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Independent double Roman domination in graphs

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ABSTRACT

For a graph $G = (V,E)$, a double Roman dominating function $(DRDF) f : V \to \{0, 1, 2, 3\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N(v)$, with $f(u) = 3$, or at least two neighbors $x, y \in N(v)$ having $f(x) = f(y) = 2$, and every vertex with value 1 under f has at least a neighbor with value 2 or 3. The weight of a *DRDF* is the sum $f(V) = \sum_{v \in V} f(v)$. A DRDF f is called independent if the set of vertices with positive weight under f, is an independent set. The independent double Roman domination number $i_{dR}(G)$ is the minimum weight of an independent double Roman dominating function on G. In this paper, we show that for every graph G of order n, $i_{r3}(G) - i_{dR}(G) \le n/5$ and $i(G) + i_R(G) - i_{dR}(G) \le n/4$, where $i_{r3}(G), i_R(G)$ and $i(G)$ are the independent 3-rainbow domination, independent Roman domination and independent domination numbers, respectively. Moreover, we prove that for any tree G, $i_{dR}(G) \ge i_{r3}(G)$.

KEYWORDS

Independent double Roman domination; independent 3-rainbow domination; independent Roman domination

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ and size $|E|$ of G is denoted by $n(G)$ and $m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A *strong support vertex* is a support vertex adjacent to at least two leaves. The *distance* between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *vertex independence number* or simply the *independence number*, of a graph G, denoted by $\alpha(G)$ is the maximum cardinality among the independent sets of vertices of G. We write P_n for the path of order n, C_n for the cycle of order n and K_n for the complete graph of order n. The *join* of two graphs G and H, $G \vee H$, is the graph with vertex-set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

An independent set $S \subseteq V$ in a graph G is called an *independent dominating set* if every vertex of G is either in S or adjacent to a vertex of S. The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set on G.

An independent subset $S \subseteq V$ is an *independent k-dominating set* if every vertex of $V - S$ has at least k neighbors in S. The *independent k-domination number* $i_k(G)$ is the minimum cardinality of an independent k-dominating set in G. This parameter has been studied in [14, 22].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function f is the value $\omega(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination was introduced by Cockayne et al. in [8] and was inspired by the work of ReVelle and Rosing [16] and Stewart [18], and has been extensively studied by researchers (see for example [5, 9, 13]). Here we recall some new variants of Roman domination.

A Roman dominating function $f : V(G) \to \{0, 1, 2\}$ is called an *independent Roman dominating function* (IRDF) if the set $\{u \in V(G) \,|\, f(u) \ge 1\}$ is an independent set. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an IRDF on G. The concept of independent Roman dominating function was first defined by Cockayne et al. in [8] and has been studied by several authors [7, 19].

A *Roman $\{2\}$-dominating function* is a function $f : V \to \{0, 1, 2\}$ with the property that for every vertex $v \in V$ with $f(v) = 0, f(N(v)) \ge 2$, that is, there is a vertex $u \in N(v)$ with $f(u) = 2$, or there are two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The minimum weight of a Roman $\{2\}$-dominating function is called the *Roman $\{2\}$-domination number* and denoted by $\gamma_{\{R2\}}(G)$. This concept was investigated in [6, 11]. A Roman $\{2\}$-dominating function $f = (V_0, V_1, V_2)$ is called an *independent Roman $\{2\}$-dominating*

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function (IR2DF) if $V_1 \cup V_2$ is an independent set. The *independent Roman {2}-domination number* $i_{\{R2\}}(G)$ is the minimum weight of an IR2DF on G. This concept was introduced by Rahmouni and Chellali [15].

A *double Roman dominating function* (DRDF) on a graph G is a function $f : V \rightarrow \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex v has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then vertex v has at least one neighbor u with $f(u) \geq 2$. The *double Roman domination number* $\gamma_{dR}(G)$ of a graph G is the minimum weight of a DRDF on G. For a DRDF f, let $V_i = \{v \in V | f(v) = i\}$ for $i = 0, 1, 2, 3$. Since these four sets determine f, we can equivalently write $f = (V_0, V_1, V_2, V_3)$ (or $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ to refer f). We note that $\omega(f) = |V_1| + 2|V_2| + 3|V_3|$. The concept of double Roman domination in graphs was introduced by Beeler, Haynes and Hedetniemi in [4] and has been studied in [1–3, 20, 21, 23]. In [4], it is proved that for any connected graph G of order $n \geq 3$, $\gamma_{dR}(G) \leq \frac{5n}{4}$ and all extremal graphs are characterized. Abdollahzadeh Ahangar et al. in [2] proved that every connected graph G having minimum degree at least 3 with order n, satisfies the inequality $\gamma_{dR}(G) \leq n$.

A DRDF $f = (V_0, V_1, V_2, V_3)$ is called an independent double Roman dominating function (IDRDF) if $V_1 \cup V_2 \cup V_3$ is an independent set. The *independent double Roman domination number* $i_{dR}(G)$ is the minimum weight of an IDRDF on G, and an IDRDF of G with weigth $i_{dR}(G)$ is called an i_{dR}-function. By definition we have

$$\gamma_{dR}(G) \leq i_{dR}(G). \tag{1}$$

Let S be a maximal independent set of graph G. It is clear that, a function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = V \setminus S$, $V_1 = V_2 = \emptyset$ and $V_3 = S$ is an IDRDF function. Hence $i_{dR}(G) \leq 3\alpha(G)$, where $\alpha(G)$ is the independence number of G.

The concept of independent double Roman domination introduced by H. R. Maimani and et al. in [12]. They show that the decision problem associated with $i_{dR}(G)$ is NP-complete for bipartite graphs.

In this paper, we first present some sharp bounds on the independent double Roman domination number and in last section Relationship between independent double Roman domination and independent 3-rainbow domination is investigated.

Proposition 1. [12] *Let G be a graph. There exists an i_{dR}-function $f = (V_0, V_1, V_2, V_3)$ such that $V_1 = \emptyset$.*

By Proposition 1, we assume no vertex needs to be assigned the value 1 for any $i_{dR}(G)$-function f. Thus according to this proposition, it is sufficient in the definition of an independent double Roman dominating function f that $V_2 \cup V_3$ is independent.

Lemma 2. [15] *For every connected graph G of order n, $i_R(G) - i_{\{R2\}}(G) \leq \frac{n}{4}$, and this bound is sharp.*

Proposition 3. [2] *For $n \geq 1$,*

$$\gamma_{dR}(P_n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3, \\ n+1 & \text{if} \quad n \equiv 1, 2 \pmod 3). \end{cases}$$

Corollary 4. [12] *For $n \geq 1$, $i_{dR}(P_n) = \gamma_{dR}(P_n)$.*

Proposition 5. [2] *For $n \geq 3$,*

$$\gamma_{dR}(C_n) = \begin{cases} n & \text{if} \quad n \equiv 0, 2, 3, 4 \pmod 6 \\ n+1 & \text{if} \quad n \equiv 1, 5 \pmod 6). \end{cases}$$

Corollary 6. [12] *For $n \geq 3$, $i_{dR}(C_n) = \gamma_{dR}(C_n)$.*

Proposition 7. [2] *Let G be a connected graph of order $n \geq 3$. Then*

1. *$\gamma_{dR}(G) = 3$ if and only if $\Delta(G) = n - 1$.*
2. *$\gamma_{dR}(G) = 4$ if and only if $G = \overline{K_2} \vee H$, where H is a graph with $\Delta(H) \leq |V(H)| - 2$.*
3. *$\gamma_{dR}(G) = 5$ if and only if $\Delta(G) = n - 2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n - 2$.*

Proposition 8. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{dR}(G) = k$ if and only if $i_{dR}(G) = k$ for $k \in \{3, 4, 5\}$.*

Proof.
1. Suppose that $\gamma_{dR}(G) = 3$. It is clear that $i_{dR}(G) \geq 3$ and by Proposition 7 (1), $\Delta(G) = n - 1$. Suppose that $d(v) = n - 1$. Assigning 3 to v and 0 to other vertices of G, introduces an IDRDF of G implying that $i_{dR}(G) \leq 3$. Conversely, Suppose that $i_{dR}(G) = 3$. It is easy to see that $\gamma_{dR}(G) = 3$.
2. Suppose that $\gamma_{dR}(G) = 4$. By Proposition 7 (2), $G = \overline{K_2} \vee H$, where H is a graph with $\Delta(H) \leq |V(H)| - 2$. This means there exist vertices x, y that $xy \notin E(G)$ and $d(x) = d(y) = n - 2$. Assigning 2 to x, y and 0 to other vertices of G introduces an IDRDF of G yielding $i_{dR}(G) \leq 4$ and it follows from (1) that $i_{dR}(G) = 4$. Conversely, Suppose that $i_{dR}(G) = 4$. It is easy to see that $3 \leq \gamma_{dR}(G) \leq 4$, an by above item we have $\gamma_{dR}(G) = 4$.
3. Suppose that $\gamma_{dR}(G) = 5$. It is clear that $i_{dR}(G) \geq 5$ and by Proposition 7 (3), $\Delta(G) = n - 2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n - 2$. This means there exist vertices x, y that $xy \notin E(G)$ and $d(x) = n - 2$, $d(y) < n - 2$. Assigning 3 to x, 2 to y, and 0 to other vertices of G introduces an IDRDF of G and so $i_{dR}(G) \leq 5$. Conversely, Suppose that $i_{dR}(G) = 5$. It is easy to see that $3 \leq \gamma_{dR}(G) \leq 5$, if $\gamma_{dR}(G) = 3$ or $\gamma_{dR}(G) = 4$ then by 1,2, $i_{dR}(G) = 3$ and $i_{dR}(G) = 4$ a contradiction. So $\gamma_{dR}(G) = 5$. □

Proposition 9. *If G is a connected graph with order n and size m, then for some spanning tree T_G of G, $i_{dR}(T_G) \leq i_{dR}(G) + m - n + 1$.*

Proof. If G is a tree, the result is trivial. Assume that G is not a tree. Let C be a cycle in G. Suppose that f is an i_{dR}-function on G such that $V_1 = \emptyset$. Obviously at least one vertex on C say v has the value 0. We consider two cases.
Case 1. One of two vertices that are adjacent with v on C say w, has the value 0. In this case we let $G' = G - vw$ and define $g = f$ on G'.
Case 2. Each of two vertices that are adjacent with v on C, has the value 2 or 3. Suppose that w, z are two adjacent

vertices with v on C. Let $G' = G - vz$ and define a function g on G' as $g(w) = 3$, $g = f$ for other vertices of G'.

Thus we obtain a graph G', with at least one cycle less than G and independent double dominating function g of weight at most $w(f) + 1$ on G'. By repeating this procedure we obtain a spanning tree T_G of G and an independent dominating function on T_G of weight at most $w(f) + m - (n-1)$. Thus $i_{dR}(T_G) \leq i_{dR}(G) + m - n + 1$. □

2. Some upper and lower bounds of $i_{dR}(G)$

Proposition 10. *Let G be a graph and $f = (V_0, V_1, V_2)$ an i_R-function of G. Then $i_{dR}(G) \leq 2|V_1| + 3|V_2|$.*

Proof. Let G be a graph and $f = (V_0, V_1, V_2)$ be an i_R-function of G. We define a function $g = (V'_0, V'_1, V'_2, V'_3)$ as follows: $V'_0 = V_0$, $V'_1 = \emptyset$, $V'_2 = V_1$, and $V'_3 = V_2$. Note that under g, every vertex assigned a 0 has a neighbor assigned 3, and no vertex is assigned 1. Hence, g is an independent double Roman dominating function. Thus, $i_{dR} \leq 2|V'_2| + 3|V'_3| = 2|V_1| + 3|V_2|$, as desired. □

Corollary 11. *If G is a nontrivial connected graph, and $f = (V_0, V_1, V_2)$ is an i_R-function of G that maximizes the number of vertices in V_2, then $i_{dR}(G) \leq 2i_R(G) - |V_2|$.*

Proposition 12. *For every graph G, $i_R(G) < i_{dR}(G)$.*

Proof. Let $f = (V_0, V_2, V_3)$ be any i_{dR}-function of G, where $V_1 = \emptyset$ (by Proposition 1 such a function exists). If $V_3 \neq \emptyset$, then every vertex in V_3 can be reassigned the value 2 and the resulting function will be an independent Roman dominating function, that is, $i_R(G) < i_{dR}(G)$. Assume that $V_3 = \emptyset$. Since $V_2 \cup V_3$ dominates G, it follows that $V_2 \neq \emptyset$. Thus, all vertices are assigned either the value 0 or the value 2, and all vertices in V_0 must have at least two neighbors in V_2. In this case one vertex in V_2 can be reassigned the value 1 and the resulting function will be an independent Roman dominating function, that is, $i_R(G) < i_{dR}(G)$. □

Corollary 13. *If $f = (V_0, V_2, V_3)$ is any i_{dR}-function of a graph G, then $i_R(G) \leq 2(|V_2| + |V_3|) = i_{dR}(G) - |V_3|$.*

Corollary 14. *For any nontrivial connected graph G, $i_R(G) < i_{dR}(G) < 2i_R(G)$.*

Proposition 15. *For any graph G, $2i(G) \leq i_{dR}(G) \leq 3i(G)$.*

Proof. Let S be an $i(G)$-set. Note that $(\emptyset, \emptyset, S)$ is an independent double Roman dominating function. This yields the upper bound of $i_{dR}(G) \leq 3i(G)$. For the lower bound, let $f = (V_0, V_2, V_3)$ be an i_{dR}-function of a graph G. Note that, $V_2 \cup V_3$ is an independent dominating set for G. Thus, $i(G) \leq |V_2| + |V_3|$. Using this observation, we can obtain the lower bound, $i_{dR}(G) = 2|V_2| + 3|V_3| \geq 2(|V_2| + |V_3|) \geq 2i(G)$. □

Both the bounds of Proposition 15 are sharp. For the upper bound, as we have seen, the family of non-trivial stars $K_{1,n-1}$ has $i(K_{1,n-1}) = 1$ and $i_{dR}(K_{1,n-1}) = 3$. For the lower bound, we also recall that the family of complete bipartite graphs $K_{2,k}, k \geq 2$, has $i(K_{2,k}) = 2$ and $i_{dR}(K_{2,k}) = 4$. The independent 2-domination number $i_2(G)$ equals the minimum cardinality of an independent set S such that every vertex in $V - S$ is adjacent to at least two vertices of S.

Proposition 16. *For any graph G, $2i(G) = i_{dR}(G)$ if and only if $i(G) = i_2(G)$.*

Proof. Let $f = (V_0, V_2, V_3)$ be an i_{dR}-function on G. Note that $2i(G) \leq 2(|V_2| + |V_3|) \leq 2|V_2| + 3|V_3| = i_{dR}(G)$. If $2i(G) = i_{dR}(G)$, then both inequalities must be equalities. The first inequality is an equality if and only if $V_2 \cup V_3$ is a minimum independent dominating set. The second inequality is an equality if and only if $V_3 = \emptyset$. This happens if and only if V_2 is a minimum independent dominating set and $(V - V_2, V_2, \emptyset)$ is an i_{dR}-function of G. Hence by definition, every vertex in $V - V_2$ must have at least two neighbors in V_2. Therefore, $i(G) = i_2(G)$. □

Proposition 17. *If G is a connected graph of order n and maximum degree $\Delta(G) = \Delta$, then for any i_{dR}-function $f = (V_0, V_2, V_3)$, $i_{dR}(G) \geq \frac{3(n+|V_3|)}{\Delta+2}$, and this bound is sharp.*

Proof. Assume first that $\Delta \leq 2$. In this case G is a path or a cycle and by Corollaries 4 and 6, it is not difficult to see that, $i_{dR}(G) \geq \frac{3(n+|V_3|)}{\Delta+2}$. Assume now that $\Delta \geq 3$, and let $f = (V_0, V_2, V_3)$ be an i_{dR}-function. Let $V'_0 = \{x \in V_0 : N(x) \cap V_3 \neq \emptyset\}$ and $V''_0 = V_0 - V'_0$. Since every vertex of V_3 can have at most Δ neighbors in V'_0, we obtain that $|V'_0| \leq \Delta|V_3|$. On the other hand, using the facts that every vertex of V''_0 has at least two neighbors in V_2 and every vertex of V_2 has at most Δ neighbors in V''_0, we deduce that $2|V''_0| \leq \Delta|V_2|$. Therefore, we obtain

$$\begin{aligned}
n &= |V'_0| + |V''_0| + |V_2| + |V_3| \\
&\leq \Delta|V_3| + 1/2\Delta|V_2| + |V_2| + |V_3| \\
&= (\Delta+1)|V_3| + 1/2(\Delta+2)|V_2| \\
&= (\Delta+2)|V_3| + 1/2(\Delta+2)|V_2| - |V_3| \\
&= (\Delta+2)(|V_3| + 1/2|V_2|) - |V_3| \\
&\leq (\Delta+2)(|V_3| + 2/3|V_2|) - |V_3| \\
&= (\Delta+2)(i_{dR}(G)/3) - |V_3|,
\end{aligned}$$

and this leads to the desired bound.

That this bound is sharp may be seen by the graph of order 8 obtained from path $P_4 : v_1 - v_2 - v_3 - v_4$ by adding new vertices x, y, z, t and adding edges xv_1, yv_1, zv_4, tv_4. □

Proposition 18. [12] *For any graph G of order $n \geq 1$,*

$$i_R(G) + 1 \leq i_{dR}(G) \leq 2i_R(G).$$

The equality in upper bound holds if and only if $G = \bar{K}_n$.

Next result present bounds on $i_{dR}(G)$ in terms of domination, independent domination and independent Roman {2}-domination numbers.

Proposition 19. *For every connected graph G of order n,*

$$i_{\{R2\}}(G) + i(G) \leq i_{dR}(G) \leq n + i_{\{R2\}}(G) - \gamma(G).$$

Proof. First we prove the lower bound. Clearly the result is valid for $n = 1, 2$. Let $n \geq 3$ and $f = (V_0, \emptyset, V_2, V_3)$ be an

i_{dR}-function of G. Note that $V_2 \cup V_3$ dominates V_0 and so $i(G) \le |V_2| + |V_3|$. Now define the function g on G as follows: $g(x) = f(x) - 1$ for all $x \in V_2 \cup V_3$ and $g(x) = f(x)$ for all $x \in V_0$. Clearly, g is an independent Roman {2}-dominating function on G, and so $i_{\{R2\}}(G) \le |V_2| + 2|V_3|$. Therefore, $i_{dR}(G) = 2|V_2| + 3|V_3| = |V_2| + |V_3| + |V_2| + 2|V_3| \ge i(G) + i_{\{R2\}}(G)$.

Now we prove the upper bound. Let $f = (V_0^f, V_1^f, V_2^f)$ be an $i_{\{R2\}}$-function of G such that V_2 is as large as possible. It is easy to see that V_0^f is a dominating set of G. Define $h : V(G) \to \{0, 1, 2, 3\}$ by $h(u) = f(u)$ if $u \in V_0^f$, and $h(u) = f(u) + 1$ for $u \in V_1^f \cup V_2^f$. Clearly, h is an IDRDF on G and this implies that

$$\begin{aligned} i_{dR}(G) &\le 2|V_1^f| + 3|V_2^f| \\ &= i_{\{R2\}}(G) + |V_1^f| + |V_2^f| \\ &= i_{\{R2\}}(G) + n - |V_0^f| \\ &\le i_{\{R2\}}(G) + n - \gamma(G), \end{aligned}$$

as desired. □

Next result is an immediate consequence of Proposition 19 and Lemma 2.

Corollary 20. *For every connected graph G of order n, $i_{dR}(G) \ge i(G) + i_R(G) - \frac{n}{4}$.*

3. Relationship between independent double Roman domination and independent 3-rainbow domination

Let f be a function that assigns to each vertex a subset of colors chosen from the set $\{1, \dots, k\}$; that is $f : V(G) \to P(\{1, \dots, k\})$. If for each vertex $v \in V(G)$ with $f(v) = \emptyset$, we have $\cup_{u \in N(v)} f(u) = \{1, \dots, k\}$, then f is called a *k-rainbow dominating function* (kRDF) of G. The weight of a kRDF f is defined as $f(V) = \sum_{v \in V(G)} |f(v)|$. For a sake of simplicity, a 3RDF f on a graph G will be represented by the ordered partition $(V_\emptyset^f, V_{\{1\}}^f, V_{\{2\}}^f, V_{\{3\}}^f, V_{\{1,2\}}^f, V_{\{1,3\}}^f, V_{\{2,3\}}^f, V_{\{1,2,3\}}^f)$ of $V(G)$ induced by f, where $V_A^f = \{u \in V | f(u) = A\}$. A function $f : V(G) \to P(\{1, \dots, k\})$ is called an independent k-rainbow dominating function (IkRDF) of G, if f is a kRDF and no two vertices in $V(G) - V_\emptyset^f$ are adjacent. The *independent k-rainbow domination number* $i_{rk}(G)$ is the minimum weight of an *IkRDF* of G. The independent k-rainbow domination number was introduced by Shao et al. [17].

In this section, we present bounds on $i_{dR}(G)$ in terms of $i_{r3}(G)$. For any $i_{dR}(G)$-function $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ of a graph G, let $V_{0,1}^f$ be the set of all vertices of V_0^f having at least one neighbor in V_3^f and $V_{0,2}^f = V_0^f - V_{0,1}^f$. Clearly, each vertex of $V_{0,2}^f$ has at least two neighbors in V_2^f. Let R be a minimal spanning subgraph of the induced subgraph $G[V_{0,2}^f \cup V_2^f]$ such that each vertex in $V_{0,2}^f$ is adjacent to two vertices in V_2^f. By the choice of R, we have $\deg_R(v) = 2$ for each $v \in V_{0,2}^f$. Suppose H_f is the graph with vertex set $V(H_f) = V(V_2^f)$ and edge set $E(H_f) = \{xy | x$ and y have a common neighbor in V_0^f in $R\}$. Clearly, if $V_{0,2}^f \ne \emptyset$, then $|E(H_f)| \ge 1$. The graph H_f will be called an associated graph with respect to f.

For every graph G, let $\pi(G) = \{S_1, S_2, \dots, S_k\}$ be a partition of $V(G)$ such that S_i is a maximum independent set of the subgraph induced by $S_i \cup S_{i+1} \cup \dots \cup S_k$ for every $i \in \{1, \dots, k\}$. This concept was introduced by Fradkin [10] who defined a greedy independent decomposition of G. According to the definition of $\pi(G)$, we have $|S_i| \ge |S_j|$ and every vertex of S_j has at least one neighbor in S_i for all i, j with $i < j$. Now let us define the sets A, B and C as follows: A is the set of all vertices of $V_{0,2}^f$ having neighbors in both S_1 and S_2, C is the set of all vertices of $V_{0,2}^f$ having no neighbor in $S_1 \cup S_2$, and $B = V_{0,2}^f - (A \cup C)$. Let $X_f = \cup_{i=3}^t S_i = V(H_f) - (S_1 \cup S_2)$.

Obviously we have the following proposition.

Proposition 21. *Every graph G has an independent 3-rainbow dominating function.*

Proposition 22. *Let $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ be an $i_{dR}(G)$-function and H_f an associated graph with respect to f whose vertex partition is $\pi(H_f) = \{S_1, S_2, \dots, S_t\}$. Then*

(i) *if H_f is a bipartite graph, then $i_{r3}(G) \le i_{dR}(G)$.*
(ii) *$i_{r3}(G) \le i_{dR}(G) + |X_f|$.*
(iii) *Every vertex of X_f (if any) has at least two neighbors in B and $|B| \ge 2|X_f|$.*
(iv) *$|A| \ge |S_2|$.*
(v) *$|C| = 0$ if $t \le 3$.*
(vi) *$|C| \ge |X_f| - |S_2|$ if $t = 4$.*
(vii) *$|C| \ge 2|X_f| - 3|S_2|$ if $t \ge 5$.*

Proof.

(i) Assume that H_f is a bipartite graph (possibly $E(H_f) = \emptyset$ when $V_{0,2}^f = \emptyset$) and let U and W be the partite sets of H_f. By definition of H_f, every vertex of $V_{0,2}^f$ (if any) has a neighbor in U and another one in W. Define the function $g : V(G) \to \mathcal{P}(\{0, 1, 2, 3\})$ by $g(x) = \emptyset$ if $x \in V_0^f, g(x) = \{1, 2, 3\}$ if $x \in V_3^f, g(x) = \{1, 2\}$ if $x \in U$ and $g(x) = \{2, 3\}$ if $x \in W$. Clearly, g is an I3RDF on G implying that $i_{r3}(G) \le \omega(g) = i_{dR}(G)$ as desired.

(ii) If H_f is a bipartite graph, then the result follows from (i). Assume that H is not a bipartite graph. Then $t \ge 3$ and the function $g : V(G) \to \mathcal{P}(\{0, 1, 2, 3\})$ by $g(x) = \emptyset$ if $x \in V_0^f$, $g(x) = \{1, 2, 3\}$ for $x \in V_3^f \cup X_f$, $f(x) = \{1, 2\}$ for $x \in S_1$ and $f(x) = \{2, 3\}$ for $x \in S_2$, is an I3RDF on G of weight $i_{dR}(G) + |X_f|$ yielding $i_{r3}(G) \le \omega(g) = i_{dR}(G) + |X_f|$ as desired.

(iii) Let u be an arbitrary vertex of X_f. By the definition of $\pi(H_f)$, u has a neighbor in S_1, say v, and a neighbor in S_2, say w. Let a be the common neighbor of u and v in R, and a' be the common neighbor of u and w in R. Since a and a' do not belong to A (because of $u \in X_f$) and $v, w \in S_1 \cup S_2$, we conclude that $a, a' \in B$. Since each vertex of B has exactly one neighbor in X_f and exactly one neighbor in $S_1 \cup S_2$, we deduce that $|B| \ge 2|X_f|$.

(iv) Let $S_2 = \{x_1, ..., x_k\}$. By definition of $\pi(H_f)$, there is an edge $x_iy_i \in E(H_f)$ where $y_i \in S_1$ for each i. By the construction of H_f, x_i, y_i must have a common neighbor $z_i \in A$ in R for each i. Since $\deg_R(z_i) = 2$ for each i, we conclude that $z_i \neq z_j$ if $i \neq j$. It follows that $|A| \geq |S_2|$.

(v) If $t \leq 3$, then X_f is either empty or independent, and thus $C = \emptyset$.

(vi) Let $t = 4$. Let $S_4 = \{x_1, ..., x_k\}$. By definition of $\pi(H_f)$, there is an edge $x_iy_i \in E(H_f)$ where $y_i \in S_3$ for each i. By the construction of H_f, x_i, y_i must have a common neighbor $z_i \in C$ in R for each i. Since $\deg_R(z_i) = 2$ for each i, we conclude that $z_i \neq z_j$ if $i \neq j$. It follows that $|C| \geq |S_4| = |X_f| - |S_3| \geq |X_f| - |S_2|$.

(vii) Let $t \geq 5$. Let $C_1 = \{x \in C | N_R(x) \cap S_3 \neq \emptyset\}$ and $C_2 = \{x \in C | N_R(x) \cap S_3 = \emptyset\}$. As in the (v), we can see that $|C_1| \geq |X_f| - |S_3|$ and $|C_2| \geq |X_f| - |S_3| - |S_4|$. Since $|C| = |C_1| + |C_2|$, we obtain

$$\begin{aligned}|C| &\geq (|X_f| - |S_3|) + (|X_f| - |S_3| - |S_4|)\\ &= 2|X_f| - 2|S_3| - |S_4|\\ &\geq 2|X_f| - 3|S_2|,\end{aligned}$$

and the proof is complete. □

Proposition 23. *If G is a C_{4k+2}-free connected graph for an integer $k \geq 1$, then*

$$i_{dR}(G) \geq i_{r3}(G).$$

Proof. Let $f = (V_0^f, \emptyset, V_2^f, V_3^f)$ be an $i_{dR}(G)$-function. If $V_{0,2}^f = \emptyset$, then clearly the function $g : V(G) \to \mathcal{P}(\{0,1,2,3\})$ defined by $g(x) = \{1,2,3\}$ for $x \in V_3$, $g(x) = \{1,2\}$ for $x \in V_2$ and $g(x) = \emptyset$ otherwise, is an I3RDF of G with weight $i_{dR}(G)$, and so $i_{dR}(G) \geq i_{r3}(G)$. Henceforth, we assume that $V_{0,2}^f \neq \emptyset$. Let H_f be the associated graph with respect to f. We claim that H_f is a bipartite graph. Suppose, to the contrary, that H_f is not bipartite. Then H_f contains an induced odd cycle $C_{2t+1} := (u_1u_2...u_{2t+1})$ for some positive integer t. By the construction of H_f, for every two consecutive vertices u_i, u_{i+1} of the cycle, there exists a vertex $w_i \in V_{0,2}^f$ such that w_iu_i and $w_iu_{i+1} \in E(G)$. Hence the subgraph of G induced by vertices $\{u_1, u_2, ..., u_{2t+1}, w_1, w_2, ..., w_{2t+1}\}$ contains a cycle C_{4k+2} which is a contradiction. Thus H_f is a bipartite graph. Now the result follows from proposition 22 (part (i)). □

Corollary 24. *If G is a tree, then $i_{dR}(G) \geq i_{r3}(G)$.*

Next we present an upper bound on $i_{r3}(G)$ in terms of $i_{dR}(G)$ and the order of G.

Proposition 25. *For every connected graph of order n, $i_{dR}(G) \geq i_{r3}(G) - \frac{n}{5}$.*

Proof. Let $f = (V_0^f, \emptyset, V_2^f, V_3^f)$ be an $i_{dR}(G)$-function, H_f an associated graph with respect to f and $\pi(H_f) = \{S_1, S_2, ..., S_k\}$ be the independent partition of $V(H_f)$. If H_f is bipartite, the result is valid by Proposition 22 (part (i)). Assume that H_f is not bipartite. Then $t \geq 3$. By Proposition 22 (part (ii)), $i_{r3}(G) \leq i_{dR}(G) + |X_f|$. Hence, we need only to show that $|X_f| \leq \frac{n}{5}$. Recall that $|V_2^f| = |X_f| + |S_1| + |S_2|$. Consider the following cases.

Case 1. $t = 3$.

Then $X_f = S_3$ and $|X_f| = |S_3| \leq |S_2|$ by the definition of $\pi(H_f)$. It follows from Proposition 22 (parts (ii)-(iv)) that

$$\begin{aligned}n &= |V_3^f| + |V_2^f| + |V_0^f|\\ &\geq |V_2^f| + |V_{0,2}^f|\\ &= (|X_f| + |S_1| + |S_2|) + (|A| + |B| + |C|)\\ &\geq 3|X_f| + (|A| + |B|)\\ &\geq 3|X_f| + |S_2| + 2|X_f| \geq 6|X_f|,\end{aligned}$$

and so $|X_f| \leq \frac{n}{6} < \frac{n}{5}$.

Case 2. $t = 4$.

Clearly, $|X_f| = |S_3| + |S_4| \leq |S_1| + |S_2|$. Applying Proposition 22 (parts (ii), (iii) and (v)), we have

$$\begin{aligned}n &= |V_3^f| + |V_2^f| + |V_0^f|\\ &\geq |V_2^f| + |V_{0,2}^f| = (|X_f| + |S_1| + |S_2|) + (|A| + |B| + |C|)\\ &\geq 2|X_f| + |S_2| + 2|X_f| + |X_f| - |S_2| = 5|X_f|\end{aligned}$$

and so $|X_f| \leq \frac{n}{5}$.

Case 3. $t \geq 5$.

Using parts (ii), (iii) and (vi) of Proposition 22, we obtain

$$\begin{aligned}n &= |V_3^f| + |V_2^f| + |V_0^f|\\ &\geq |V_2^f| + |V_{0,2}^f| = (|X_f| + |S_1| + |S_2|) + (|A| + |B| + |C|)\\ &\geq (|X_f| + 2|S_2|) + (|S_2| + 2|X_f| + 2|X_f| - 3|S_2|) = 5|X_f|\end{aligned}$$

and so $|X_f| \leq \frac{n}{5}$. This completes the proof. □

For any $i_{dR}(G)$-function $f = (V_0, \emptyset, V_2, V_3)$, the function $g : V(G) \to \mathcal{P}(\{1,2,3\})$ defined by $g(x) = \{1,2,3\}$ for $x \in V_2 \cup V_3$ and $g(x) = \emptyset$ for $x \in V_0$, is an I3RDF on G and so

$$i_{r3}(G) \leq \frac{3}{2} i_{dR}(G). \tag{2}$$

Now we improve the upper bound (2) considerably.

Proposition 26. *Let G be a connected graph with maximum degree $\Delta \geq 1$. Then*

$$i_{dR}(G) \geq \left(\frac{2\Delta + 2}{3\Delta + 1}\right) i_{r3}(G).$$

Proof. Let $f = (V_0^f, \emptyset, V_2^f, V_3^f)$ be an $i_{dR}(G)$-function and H_f be an associated graph with respect to f with vertex partition $\pi(H_f) = \{S_1, S_2, ..., S_k\}$. If H_f is bipartite, then $i_{r3}(G) \leq i_{dR}(G)$ by Proposition 22 (part (i)). Assume that H_f is not bipartite and so $|V_2^f| \geq 1$. By Proposition 22 (part (ii)), $i_{r3}(G) \leq i_{dR}(G) + |X_f|$. Thus $i_{r3}(G)/i_{dR}(G) \leq 1 + |X_f|/i_{dR}(G)$. Since $i_{dR}(G) = 2|V_2^f| + 3|V_3^f|$ and $|X_f| = |V_2^f| - (|S_1| + |S_2|)$, we obtain

$$\begin{aligned} i_{r3}(G) &\le i_{dR}(G)\left(1+|X_f|/i_{dR}(G)\right) \\ &= i_{dR}(G)\left(1+\frac{|V_2^f|-(|S_1|+|S_2|)}{2|V_2^f|+3|V_3^f|}\right) \\ &\le i_{dR}(G)\left(1+\frac{|V_2^f|-(|S_1|+|S_2|)}{2|V_2^f|}\right) \\ &= i_{dR}(G)\left(\frac{3}{2}-\frac{|S_1|+|S_2|}{2|V_2^f|}\right). \end{aligned}$$

Moreover, since $|V_2^f| = |S_1|+|S_2|+\cdots+|S_t|$ and $|S_i| \le |S_j|$ for all $i \ge j$, we have

$$\begin{aligned} 2|V_2^f| &= (|S_1|+|S_t|)+(|S_2|+|S_{t-1}|)+\cdots+(|S_t|+|S_1|) \\ &\le (|S_1|+|S_2|)+(|S_1|+|S_2|)+\cdots+(|S_1|+|S_2|) \\ &= t(|S_1|+|S_2|). \end{aligned}$$

and so $(|S_1|+|S_2|)/|V_2| \ge 2/t$. Now using the fact that $t \le \Delta(H)+1 \le \Delta(G)+1$ and combining the previous results, we obtain

$$\begin{aligned} i_{r3}(G) &\le i_{dR}(G)\left(\frac{3}{2}-\frac{|S_1|+|S_2|}{2|V_2^f|}\right) \\ &\le i_{dR}(G)\left(\frac{3}{2}-\frac{1}{t}\right) \\ &\le i_{dR}(G)\left(\frac{3}{2}-\frac{1}{(\Delta(G)+1)}\right). \end{aligned}$$

Then by a straightforward calculation, we have

$$i_{dR}(G) \ge \left(\frac{2\Delta+2}{3\Delta+1}\right) i_{r3}(G),$$

and the proof is complete. □

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