of correlation. No statistical connection between the MKT codes and the categories meant that I could not support a conception of MKT with interconnected categories. However, as I looked across conversations it was clear that there were common MKT themes throughout the data. I made changes during the analysis process to highlight and categorize the type of MKT that was being discussed between the CTs and STs. These changes consisted of grouping like codes together in clusters. Examples of similar codes were the second flow and first unit code. Since these codes both dealt with prior student knowledge I looked across the eight discussions and found that the CTs and STs often accessed MKT that dealt with student's prior knowledge. I used this same process and arrived at the categories of knowledge that I presented below in Figure 6. In this manner the chart in Figure 6 reflected common MKT categories that frequently arose throughout all of the conversations because the codes that composed the clusters came up frequently and in every conversation. It is important to note that this was the process by with the 27 codes above were condensed into the three categories of MKT presented in the results section. As a final note, it became clear that this clustering was necessary in the preliminary coding when I looked back and discovered that single parts of a conversation contained multiple codes. This meant that the codes were too similar to each other to distinguish the difference, therefore, they were combined in a cluster.

For each of the subcategories I used a cognitive view of knowledge. This meant that the knowledge that was displayed throughout the conversations was seen as originating from a preexisting structure of knowledge either on the part of the ST or the CT. Learning did take place, but it was done by the ST when the CT demonstrated his or her MKT.

The categories presented below were not knowledge of *how* to teach well or teaching actions. Rather, they represented a summary of the knowledge structures that were accessed as the CTs taught the STs or the STs demonstrated mastery of a certain MKT relevant task. In other words, when the teachers demonstrated how to teach well I interpreted it as the teacher accessing knowledge from a preexisting structure. For example, knowledge of how to create a task was not included in the results below. This was because, in this study, the knowledge needed to know how to create a good task required knowledge from a preexisting knowledge structure, in this case, task knowledge (knowledge of a pool of relevant tasks that the teachers drew upon).

Results

I classified the common MKT categories then realized that they could be placed into three overarching groups. The yellow parts represented the MKT clusters and the blue a common category to which they belonged. The grouping of the yellow subcategories was done based on commonalities between subcategories. They were knowledge about students, knowledge about mathematics, and knowledge about the practice of teaching. The common MKT categories, mentioned earlier, then became subcategories to these three overarching groups. The following figure is a map of the MKT mentioned by the Japanese teachers in this study.

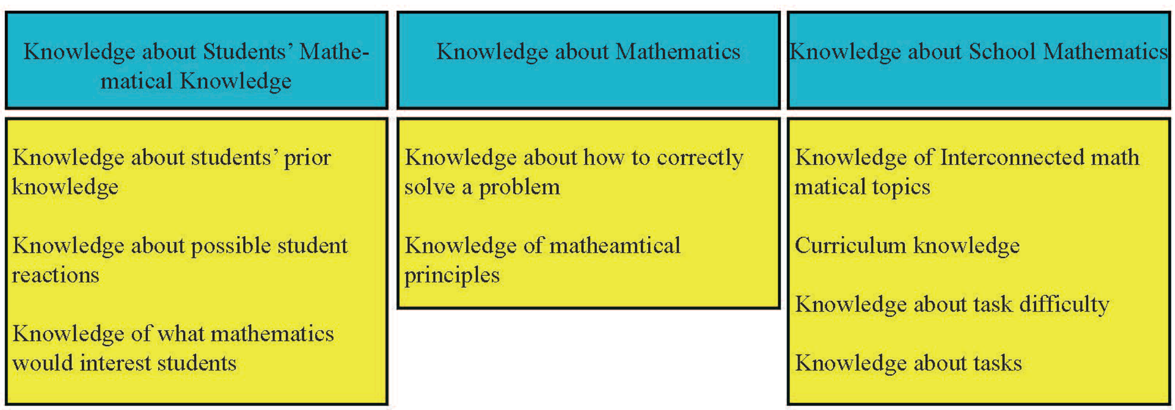

Figure 6. Map of Japanese MKT. The three categories of MKT are in blue with their corresponding subcategories in yellow.

In the following section I have included excerpts from the data that captured the essence of each subcategory of knowledge so that the reader could get a better feel for each subcategory. The reader should take note that an excerpt may contain elements of other MKT subcategories. I cut and included the excerpts in such a way so as to allow the reader to understand the context in which the MKT subcategory was discussed while at the same time highlighting that specific knowledge subcategory.

Knowledge about students' mathematical knowledge

Knowledge about students was based on the teachers' familiarity of students' mathematical preferences and abilities. The STs and CTs made this familiarity evident when they considered how the mathematics they were assigned to teach would affect the students. It also covered familiarity with the students' pre-knowledge. Like the two other main categories, elements of this category were evident in all eight conversations. The subcategories that made up knowledge about students were knowledge about students' prior knowledge, knowledge of possible student reactions, and knowledge about what mathematics would interest students.

Knowledge about students' prior knowledge. The knowledge about students' prior knowledge dealt with an awareness of the mathematical knowledge students were bringing into class. When accessing this knowledge the teachers mentioned both the students' in-class (based on the teachers' previously taught lesson) and out-of-class (mathematical lessons not taught by the teacher in class) mathematical experiences. Evidence of the teachers' awareness of the students' previous mathematical knowledge was seen in the conversation between ST Tomoko and CT Sasaki. In that conversation ST Tomoko was preparing to teach a lesson about simultaneous equations. During the conversation ST Tomoko was having issues with the introduction of the lesson. She shared with CT Sasaki her idea for how to start the lesson to which CT Sasaki stated that ST Tomoko should keep the introduction open ended and should let the students explore.

> CT Sasaki: I don't think you should explain everything to them in the introduction. You should insert this introduction for the purpose of checking students' pre-knowledge.

CT Sasaki went on to state that if the students were left to explore they would use a method of which they were familiar. He said some would use a table and others might make a list to solve the task. He also explained that some students may be familiar with simultaneous equations and use it right away.

By suggesting that ST Tomoko leave the introduction open to check for students' pre-knowledge highlighted the importance of teachers to account for the lesson the students brought into the lesson. It was clear to CT Sasaki that the students knew certain elements of the lesson and that ST Tomoko was not aware and did not prepare his instruction based on what students already knew. CT Sasaki went on to discuss the different methods students might use to solve the task and suggested that ST Tomoko be prepared for the varying responses. This was evidence to the varying nature of students' pre-knowledge. This was evident because CT Sasaki explicitly stated that ST Tomoko should leave the task more open in order to check the students' pre-knowledge. Therefore, it was clear that CT Sasaki did not assume that students' knowledge would be uniform and consistent with what the curriculum suggested. Rather it highlighted that he viewed students' pre-knowledge as broad and not homogeneous.

The CT Sasaki and ST Tomoko conversation also highlighted another aspect of the knowledge of students' prior knowledge that was also prevalent in other conversations. This was the awareness that students' mathematical knowledge was not homogeneous. On frequent occasion CTs would address the varying degree of knowledge about particular mathematical topics to which students would enter the lesson. There was an example of this in the discussion between CT Ueno and ST Motori. After ST Motori had explained the task he was planning to present to the students it was evident that CT Ueno was not satisfied that ST Motori had not taken into account the varying level of student understanding. CT Ueno then gave the following council to ST Motori.

CT Ueno: For instance, if you have many students who are having difficulty in understanding, you should prepare instruction for them. For students who understand well, you should also prepare different types of instruction. This part [pointing to a section of the lesson plan] is about instruction, so you should write both teaching material and what would be appropriate instruction by considering each student's circumstances.

Here when CT Ueno anticipated that there would be students that would have difficulty and others that would understand the topic easily he displayed an understanding of the varying levels to which students understand mathematical topics.

One unique element of this subcategory that often came up in the conversations was the mention of cram schools. Cram schools were often a factor for the differing level of student mathematical understanding mentioned above by CT Ueno. Cram schools were schools set up for students as an after-school program where students could get more practice with mathematics. All of the CTs addressed cram schools and how they could possibly influence their students' prior knowledge. The CTS often reminded their STs to be aware of the type of mathematics their students encountered at these cram schools. This was because the lessons at the cram schools introduced the students to topics that were not covered in the classes. In the previously cited conversation CT Sasaki asked ST Tomoko to consider the possibility of the students in her class that attended a cram school. CT Sasaki mentioned that the lesson would be different for them as compared to students that had not previously encountered the mathematics.

CT Sasaki: Or what if students have already learned simultaneous equation at a cram school and easily solve that problem?

ST Tomoko: I don't know exactly how cram schools teach their students since I myself have never gone to a cram school. My impression is that they teach students how to solve problems quickly and accurately like a machine. That is why I want to teach them something they have never learned at cram school.

CT Sasaki goes on to give suggestions to ST Tomoko how she can create a lesson that is fresh and new to the students that did attend a cram school by helping them see the convenience of using simultaneous equations when solving multivariable equations. CT Sasaki also mentioned that at a cram school they may have learned only one way to solve the problem. Here ST Tomoko acknowledged that cram schools were based on building students procedural based knowledge. By suggesting ST Tomoko help the students see the usefulness of simultaneous equations showed that CT Sasaki knew the type of knowledge that students had when they entered the classroom. Both CT Sasaki and ST Tomoko also knew that they cram school did not cover connection between the procedure, such as elimination, and other methods such as tables or charts. That way students could see the usefulness of simultaneous equations.

This excerpt from CT Sasaki and ST Tomoko highlighted the awareness that Japanese teachers have about students' prior knowledge. Here CT Sasaki mentioned that ST Tomoko's students who had attended a cram school may have already learned about the simultaneous equation method of solving systems of equations. Because this would affect the way those students viewed the mathematics of the lesson CT Sasaki offered a bit of advice in changing the lesson so that it met those students' background as well. Because cram schools came up so frequently in the other conversations it was clear that the Japanese teachers took it into account when considering the students' pre-knowledge.

Knowledge about students' prior mathematical knowledge was important to the Japanese teachers when they considered how their students would interact with the planned task. It is important to point out that this knowledge varied based on the topic that the ST was planning to teach. This meant that if the lesson was on transposition the pre-knowledge that they addressed dealt with transposition also much like in the excerpt above that dealt with the topic of simultaneous equations.

Knowledge of possible student reactions. The second subcategory, knowledge of possible student reactions, represented the awareness of teachers about how students would engage with the task. Knowledge of possible student reactions was a mixture of knowledge about student abilities and student preferences. This subcategory included the teachers' mention or use of strategies which anticipated the student thinking. It was limited to the students' possible mathematical responses and therefore did not include student's dispositions to possible tasks.

This was an intricate subcategory due to the level of metacognition used by the Japanese teachers. The CTs frequently urged the STs to imagine the thought process and strategies that students would use in order accomplish the task they were planning to present to the students. However, it did not end there—the teachers also discussed multiple lines of thinking that would need to take place as the lesson progressed. The following excerpt between CT Kimura and ST Akahiko captured well the level of intricacy of this subcategory. In this conversation ST Akahiko was preparing a lesson on transposition in solving linear equations and was explaining to CT Kimura his plan for the lesson.

ST Akahiko: I want students to make their own problems and let them come up with their own equation. This matches question is what I have found in a book. The problem says to use matches like this and connect to these lines on triangles.

CT Kimura: Oh, making triangles.

ST Akahiko: Connect these lines on triangles. And then ask "when there are six triangles, how many matches are being used?" After asking this question, some students may count like this. They might focus on this one match and think "the pattern is that two matches are increasing." Some may say there are six perfect triangles and because the five matches are being used, they might focus on these five matches. You see, there is a perfect triangle right here and some might think that two matches are increasing. Then, I won't ask "how many perfect triangles are there" but I will ask it this way; when there are 55 matches to make perfect triangles, how many perfect triangles are there? The number of triangles can be replaced for "x" and then I would let them solve this question. Some students might start thinking of an expression and a diagram. That is why I set a big number. In that case, it will be $2x + 1 = 55$, right? So the answer of this problem is 27. I want to explain this process when we learn about transposition.

As seen here ST Akihiko laid out the initial task and then expressed how he felt students would respond to the task. After that he anticipated his reaction to the students' initial strategy and the question he would then present to the students. After that he anticipated the students' response to the new question when he stated that some students might start thinking by using an expression or a diagram. In doing that he delved one level deeper by anticipating the students' response to his response which came up from the students' possible strategy on the initial task he posed.

In the above excerpt CT Kimura praised ST Akahiko for his in depth planning. ST Akahiko knew the students' abilities and preferences in a deep way and was able to construct that possible line of student engagement. Though this was a manifestation of how knowledge of possible student reactions came up in the data it was clear that this unique blend of knowledge of student abilities and preferences deserved its own place in this paper as a subcategory of knowledge.

Knowledge of what mathematics would interest students. This subcategory of knowledge about the practice of teaching represented the teachers' awareness of what mathematics and contexts of tasks would connect with students curiosity. This category was based on the teachers' knowledge of students' mathematical interests. Finding an interesting task was very important to all of the teachers and the strategies that the CTs shared with the STs demonstrated knowledge of students' mathematical interests.

There were two ways that this knowledge came up in the conversations. Both had to do with the design of the task. The first was knowledge that dealt with the type of task and the second was the knowledge of the lack of interesting mathematics that dealt with practicing procedural mathematics. The type of lesson the ST was planning determined the type of knowledge the teachers drew upon in the pre-lesson conversation. If the lesson was an introduction lesson the teachers drew upon the first type of knowledge. If the lesson was a follow-up lesson the second type of knowledge came up.

In order for the reader to better understand this knowledge subcategory it was important to briefly discuss introduction and follow-up lessons. Introduction lessons were the first lesson in a new unit. They were designed for students to engage in one large task. The task needed to be engaging to students and help them explore and discover introductory principles about a topic. This was why introductory lessons drew upon the type of task the teachers chose in order to spark their students' mathematical interest. Follow-up lessons were designed to allow students a more in-depth practice on the topic that was not covered in the introduction lesson. There were more problems for the students to work on but these problems were less interesting than the introduction task. This was why the teachers discussed the lack of interesting mathematics in follow-up lessons and suggested strategies to make the mathematics more interesting.

First, the knowledge that dealt with the design of the task covered knowledge of a set of strategies that could be used to interest students in the task. In the data there were two main strategies that CTs suggested to the STs to consider. One way was captured in the conversation between CT Ueno and ST Tomoko. In that conversation ST Tomoko was planning to teach a lesson on second degree equations but was having a hard time finding a good introduction task. CT Ueno suggested that ST Tomoko's lesson would connect with the students better if she design a lesson that connected with students' everyday lives.

> CT Ueno: They need to be able to image acceleration of the ball in their heads. If a free fall is without acceleration, a man falling down from a tall building would be smiling.

Later ST Tomoko made the following comment.

ST Tomoko: I wanted to plan something that helps students to be interested in this unit, but ideas did not come up to my mind immediately.

CT Ueno: I think it is good enough if you can come up with an idea that can relate second degree proportional functions to daily life or that can help students be interested in math.

Here one of the strategies that CT Ueno suggested ST Tomoko use was to related second degree equations to the students' everyday life.

Another way teachers were instructed to make introduction lessons interesting was to choose a task that had an interesting outcome such as an outcome that students would not expect. In the following excerpt CT Sasaki taught ST Akahiko how an interesting outcome could spark the students' curiosity. Here ST Akahiko felt that his lesson would not engage the students. CT Sasaki suggested that it might be the choice of the task and offered the following example.

CT Sasaki: So I used a problem in the introduction that deals with installing an electric cable one meter above the ground all around the earth and they need to find out the difference in length between that and the surface of the earth… If it would be the earth, what's the radius of the earth? 6400 kilometers?

ST Akahiko: Something like that.

CT Sasaki: O.K. That's a pretty big number, but the actual difference is only 6 meters, right?

ST Akahiko: Yes.

CT Sasaki: Students would be surprised by that answer. They would see the huge gap between tens of thousands kilometer and a few kilometers. Then they will be interested and probably would feel like the radius is not necessary to solve the problem or something, but it helps to catch their attention and interest.

Here CT Sasaki clearly stated that a problem with an interesting answer can help students explore the nature of the mathematics behind it. This technique was therefore included in the set of strategies used by teachers to help the students find interest in the introduction problem.

Second, in follow-up lessons the Japanese teachers accessed knowledge of teaching moves that could make a task more interesting for students. Follow-up lessons were a bit harder for the STs to make interesting. This was because follow-up lessons were often intended for students to get more practice with the topic. Therefore, teachers tried to plan lessons that gave students a wider sample of problems that dealt with topic of which the introduction lesson covered. In short introductions lessons focused on one main task whereas follow-up lessons consisted of a larger range of problems.

The CTs suggested many techniques that STs could use to make follow-up lessons interesting. In the conversation between CT Kimura and ST Akihiko CT Kimura shared three such techniques.

> CT Kimura: Lessons are not always interesting. Student teachers usually teach interesting parts [referring to introduction lessons] and what we regularly do as teachers are really normal lessons. But, if you become creative, it's possible to make a normal lesson interesting. For example, create small groups, make them come up with their own questions, and solve those questions as groups.

This suggestion that CT Kimura offered were three teacher moves or decisions in his pool of techniques that he could draw from in time of need. It was clear in the conversation that ST Akihiko did not have knowledge of these same techniques or their appropriateness of their use in his lesson. Also CT Kimura recognized that procedural mathematics can be tedious for students

Further on in that conversation CT Kimura gave ST Akahiko more advice on how to make follow-up lessons interesting. In this particular part of the conversation ST Kimura had instructed ST Akahiko that individualizing instruction may be a way to help the students connect with the mathematics. As a technique to accomplish this CT Kimura offered the following example.

> CT Kimura: For example, you are teaching solution of equation by using transposition and the lesson is not interesting. Remember that class is not always interesting. However, here you could make handouts one, two, and three and have them compete against each other. Then you could say "when you finish this [handout 1] then you can do the next [handout 2] and the next [handout 3]. But if your answers are wrong more than three times, you have to come to my office."
>
> ST Akahiko: Wow.

CT Kimura: In that way, you can see what mistakes they make and what transpositions are difficult. You can teach students individually. In reality, you need to be creative like that.

By using handouts CT Kimura sought to individualize instruction and monitor students' engagement. Based on his knowledge about the nature of the mathematics and the students mathematical preferences CT Kimura knew that the problems by themselves might not interest students. That was why he decided to be creative and make solving the problems a competition. Perhaps one of the interesting parts of this excerpt was when CT Kimura said to be creative. This could suggest that the pool of techniques that could be used to make a follow-up lesson more interesting could be limited only by a teacher's creativity.

CT Kimura further displayed his creativity by suggesting that ST Akahiko could make the students that completed the handouts "small teachers."

CT Kimura: One teaching technique is called "small teacher". All those who pass this evaluation test become a teacher in each group and you can have them grade other students' problems. I thought you could implement that technique as well.
ST Akahiko: Oh, I see.
CT Kimura: What is more important is how you help students familiarize themselves with these concepts and help them understand the principles.

Here CT Kimura demonstrated another technique at his disposal to help enliven the mathematics to a follow-up lesson. He knew that the mathematics itself would have a hard time capturing the students' interest and motivation. This was why CT Kimura stated that *how* you help students familiarize themselves with the concepts can be very important and urged ST Akahiko to be creative.

Knowledge about mathematics

This category of knowledge encompassed the teachers' knowledge about the mathematics they would present to the students. This category consisted of elements of MKT that only dealt with the teachers' knowledge dealing with mathematics and had little to do with students thinking or knowledge of mathematics. It is important to note that this category covered knowledge that was not specific to teaching mathematics. That is other mathematics professionals could also be required to have this type of knowledge. The subcategories that made up this section of knowledge were knowledge about how to correctly solve a problem and knowledge of mathematical principles.

Knowledge about how to correctly solve a problem. This subcategory encompassed the teachers' ability to successfully complete the process of solving a task. There were many times when this knowledge was both used and tested. It was used by both the CT and the ST as they demonstrated the steps taken when solving a task. It was tested when CTs were not sure if their STs thought through their task and for one reason or another wanted to make sure the STs had thought about the right problem to give to the students.

An example of a CT demonstrating this knowledge was found in CT Ueno's conversation with ST Motori. In this conversation ST Motori was planning to teach a lesson on variables. CT Ueno was worried that the students would not complete the aims of the goal with the task that ST Motori had planned to present to the students. CT Ueno then offered an option of a different task that dealt with Go, a Japanese chess game. Since ST Motori had not thought of this task CT Ueno proceeded to explain this task to ST Motori. The task consisted of the teacher placing Go tiles in a square formation and asking how many tiles were there. Since the tiles were put in a square pattern with a square pattern on the inside "missing" there were many ways the problem could be solved. CT Ueno arranged the Go tiles in the manner of the figure below for the first example which he posed to ST Motori. He then explained to ST Motori the different ways to solve the task.

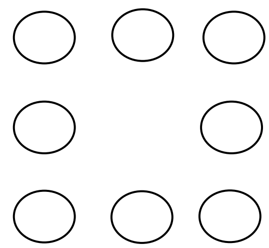

Figure 7. Representation of the figure that CT Ueno drew for ST Motori when he presented the Go task as an option for a task that ST Motori could use.

CT Ueno: You might also do this as another option…How many goishis are there?
ST Motori: There are eight checkers.
CT Ueno: Right. Some may think from 2 x 4. But this is coming from this way of thinking. Others may think 3 x 4 – 4.
ST Motori: 3 x 4 – 4?
CT Ueno: What is this three?
ST Motori: Well, I am still thinking about it. This three?
CT Ueno: Yeah. This is the only three I can think of from this diagram [pointing at the three Go tiles on one side]. Then, what is this four?
ST Motori: I am also still thinking about that… could it be adding one with it?
CT Ueno: No, that would give an incorrect answer. The four would be there because there are four sides.
CT Ueno: There are also many other ways for solution. For example, you can solve it like this: 1 x 4 + 4.
ST Motori: 1 x 4?
CT Ueno: The one is this right here [pointing at the middle dots on each side]. You can solve it like this [he points to the paper]. This is a solution which the students learned in algebra. I think they studied it in the fourth grade in elementary school. They solved a problem of Go that had five Go pieces on one side.

Here CT Ueno demonstrated many correct ways to think about and solve the task that he presented to ST Motori. It is interesting to note that the CT Ueno did not consider the problem solved until he explained how the number sentence he presented connected to the problem he presented. In this example CT Ueno made it clear that there were many different ways to solve this problem. He showed how to correctly solve each problem as well as explain the thinking behind each solution strategy. This excerpt also stressed the need of the teacher to know how to solve a problem using many different strategies. This was evident throughout the discussions where this category of knowledge came up. As in this occurrence, the CT stressed the importance of knowing various ways to correctly solve the task.

Another type of interaction where knowledge of how to correctly solve a problem came up was when the ST clarified how to solve a task they were planning to present to the students. One such occurrence was evident in the conversation between CT Sasaki and ST Tomoko. In this part of the conversation ST Tomoko explained a way she would solve the task that she was planning to pose in class.

> ST Tomoko: If I were them, I would place numbers anyway.
> CT Sasaki: Place numbers?
> ST Tomoko: Yes.
> CT Sasaki: How?
> ST Tomoko: For example, $x + y = 35$, then I can guess the numbers of pheasant and rabbit are almost half and half, then think the numbers of legs of each animals. Rabbit has 4 legs so I will eliminate the numbers of rabbit.
> CT Sasaki: That is right.

In this part of the conversation ST Tomoko demonstrated knowledge to work through the task in a correct way. Though, in this example, she solved it as a student would she displayed her knowledge of how to correctly solve the task. This was an example of an excerpt that could have gone in two places in this paper both in this section and knowledge of students' reactions.

The final type of interaction where knowledge of how to correctly solve a problem came up was when the CT quizzed the ST about how to solve a problem. This usually happened when the CT was unclear that the ST could correctly solve the task or that the ST had planned to give a task that the CT could see lacked forethought from the ST. An example of this was found in the conversation between CT Sasaki and ST Motori. In this part of the conversation CT Sasaki was concerned that ST Motori did not put enough thought into his task. In his task he was planning on asking the students to guess what card he was holding. As a hint he would give the students a number sentence to solve in order to say the correct number. All was fine until CT Motori shared a problem that dealt with a quadratic number sentence.

Kawasaki: To tell you the truth, it is not right.

CT Kimura: Where is the wrong answer? Why?
ST Tomoko: My answer is wrong?
CT Kimura: Yes. It's obvious.
ST Tomoko: This is right.
CT Kimura: That is right.
ST Tomoko: This is also right.
ST Tomoko: Two divided by zero, and zero divided by two [She is thinking].
CT Kimura: One of them is wrong. Which one?
ST Tomoko: This one?
CT Kimura: Your answer is the same as the students'.
ST Tomoko: Two dived by Zero?
CT Kimura: The answer should be?
ST Tomoko: Doesn't exist?
CT Kimura: Why? …What about six divided by two?
ST Tomoko: 3.
CT Kimura: Division is quotient times the divisor [writes $\frac{6}{2} = 3,\ 3 * 2 = 6$]. How about this [writes $\frac{2}{0} = 0,\ 2 * 0 = 0$ 2/0=0, 0*0=2)? If you do it like this you can see that a number can't be divided by 0. Can you divide 0?
ST Tomoko: Yes you can.

Here CT Kimura then continues to question ST Tomoko leading her to see the connection between multiplication and division so that ST Tomoko can understand why two divided by zero does not exist. CT Kimura asked ST Tomoko a series of questions to make sure that ST Tomoko understood the principle and was able to use that same MKT when teaching the students.

Knowledge of mathematical principles. This knowledge accounted for the teachers' deep understanding of the mathematics that was being discussed. Whenever the teachers talked about mathematical principles behind the lesson the ST was planning on teaching I coded it as knowledge of mathematical principles. For this reason, if a part of the discussion dealt with a struggle for how to teach the mathematical content or possible student struggles with the content I placed it in a different category. Knowledge of mathematical principles came up when the CT wanted to make sure that the ST understood the mathematical principles behind the lesson he or she was teaching. If the ST did not demonstrate a deep understanding of the mathematics the CT would then expound upon the mathematics of the lesson.

One such occurrence came up in the conversation between ST Motori and CT Sasaki. In this particular part of the CT Sasaki expressed further concern with ST Motori's lesson plan. In this case CT Sasaki was concerned that ST Motori lacked knowledge about the connections between completing the square and quadratic equation.

CT Sasaki: Are you ok with the complete the square?
ST Motori: What do you mean by ok?
CT Sasaki: You understand it, don't you?
ST Motori: Yes, I do.
CT Sasaki: Can you derive a quadratic formula using the complete the square method? Do you understand the process of how $ax^2 + bx + c = 0$ can be changed to $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
ST Motori: Hold on one second, please.

CT Sasaki went on to instruct ST Motori on the differences between the complete the square method and the quadratic equation how the differences determined what should be taught to students. CT Sasaki demonstrated knowledge of mathematical principles by first of all by recognizing and informing ST Motori of the process by which $ax^2 + bx + c = 0$ could become $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by completing the square. It was clear that ST Motori was not aware of any connection as he was trying to figure it out when CT Sasaki brought up the question. This excerpt dealt with knowledge of mathematical principles because this dealt with the teachers' knowledge of solving second degree equation and mathematical connections between two distinct methods.

Knowledge of mathematical principles was different depending on the topic that was being discussed by the teachers. However, this type of conversation about the mathematics was prevalent throughout all of the conversations. Many of them were the same as the excerpt above in that they dealt with mathematical connections between methods and solution strategies.

Knowledge about school mathematics

This category encompassed the pool of knowledge that dealt with two main themes. The first was the connection of math topics across lessons, units, and grade levels and the second was the tasks used to teach a topic. This category of knowledge was specific to teaching and was highly situated within the Japanese teaching system. That was why this category of knowledge was named knowledge about school mathematics. The subcategories that composed this category were knowledge of interconnected math topics, curriculum knowledge, knowledge about task difficulty, and knowledge about tasks.

Knowledge of interconnected mathematical topics. This dealt with the teachers' familiarity of the mathematical connections between topics. It also included the teachers' knowledge of what topics were necessary in order for students to make connections to more complicated topics. This knowledge dealt with the mathematical topics across lessons, units, and even grade levels. This subcategory addressed the general connections between topics.

In these occurrences there was no mention made of specific curricula or mathematical topics, but rather a demonstrated awareness that the success of future lessons depended on the lesson being taught. These occurrences of the knowledge of the interconnectedness of mathematical topics were general statements from the CT to the ST which encouraged them to be aware that lessons should build on one another.

One such example took place between CT Ueno and ST Motori. As stated before, in this discussion ST Motori was planning to teach a lesson introducing the students to variables. In this particular section of the conversation CT Ueno felt that ST Motori did not know how the lesson he was teaching would influence future lessons.

CT Ueno: When you become a real teacher, what would you teach next? Which path would you follow in your lesson?
ST Motori: In the next lesson?
CT Ueno: When you become a real teacher, you won't teach only this lesson, right? You have 365 days in one year from seventh grade to ninth grade. If you teach a math class well, you must have an image of what you would do next in your mind.

For the next page of conversation CT Ueno then proceeded to walk ST Motori through other topics that were connected to the use of the variable. They spoke about equality, equations, and systems of equations and how variables were used in each of those topics as well. Here CT Ueno mentioned that teachers need to have an image of what comes next. In this excerpt the knowledge that CT Ueno is referring to is knowledge of what topics are coming up. It is interesting to note that CT Ueno said "you must have an image of what you would do next in your mind." This suggested that lessons should build on one another and the teacher must know what is coming up next in order to properly teach a lesson that prepares for future lessons. He was also clearly disturbed that ST Motori's lesson plan was too short sighted and lacked demonstration of the knowledge of what lessons were in the students' future. Because CT Ueno didn't mention the grade level of the content I interpreted this as a general connection between mathematical topics.

Curriculum knowledge. This subcategory covered the teachers' awareness of the topics as determined by the nationwide curriculum used by the Japanese. In the conversations this knowledge usually came up when the Japanese teachers discussed the mathematics of a lesson in the context of both past and future topics. I only considered MKT relevant portions of conversations that contained a specific mention of either past or future mathematical topics as curriculum knowledge. The conversation had to be more than cursory mention of future mathematics such as a teacher mentioning the importance of "being prepared for future lessons." In order to be considered curriculum knowledge the excerpt had to include a mention of a specific mathematical topic and the grade to which it pertained.

The conversation between CT Sasaki and ST Akahiko had examples of curriculum knowledge in the past context. In this conversation ST Akahiko was planning a lesson on solving a system of equations. Throughout the conversation ST Akahiko was afraid that he was missing something in his lesson plan. CT Sasaki went over various aspects of a lesson that would make ST Akahiko feel better about his lesson. One of those aspects was curriculum knowledge that dealt with past lessons.

> CT Sasaki: What did they learn in seventh grade? They learn simultaneous equation in 8th grade. How about the seventh grade?
> ST Akahiko: Linear equations.
> CT Sasaki: So technically they can solve this with any of these [pointing at the problems of the task].
> ST Akahiko: Yes.

Both CT Sasaki and ST Akahiko knew that linear equations were covered in 7th grade. Because of this knowledge CT Sasaki suggested that the students could complete the task without knowing simultaneous equations. Because the teachers mentioned both the topic and the grade level this excerpt demonstrated the knowledge that Japanese teachers have about their curriculum.

Curriculum knowledge also came up when the teachers discussed lessons that would be presented in the future to students. This happened in the conversation between CT Kimura and ST Akahiko. In that conversation ST Akahiko was planning to teach a lesson on transposition and was going to use the students' idea of equality to help the students understand it. CT Kimura thought that that was a good idea but then reminded ST Akahiko that the lesson on transposition had connections to future lessons.

> CT Kimura: The seventh grade equation has a connection to the equation used in eighth grade but there is no connection between the eighth grade equation and the ninth grade equation.
> ST Akahiko: Yes.
> CT Kimura: So, students cannot solve simultaneous equations if they don't understand the seventh grade equation.
> ST Akahiko: That's right.
> CT Kimura: In other words, this study of equation will lead deeper understanding for the eighth grade and ninth grade equations. Therefore, this is an important unit and you will want to teach it well.

CT Kimura was clear that the idea of equation that ST Akahiko was planning to use in his lesson would come up in future lessons. Though it is unclear what the ninth grade equation was it was clear that CT Kimura knew that ST Akahiko's lesson was connected to simultaneous equations that the students would see in eighth grade.

Knowledge about task difficulty. This area of knowledge dealt with the teachers' awareness of how hard a task would be to complete. It came up in two types of occasions. The first was in the context of how long it would take the students to solve a task. The second happened when CTs suggested a certain progression of the lesson from easier problems to harder ones. I have also expounded upon how this subcategory of knowledge was different than knowledge of how students will respond to the task and knowledge of student reactions.

First, knowledge about task difficulty came up when teachers discussed the time it would take for students to finish a task or a part of a task. This discussion took place in order to help the ST plan out his or her lesson. An example of this form of the knowledge about task difficulty came up in the conversation between CT Kimura and ST Motori. In this conversation ST Motori was planning to teach a lesson on linear equations. He was not sure about what task he was going to use to introduce the topic. He had a set of four tasks and wanted to present the students with two of them. CT Kimura expressed the need for ST Motori to stick to one of the tasks. He agreed and the next day he came with a lesson plan written ready to discuss with CT Kimura. Below is that part of the conversation that took place when CT Kimura had discovered that ST Motori had expanded the task he said he would focus on

CT Kimura: I don't think you have enough time to cover this.
ST Motori: You don't think so? But if I just do this part, I don't think I can use up all 50 minutes.
CT Kimura: I think you can.
ST Motori: Really? I wondered if it would take 50 minutes for just this part. But at the same time, if I do both, 50 minutes is not enough.
CT Kimura: But, you think students can solve this problem in 15 minutes including graphing as well as the other methods?
ST Motori: They can't.
CT Kimura: If they are given the function, it's possible. But I think it's impossible to do this in 15 minutes if they are to examine this by graphing, making a table, finding the equation, and so forth.

Here the extent of the task that ST Motori had planned out for his students was too difficult to complete in the allotted time. CT Kimura knew how difficult it would be to graph, make a table, and find an equation of the task and expressed that to ST Motori. This part of the conversation helped ST Motori restructure his lesson in an attempt to help his students spend time doing worthwhile mathematics.

Second, knowledge of task difficulty came up when CTs suggested a proper flow or progression of mathematical topics within a lesson. When it came up the CTs used their knowledge of how topics within a lesson were connected and could build on each other. An example of this came up in the conversation between CT Kimura and ST Akahiko. In this part of the conversation ST Akahiko finished sharing his plan for the first task of his lesson to which CT Kimura offered the following advice about how to structure the lesson.

CT Kimura: In practice questions, there are questions that require them to think whether transposition of variables is possible. See this part?
ST Akahiko: I see.

CT Kimura: Students understand transposition of value from the characteristics of equality. Later on, the practice questions get gradually difficult. A variable is on the left side and this variable is on the right side. In this problem, here there is a variable and a value and this problem has also a variable and value. The next problem has parentheses and the following problem has a fraction. See that, it's getting more and more complicated.

CT Kimura displayed his knowledge about how mathematics problems within a lesson can build on each other. The goal was to help students question the nature of transposition in different circumstances. CT Kimura knew that some of the questions would be more complicated than others and that ST Akahiko needed to be aware of that.

Knowledge about tasks. This subcategory dealt with the teacher's awareness of a pool of relevant tasks that could be presented to the students. Throughout the data this knowledge came up in two ways. The first was when STs presented their task to CTs for their approval. The second was when CTs offered a different task to the STs than the one they planned on presenting in class.

First, at the beginning of every conversation the ST presented a task that they planned to present in class. These occasions demonstrated that the STs knew of a task that pertained to the topic they were teaching. In the conversation between CT Ueno and ST Motori CT Motori demonstrated knowledge about tasks by explaining his introduction task to CT Ueno.

CT Ueno: Please tell me your basic idea for your next lesson.
ST Motori: Well, I am going to teach an introduction lesson on variable. If I immediately start by talking about x, a, or other variables most students would not understand what I meant unless they go to a cram school. Therefore, Initially I plan to use specific values. In other words, I will talk about multiplication problems first. For example, I will say that I want eggs but I don't know how many I want. Then set a general expression to find a number of eggs. I will write like this $30 * (\quad)$. Then I will explain this expression in words. Then I will say "it's very troublesome if I explain every expression in words." And then, I will talk about variable.

As the conversation continued it was clear that CT Ueno was not impressed with the tasks that ST Motori had planned to present. Nevertheless, in this example ST Motori did display his knowledge about a task that dealt with variables, however limited this knowledge may have been. This type occurrence was typical in all the conversations in that the ST always started the discussion out by sharing a proposed task.

Second, CTs often suggested different tasks that STs could present in class. This usually happened if the CT was not satisfied with the task that the ST presented to the CT. An example of this was seen in the knowledge of how to correctly solve a task subcategory. In that example CT Ueno mentioned the Go task as an example of a task to replace the one above. When he did that he displayed task knowledge. He drew upon his knowledge of tasks and offered ST Motori with an example of a better task for the students.

The conversation where task knowledge came up was shared above between CT Sasaki and ST Akahiko. When ST Akahiko was struggling to find an interesting task CT Sasaki drew upon her knowledge of tasks. She shared the task about the wire one kilometer off the Earth's surface. She also shared the following task shortly before the wire one.

> CT Sasaki: We need to help them to identify their items and hopefully that can be expanded to real life situation.
> ST Akahiko: I see.
> CT Sasaki: So the reason why I inserted a problem with track field, is because the sports festival is coming up soon. Some students are drawing track lines, but I don't think they would think about this math problem as they do it.

It was clear here that CT Sasaki had a wide range of tasks at her disposal. One interesting aspect to note is that the CT usually presented a task that dealt with the topic the ST was preparing to teach, however, here CT Sasaki presented this task to ST Akahiko as a task that was interesting. CT Sasaki did this to give ST Akahiko a feel for the types of tasks she would use in order to catch the students' interests.

Discussion

It was mentioned in the literature review that Japanese teachers have a highly shared and refined view of what makes good instruction. It was a goal of this research to examine the mathematical knowledge that made this type of instruction possible. The map of Japanese MKT presented in the results represented the fulfillment of this goal. I do not claim that this map represents the only MKT that could arise from studying Japanese math teachers. On the contrary, the results were highly influenced by the nature of the conversations and the somewhat strict format of a common Japanese lesson plan structure. However, the map (Figure 6) presented above highlighted important and honest episodes of MKT used by Japanese teachers and is the first of its kind.

In the following portion of the discussion I have first addressed the limitations of this study. Second, I discussed the contributions of this work to current field of MKT. Third, I addressed the question of how this map fits in with the models presented in the literature review. Finally, I addressed other questions that came about due to this study.

Limitations

There were five main limitations to this study. First the sample size was rather small, only consisting of the pre-lesson conversations between three CTs and three STs. Although there were strong connections among MKT themes across the data set it would be difficult to apply this map of MKT to all teachers in Japan due to its limited sample size. However, it was documented by Jacobs and Morita (2002) that Japanese teachers shared a conception of what constitutes good mathematics instruction. Therefore, it is not unreasonable to suggest that other Japanese mathematics teachers would demonstrate similar elements of MKT.

Second, the map of Japanese MKT was influenced by the structure of Japanese lesson plans. Therefore, the possibility of teacher mentioning elements of MKT off topic from the topic the ST was preparing to teach was very low. For example, none of the STs presented a lesson that dealt with a formal assessment; therefore, the MKT that would be used in creating a formal assessment is missing from this study.

The third limitation of this study was that the teachers in this study were not directly questioned about the knowledge they referenced both implicit and explicitly. Though it was clear that direct questioning could have led to further illumination of MKT topics it was also clear that MKT relevant topics came up both frequent and naturally. Within the data of the eight conversations there were 389 codeable parts dealing with at least one component of MKT. This frequency of MKT topics without direct questioning suggested that Japanese teachers were open to share their knowledge of teaching both freely and openly. However, it would have been helpful if the CTs answered questions that dealt with the students' knowledge. This was because the mention of students' mathematical knowledge dealt with a party that was not in the conversation. Clarification of how they knew about students' mathematical interests, ability, and preferences would have been helpful in this study.

Forth, the conversations took place between experienced teachers and novice teachers. This meant that the MKT that arose in the conversations could have been simply entry level MKT. Furthermore, it could have been possible that expert teachers would have mentioned more in-depth or different elements of MKT if they had been discussing their lessons with other expert teachers. However, the opposite could be true. Conversations between expert teachers could contain fewer explicit examples of MKT relevant conversation because the experienced teachers might have a larger set of shared MKT so they would make less of it explicit. Only a study involving the conversations between expert Japanese teachers could illuminate the actuality of this limitation.

The final limitation of this study was that the data consisted of teachers preparing to teach a lesson. Because the act of teaching and preparing to teach a lesson are different it is possible that a researcher, even with the same framework, could find new and different conceptions of Japanese MKT. Therefore, it could have been possible that different elements of MKT could have been observed if the data set included the actual lessons and/or post-lesson conversations. Though it could be true that other elements of MKT arise when other parts of the lesson is studied this study was a valuable starting point. Furthermore, during the lessons the STs taught the lesson they prepared and post-lesson conversations were run by another student teacher. This meant that the opportunity for the CT to demonstrate his or her MKT was not as prevalent as it was in the pre-lesson conversations.

Despite these limitations there were three reasons the pre-lesson conversations between CTs and STs was sufficient for the purpose of this study. First, more conversations were not used because if they were included the amount of work to classify and code MKT in those situations would have far exceed the work required of a master's level thesis. Second, preliminary analysis showed the pre-lesson conversations were full of MKT relevant conversations. Third, the pre-lesson conversations were one-on-one conversations where the CTs could directly instruct STs on what to improve. This meant that these conversations were occasions where CTs could demonstrate and make explicit their MKT to the STs.

Contributions

Despite the many limitations this study adds to the field of MKT in a unique way. This study was able to illuminate the nature of MKT used by Japanese teachers in a novel way. In the following section I have outlined the contributions of this study to the field of MKT in both the methodology and results.

First, the methodology of this study led to meaningful contributions to the field of research on MKT. Until now no one has attempted to map the MKT of a set of teachers based on the conversations of teachers from a different country. Though there have been few studies on the MKT of teachers in a different country none have looked into Japan. This was the first study of its kind.

As stated in the literature review, until now most international MKT studies have all focused their efforts on documenting and overcoming the difficulties of adapting MKT test materials to different languages (Delaney et al., 2008; Kwon et al., 2012). No MKT tests were given to the teachers in this study. Rather, I focused on the conversations which were recorded in a setting to which both the CTs and STs were accustomed. Therefore, I was able to capture honest and natural examples of MKT. They were honest in the sense that they were not coerced to talk about a specific topic. This coupled with the fact that there were almost 400 units of analysis spoke to level to which these Japanese teachers valued MKT. This meant that when given the freedom to talk about anything they chose to talk about MKT relevant topics.

The approached I used in the methodology in this study differed from that of Delaney et al. (2008) and Kwon et al. (2012) in that it did not pre-suppose the existence of any MKT model. This allowed the model of Japanese MKT presented in the results to reflect the elements of MKT that Japanese teachers mentioned most frequently and in most depth.

This same methodology could be applied to teachers from different countries. A similar question could be raised of teachers of other countries. That is, what kind of MKT arises when teachers talk about the work of teaching and they do not know what the researcher is looking for? It is possible that this pursuit could yield slightly different conceptions in our view of MKT.

Second, the results of this study lead to meaningful contributions to the field of MKT research. The three categories of MKT delineated in the results section provide insight into the MKT of teachers that share a high quality view of mathematical instruction (Corey et al., 2010). In the following portion I will discuss the contributions of each MKT category to the field of MKT.

The results included knowledge of students' mathematical knowledge. This data set provided interesting insight into the level to which teachers can know their students and how they will respond to mathematical tasks. The sub-category of knowledge of possible student reactions gave an example of a line of thought or a path that both the student and the teacher would take from the beginning to the end of the task. This level of forethought and knowledge of their students' knowledge and preferences was rare in the existing literature on MKT. Though Ball et al. (2008) accounted for knowledge of content and students this knowledge was limited to common student misconceptions and student solution patterns. In this study MKT that dealt with knowledge of students' mathematical preferences as well as ability. This could have been due to the teachers' focus on engaging students with fresh tasks that was found as a main factor in Japanese instruction (Stigler and Hiebert, 1999a).

The fact that the teachers considered students' knowledge showed that they did not only tend to the mathematics or the topics they were planning on teaching. The Japanese MKT witnessed in this study suggested that the Japanese teachers valued positive student outlooks on mathematics. This separated the Japanese model from the Marks (1990), Davis and Simmt (2006), and Ball et al. (2008) models because none dealt with student dispositions towards mathematical tasks. Though it was not the focus of this study the attention on students' knowledge could suggest an underlying reason that Japanese teachers viewed teaching fundamentally different than the teachers of other countries.

The results also included knowledge about mathematics. The contributions that came from this category were twofold. First, the ability of teachers to correctly solve a task was very important. The excerpt that was shared in that section showed CT Ueno solving a task in multiple ways. It was important to the CTs that the STs knew how to correctly solve the task in multiple ways and using multiple strategies. This type of knowledge of how to correctly solve tasks and the stress put on correct multiple solution strategies was not covered in any of the models presented in the literature review. This aspect of correctly solving a mathematical task demonstrated the wide extent to which the Japanese teachers understood mathematics.

Second, this data set highlighted the depth to which teachers should understand mathematics. The excerpt between CT Sasaki and ST Motori showed the teachers discuss the mathematical reasons why certain strategies were connected. It is important to note that mathematical knowledge did not end with the teachers' ability to correctly solve a problem. It went deeper than that to the underlying reasons to why a strategy worked and, as stated before, how certain strategies were connected. Knowledge about the explaining mathematical connections between procedures and processes was not covered in the Ball et al. (2008) model. The closest category was SCK, however, knowledge of these types of mathematical connections was not covered in the list Ball et al. shared in Figure 3 of their study (p. 34). The illumination of real teachers mentioning this type of deep mathematical knowledge seems new.

Finally, the results included knowledge about the practice of teaching. There were many contributions to the field of MKT in this category. Perhaps one of the most interesting was how closely this category resembled elements mentioned in hypothetical learning trajectories. Nothing was included in the literature review on learning trajectories because it was not the main focus of this study, rather an interesting connection that came up after I finished the results section.

Simon (1995) stated that a hypothetical learning trajectory (HLT) was used when the teacher considered the learning goal and the thinking, learning, and learning activities that would be necessary to reach such a goal. The excerpts shared in all four subcategories of knowledge of the practice of teaching spoke to its connections to HLT. First, the knowledge of interconnected mathematical topics and curriculum knowledge displayed the teachers' ability to picture what learning would need to take place before a certain topic should be taught. The same was true about the subcategory of task difficulty. Task difficulty displayed the teachers knowledge of what tasks were harder than others as well as how to build the task towards accomplishing a learning goal. The excerpt in that subcategory provides an example of using mathematical tasks to build towards a lesson goal. This was connected to the aspect of HLT that dealt with building lessons towards accomplishing a learning goal and showed what it could look like in a teacher setting. The teachers also drew upon their knowledge of relevant tasks in order to help students learn the topic best. The connections between this category of knowledge and HLT were numerous which could suggest that this data set could be used to further the field of HLT.

As a whole the results section showed that teaching can be just as challenging as learning. The Japanese teachers showed great depth to the knowledge they accessed when preparing to teach a lesson. Perhaps this could also be a contribution to the reader. The reader could use the categories of knowledge mentioned in the results as a type of checklist to improve their own teaching and further their personal MKT.

A comparison of the map of Japanese teachers MKT to other models of MKT

Perhaps one of the biggest questions that arose from this study was how this map of MKT fit in with the other MKT models presented in the literature review. Model comparison within MKT was difficult because the field has not agreed upon a standard by which to compare two models. Therefore, I was only able to point out obvious differences.

As mentioned in the literature review, the work of past researchers has yielded many different models. Perhaps what is needed now in this field of increasing models is a way to classify or categorize them. In this section of the discussion I will refer to the place within MKT research where the Marks (1990), Davis and Simmt (2006), and the Ball et al. (2008) and other MKT models exist as the space of models.

Before presenting some possible ways to analyze this space of models it would be helpful to present a few reasons why this would be beneficial to the field. So far in this field there is no way to know that a model contains all the important aspects of MKT or if one model adds anything to another model. Therefore, analyzing the space of models could possibly lead to a more unified work in the field. Also, as mentioned earlier, there have been an abundance of models presented that outlined various forms of MKT. It is also possible that educators and mathematics education researchers could use a tool or another way to make sense of the space of models.

There could be a few ways to analyze this space of models. The following is a discussion of these ways as well as the apparent challenges of each. One way to analyze the space of models could be to look at all of them and draw connections between them. Some connections could include the showing that two categories in different models represented the same type of MKT. I included an example of what these connections could look like in Figure *8*. In this model I included themes of MKT from one model starting from the oldest and going to the newest. I summed up the elements of MKT in each model and I looked to the other models to see if that researcher accounted for the same type of MKT. I also included two other MKT models that I did not discuss in the literature review which were the Ferrini-Mundy et al. (2005) and Rowland et al. (2003) models to better demonstrate the size of the space of models.

List of Strong Connections

MKT Principle	Marks (1990)	Rowland et al. (2003)	Frinni-Mundy et al. (2005)	Davis and Simmt (2006)	Ball et al. (2008)	Japanese
Typical student proficiencies	x	x	x	x	x	x
Individual student proficiencies	x	x	x	x	x	x
Goal of the Lesson	x	x	x	x	o	x
Deep Content knowledge	x	x	x	x	x	x
Curriculum materials	x	x	x	x	x	x
Questions to students	x	x	x	x		
Assessment of Students Knowledge	x	x	x	x	x	x
Motivation of students	x	x	x	x	x	x
Lessons Sequencing (within lesson)	x	x	x	x	x	x
Pedagogical Strategies	x	x	x	x	x	x
Knowledge of Future mathematics	o	x	x	o	x	x
Basic knowledge of the Mathematics	o	x	x	x	x	x
Connections across topics	o	o	x	o	x	x
Useful presentations	o	x	x	x	x	x
Think on one's feet	o	x	x	x	o	x

Figure 8. List of the similarities between six different MKT models. On the left are categories of knowledge that were bought up by the different models. The “x” represents that the model contained an element of MKT that addressed that type of knowledge. The “o” meant that the specified model contained no strong element of that particular category.

In this chart I compared the Marks (1990), Rowland , Huckstep, and Thwaites (2003), Ferrini-Mundy et al. (2005), Davis and Simmt (2006) , Ball et al. (2008), and the Japanese model presented in this thesis. Looking across these models strong commonalities could be drawn along many themes. One example was that all the models accounted for knowledge of how students will respond and procedural aptitude of the teacher.

Drawing upon commonalities could be beneficial to the field. One reason is that it could be used to create a type of model of models. A model of models would be helpful to the field for a few reasons. One reason is that it would be easier for researchers to know how they are contributing to the field of MKT and know what work still needs to be done in the field. This could work if dimensions were placed on the space of models as I described below.

There are, however, challenges to this approach. For example, researchers used different lenses with which to view their data and create their models. This made it difficult to draw upon commonalities about the more nuanced parts of various models. An example of this could be seen in the Davis and Simmt (2006) model. Though the major categories that contained MKT were easy to pick out, however, there the categories of knowing and categories of knowledge were hard to compare to the other models.

Another way to analyze the space of models would be to assign dimensions to this space. Models would then vary from one extreme of the dimension to another. One of these dimensions could be how connected to the actual work of teaching the model is. Models like the Japanese model presented in this thesis and the Ferrini-Mundy et al. (2005) model would be placed on the connected side of the spectrum. This would be because these models both outlined the MKT used while teaching and suggested a way to apply that knowledge during the work of teaching. In that same sense, models that were less connected to the work of teaching, such as the Marks

(1990), Rowland (2003), Davis and Simmt (2006), and the Ball et al. (2008) might then be placed on the other side of the spectrum. This would be because no such connection to the time that the MKT was used within the lesson was made in those models. This would help teachers looking to for certain types of MKT models to help them in their teaching.

Another possible dimension that could be used to analyze the space of models could be how content specific an MKT model was. General models such as the Rowland (2003), Davis and Simmt (2006), and Ball et al. (2008) models might help teachers know what type of general areas they could improve upon. Whereas content specific models such as the Marks (1990) and Ferinni-Mundy (2005) models might help teachers prepare more fully for the topic they need to teach.

One challenge to assigning dimensions to the space of models is that they might seem arbitrary. These dimensions came up through my own attempt to make sense of this space of models; however work could be done to justify and solidify the use of certain dimensions. Researchers would have to substantiate certain dimensions and display their usefulness to teachers and researchers. Despite this difficulty practicing teachers and researchers could benefit from clarity and order that dimensions would provide.

It is important to note that the methods that I suggested above suggestions on meaningful work that could further the field of MKT. It could be that none of these suggestions I presented are feasible options for analyzing the space of models, however, it is clear that with the increasing amount of MKT models a way to make sense of them could be increasingly useful.

Areas for further research

Along with question of comparing models to each this study also raises other questions that could lead to further research. One such area was the connections between categories and subcategories. It was hard to envision that these categories were separate in the minds of the CTs and STs. For example, it is hard to imagine that the teachers accessed knowledge of possible students' reactions without accessing knowledge of how to solve a problem correctly. The methodology that I used in this study was designed to identify and classify categories of MKT and not to categorize connections between categories. This was made clear through lack of statistical correlation between the categories. Though I used various statistical models to find correlations between the MKT subcategories it is possible that I used the wrong one to categorize the connections between categories and subcategories. It was also possible that the nature of the discussions did not lend itself to finding connections because the teachers were not always consistent in their instructions to the STs because the CTs stressed different MKT principles depending on the ST with come they conversed. However, now that I have established categories of MKT for these teachers the stage may be set to classify the connection between the categories.

Another question that remains is if there is a hierarchy of MKT categories. It would be interesting to know if these teachers thought about one type of knowledge before any other. It would also be interesting to know if the Japanese teachers valued one type of knowledge over another. It is possible that findings in this area could lead to understanding why Japanese teachers have a different view of instruction than their out-of-country colleagues.

Conclusion

The three cooperating teachers in this study frequently addressed the type of knowledge that was expected of the student teachers in order to go about the work of teaching mathematics successfully. These episodes were coded, analyzed, and created into a map that was presented in this paper. Due to the context of the conversations the MKT was organized into three categories. The three categories were knowledge about students' mathematical knowledge, knowledge about mathematics, and knowledge about school mathematics. Each of these categories had subcategories that pertained to each. I identified the subcategories because they came up often and in multiple conversations across the set of CTs and STs. This thesis also discussed the limitations, contributions, and further areas or research that was brought up by this study.

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