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Graphs with arbitrarily large adversary degree associated reconstruction number

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ABSTRACT
A vertex deleted unlabeled subgraph of a graph is a *card*. A *dacard* specifies the degree of the deleted vertex along with the card. The *adversary degree associated reconstruction number* of a graph G, denoted $adrn(G)$, is the minimum number k such that every collection of k dacards of G uniquely determines G. A *strong double broom* is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint paths with same ends u and v by appending leaves at u and v. The strong double broom, obtained from a union of m internally vertex disjoint paths of order k with same ends u and v by appending n leaves at each u and v, is denoted by $B(n,n,mP_k)$. In this paper, we show that the *drn* of all strong double brooms is two and we determine $adrn(B(n,n,mP_k))$ for all n, m, k. For $n \geq 1$ and $m \geq 2$, usually $adrn(B(n,n,mP_3)) = n+2$ and $adrn(B(n,n,mP_4)) = n+m+2$, except $adrn(B(1,1,2P_3)) = 4$. For $n \geq 1, m \geq 2$ and $k > 4$, usually $adrn(B(n,n,mP_k)) = n+2$, except $adrn(B(n,n,mP_5)) = max\{m,n\}+2$ when $(n,m) \neq (1,2)$ and $adrn(B(1,1,2P_5)) = 5$.

KEYWORD
Reconstruction; reconstruction number; dacard

AMS SUBJECT CLASSIFICATION (2010)
Primary 05C60;
Secondary 05C05

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We shall mostly follow the graph theoretic terminology of [4]. A vertex of degree m is called an *m-vertex* and a 1-vertex is called an *end vertex*. The neighbour of a 1-vertex is called a *base* and a base of degree m is called an *m-base*. A neighbour of v with degree k is called a *k-neighbour* of v. A *double broom* is a tree obtained from a path by appending leaves at both ends of the path. A *strong double broom*, denoted by B, is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint (u, v)-paths by appending leaves at u and v. More precisely, $B(n_1, n_2, m_1P_{k_1}, m_2P_{k_2}, \ldots, m_tP_{k_t})$ denotes the strong double broom with n_1 leaves at one end u, n_2 leaves at the other end v and there are m_i internally vertex disjoint (u, v)-paths on k_i vertices for $1 \leq i \leq t$, $m_i \geq 0$ and $k_1 < k_2 < \ldots < k_t$ ($m_1 = 1$ when $k_1 = 2$). The vertices u and v are called the *hub vertices* and the 2-vertices are called *middle vertices*.

We distinguish the dacards of B into three types: a leaf dacard L, a middle dacard M and a hub dacard K are obtained, respectively, by deleting a leaf vertex, a middle vertex and a hub vertex (Figure 1).

A *vertex deleted subgraph* or a *card* $G - v$ of a graph G is the unlabeled graph obtained from G by deleting a vertex v and all edges incident with v. The ordered pair $(d(v), G - v)$ is called a *degree associated card* (or *dacard*) of the graph G, where $d(v)$ is the degree of v in G. We denote m copies of the dacard $(d(v), G - v)$ by $m(d(v), G - v)$ or simply by $m(G - v)$. The *deck* (*dadeck*) of a graph G is the collection of all its cards (dacards). The *Ulam's Conjecture* [3], also called the *Reconstruction Conjecture* (RC) asserts that every graph on at least three vertices is determined uniquely (up to isomorphism) by its deck. Graphs that obey the RC are called *reconstructible*.

For a reconstructible graph G, Harary and Plantholt [5] have defined the *reconstruction number* $rn(G)$ to be the size of the smallest subcollection of the deck of G which is not contained in the deck of any other graph H, $H \ncong G$. Myrvold [11] referred to this number as *ally-reconstruction number* of G and also studied *adversary reconstruction number* of G, which is the smallest k such that no subcollection of the deck of G of size k is contained in the deck of any other graph H, $H \ncong G$.

An extension of the RC to digraphs - the *Digraph Reconstruction Conjecture* was disproved when Stockmeyer exhibited [14] several infinite families of counter-examples. In view of this, Ramachandran [12, 13] studied the degree (degree triple) associated reconstruction of graphs (digraphs) and their reconstruction number. For a reconstructible graph (digraph) G from its dadeck, the *degree (degree triple) associated reconstruction number* of G, denoted $drn(G)$, is the size of the smallest subcollection of the dadeck of G which is not contained in the dadeck of any other graph (digraph) H, $H \ncong G$. Monikandan and Sundar Raj [10] introduced the degree associated analogue of $arn(G)$ (attributing the notion to Ramachandran). When G is reconstructible from its dadeck, the adversary degree-associated reconstruction number, denoted $adrn(G)$, is the least k such

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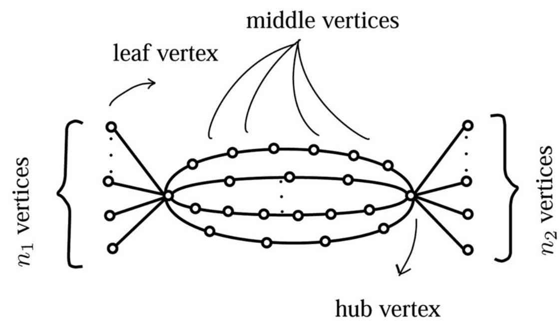

Figure 1. A strong double broom.

that every set of k dacards determines G. From the definition, $drn(G) \leq adrn(G)$. Equality holds when G is vertex-transitive, since then the dacards are pairwise isomorphic. The value of the *adrn* is known [6, 10] for complete graphs, complete bipartite graphs, cycles and wheels. In a subsequent paper, Monikandan and Sundar Raj [9] determined the *adrn* for double-stars, for subdivisions of stars, and for the disjoint union of t complete graphs of order n and s cycles of length m. If G is an r-regular of order n, then Barrus and West [1] have shown that $drn(G) \leq \min\{r+2, n-r+1\}$, which also implies that $adrn(G) \leq \min\{r+2, n-r+1\}$. (For elementary results on the edge version of adversary degree-associated reconstruction, see [8].)

Myrvold [11] showed $arn(G) = 3$ for almost every graph G. Since always $adrn(G) \leq arn(G)$, it is thus of some interest to find graphs G where $adrn(G)$ is large. Bowler et al. [2] constructed infinite families of pairs of graphs in which the pairs with n vertices have $2\lfloor\frac{n-4}{3}\rfloor$ common dacards, so $adrn(G)$ can be as large as $2\lfloor\frac{n-4}{3}\rfloor + 1$. They conjecture that this is the largest value for graphs of order n. Among vertex-transitive graphs, $G = 2K_{\frac{n}{4},\frac{n}{4}}$ in [1] achieves $adrn(G) = drn(G) = \frac{1}{4}|V(G)| + 2$. Recently Ma et al. [7] have proved that the *adrn* of most of the double brooms $B(m, n, P_k)$ is $\min\{m, n\} + 2$. In this paper, we show that the *drn* of all strong double brooms is two and we determine $adrn(B(n, n, mP_k))$ for all n, m, k. For $n \geq 1$ and $m \geq 2$, usually $adrn(B(n, n, mP_3)) = n + 2$ and $adrn(B(n, n, mP_4)) = n + m + 2$, except $adrn(B(1, 1, 2P_3)) = 4$. For $n \geq 1, m \geq 2$ and $k > 4$, usually $adrn(B(n, n, mP_k)) = n + 2$, except $adrn(B(n, n, mP_5)) = max\{m, n\} + 2$ when $(n, m) \neq (1, 2)$ and $adrn(B(1, 1, 2P_k)) = 5$.

2. *drn* and *adrn* of strong double brooms

We first determine the *drn*.

Theorem 1. *If G is a strong double broom, then* $drn(G) = 2$.

Proof. Let $G = B(n_1, n_2, m_1P_{k_1}, m_2P_{k_2}, \ldots, m_tP_{k_t})$. dacards $(1, L)$ and $(n_2 + \sum_{i=1}^{t} m_i, K)$ of G. Any leaf dacard forces G to be connected with $\binom{m_1}{2}$ cycles of length $2(k_1 - 1)$ (if $k_1 > 2$), $\binom{m_2}{2}$ cycles of length $2(k_2 - 1)$ (if $k_2 > 2$), $\ldots, \binom{m_t}{2}$ cycles of length $2(k_t - 1)$ (if $k_t > 2$) and to have m_1m_2 cycles of length $k_1 + k_2 - 2$, m_1m_3 cycles of length $k_1 + k_3 - 2, \ldots, m_1m_t$ cycles of length $k_1 + k_t - 2$, m_2m_3 cycles of length $k_2 + k_3 - 2, \ldots, m_2m_t$ cycles of length $k_2 + k_t - 2, \ldots, m_{t-1}m_t$ cycles of length $k_{t-1} + k_t - 2$. Hence to a hub dacard, add a new vertex v and join it to all isolated vertices and to $m_1, m_2, \ldots, m_t$ end vertices at distance $k_1 - 2, k_2 - 2, \ldots, k_t - 2$ respectively from a base of maximum degree if $k_1 > 2$, and joining v to m_j, $2 \leq j \leq t$ end vertices at distance $k_j - 2$ from a base of maximum degree and to this base if $k_1 = 2$. The resulting graph thus obtained is isomorphic to G. □

We denote the collection of m dacards $(d(v), G - v)$ by $m(d(v), G - v)$ or simply $m(G - v)$. An *extension* of a dacard $(d(v), G - v)$ of G is a graph obtained from the dacard by adding a new vertex x and joining it to $d(v)$ vertices of the dacard and it is denoted by $H(d(v), G - v)$ (or simply H). Throughout this paper, H and x are used in the sense of this definition.

Theorem 2. *If* $G = B(n, n, mP_3)$, *then* $adrn(G) = \begin{cases} 4 & \text{if } (n, m) = (1, 2) \\ n + 2 & \text{otherwise} \end{cases}$

Proof. The dacards of G are $2n(1, L)$, $m(2, M)$ and $2(n + m, K)$.
Lower bound: For $(n, m) = (1, 2)$, the graph G shares two middle dacards and a leaf dacard with a graph obtained from $(1, L)$ by annexing a new vertex x and joining it to the unique leaf vertex and therefore $adrn(G) \geq 4$. For $(n, m) \neq (1, 2)$, G shares $n+1$ leaf dacards with $B(n+1, n-1, mP_3)$ and therefore $adrn(G) \geq n + 2$.
Upper bound: We proceed by three cases and prove that the collection of dacards considered under each case determines G uniquely.

Case 1. Two distinct dacards, except when one is L and the other is M for $(n, m) = (1, 2)$.

Case 1.1. K and M (or L).

Since the two dacards K and L determine G uniquely (Theorem 1), we consider the two dacards K and M. The dacard M forces G to be connected and G to have every base with at most n neighbours of degree 1. Hence G can be obtained uniquely from K by annexing a vertex and joining it to n isolated vertices and to m vertices of degree 1.

Case 1.2. L and M when $(n, m) \neq (1, 2)$.

The dacard L forces G to have $\binom{m}{2}$ cycles of length $2(k - 1)$ and to have maximum degree at least $n + m$. Hence in M, the newly added vertex x must be joined to the two $(n + m - 1)$-vertices. But then the extension is isomorphic to G.

Case 2. For $(n, m) = (1, 2)$, any collection S of four dacards including at least one L and M.

In view of Case 1, we take S to contain only the dacards L and M.

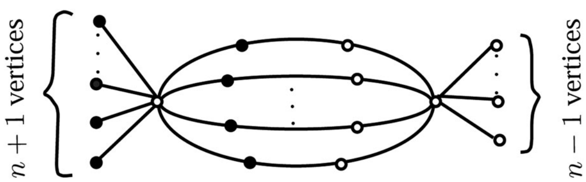

Figure 2. $B(n+1, n-1, mP_4)$.

The dacard L forces every extension to have C_4 and hence the only possibility to obtain an extension H ($\ncong G$) from M is to join the newly added vertex to the 1-vertex and to a 2-vertex at distance 2. The graph H has exactly one dacard L (obtained by removing the unique leaf) and exactly two dacards M (obtained by removing the two 2-neighbours of the unique 3-vertex). The removal of any other 2-vertex from K results in a dacard with a 3-vertex or an isolated vertex but M does not have these vertices. Hence S uniquely determines G.

Case 3. '$(n+1)L$ and D' or '$2M$ and D' for $(n,m) \neq (1,2)$.

In view of Case 1, we assume that the dacards of S are isomorphic.

Case 3.1. $(n+2)L$ (this case does not arise when $n=1$).

Now $H(1, L)$ is obtained by joining x to one vertex of L. If x is joined to a 1-vertex or a 2-vertex, then H has exactly one dacard L (obtained by removing x) and the removal of any other 1-vertex from H results in a dacard with a 2-base or two adjacent bases but L does not have these. If x is joined to the $(n+m)$-vertex, then H has exactly $n+1$ leaf dacards (obtained by removing each 1-neighbour of the $(n+m+1)$-vertex) and the removal of any other leaf from H results in a dacard with an $(n+m+1)$-vertex but L does not have an $(n+m+1)$-vertex.

Case 3.2. $3M$ (this case does not arise when $m=2$).

The extension ($\ncong G$) obtained from M, by joining x to a 1-vertex and its base, has exactly two dacards isomorphic to M (obtained by removing x and its neighbour) and the removal of any other 2-vertex results in a dacard having C_3 but M does not have C_3. For all other extensions, the removal of any 2-vertex other than x, results in a dacard having at most $\binom{m}{2} - 1$ cycles of length $2(k-1)$ or two adjacent bases or a 2-base or a base with $n+1$ neighbours of degree 1 but L does not have these. Thus, from all preceding cases, we have $adrn(B(n,n,mP_3)) = 4$ if $(n,m) = (1,2)$ (Cases 1.1 and 2) and $adrn(B(n,n,mP_3)) = n+2$ otherwise (Cases 1.2 and 3). □

Theorem 3. *If $G = B(n,n,mP_4)$, then $adrn(G) = n+m+2$.*

Proof. The dadeck of G consists of $2n(1,L)$, $2m(2,M)$ and $2(n+m,K)$. Let D denote any dacard of G.

Lower bound: The graph G shares $n+1$ leaf dacards and m middle dacards with $B(n+1, n-1, mP_4)$ (Figure 2). Hence $adrn(G) \geq n+m+2$.

Upper bound: We proceed by three cases and prove that the collection of dacards considered under each case determines G uniquely.

Case 1. M and K when $(n,m) \neq (1,2)$.

The dacard M forces G to be connected and hence K forces G to have two vertices of degree at least $n+m$ and every base with at most n neighbours of degree 1. Hence G can be obtained from M by joining x to the 1-vertex whose base has $n+1$ neighbours of degree 1 and to the $(n+m-1)$-vertex.

Case 2. L and K.

Proof follows by Theorem 1.

Case 3. M, K and one more dacard D when $(n,m) = (1,2)$.

We take D to be other than L as otherwise D and K together will determine G by Case 2.

Using M and K, we can say that every base in G has one 1-neighbour. Hence M forces every extension to have at most two 1-vertices. Therefore the only possible extension H non-isomorphic to G of K can be obtained by joining x to the isolated vertex and to the two 1-vertices at distance 3. The extension H then has exactly one dacard isomorphic to M (obtained by removing a 2-vertex adjacent to the 2-neighbour of the 3-base) as the removal of any other 2-vertex results in a dacard having every base with one 1-neighbour or an isolated vertex and exactly one dacard isomorphic to K (obtained by removing x) as the removal of any other 3-vertex results in a dacard having two non-trivial components.

Case 4. The collection S containing αL and βM where $\alpha, \beta \geq 1$ and $\alpha + \beta \leq n+m+1$, together with one more dacard.

Let us assume that S contains no K as otherwise, by Case 2, S uniquely determines G. The dacard L forces G to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in M, to get an extension non-isomorphic to G, join x to the unique $(n+m)$-vertex and to a 1-vertex at distance $2(k-2)$. The resulting extension then has exactly $n+1$ dacards isomorphic to L (obtained by removing the 1-neighbours of $(n+m+1)$-vertex), has no more leaf dacards (since there is no more leaf or the removal of any leaf results in a dacard having an $(n+m+1)$-vertex), exactly m dacards isomorphic to M (obtained by removing the 2-neighbours of the $(n+m+1)$-base), no more middle dacards (since the removal of any 2-vertex results in a dacard having an $(n+m+1)$-vertex) and no hub dacard (since there is no $(n+m)$-vertex).

Case 5. $(n+2)L$ (when $n \neq 1$) or $(m+1)M$.

Case 5.1. $(n+2)L$.

If, in L, vertex x is joined to a 1-vertex or an $(n+m)$-vertex, then the proof is similar to Case 3.1 of Theorem 2. If x is joined to a 2-vertex, then the resulting

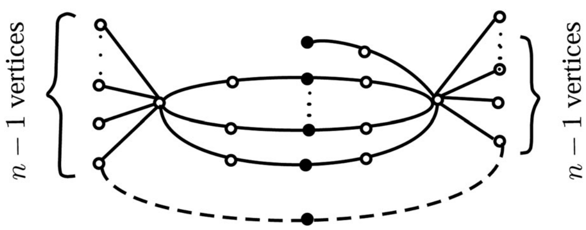

Figure 3. The extension H.

extension non-isomorphic to G has exactly one dacard isomorphic to L, since the removal of any other leaf results in a dacard having two bases at distance at most 2.

Case 5.2. $(m+1)M$.

If, in M, vertex x is joined to an $(n+m)$-vertex and a 1-neighbour of $(n+m-1)$- base, then the resulting extension has exactly m dacards isomorphic to M (obtained by removing the 2-neighbours of an $(n+m+1)$-base) and no more middle dacards, since the removal of any other 2-vertex results in a dacard with an $(n+m+1)$-base. If x is joined to a 1-vertex and its base, then the resulting extension has exactly two dacards isomorphic to M (obtained by removing x and its 2-neighbour) or if x is adjacent to a 1-vertex and a 2-vertex when $(n,m)=(1,2)$, the resulting extension has two dacards isomorphic to M (obtained by removing the 2-neighbours of 3-vertex lying on a cycle) and the removal of any other 2-vertex results in a disconnected dacard (when $(n,m)=(1,2)$) or in a dacard having a cycle C_3 or two bases at distance at most 2 (when $(n,m)=(1,2)$). All other extensions of M has at most two dacards isomorphic to M, since the removal of any 2-vertex other than x results in a dacard having two bases at distance at most 2 or at most $\binom{m}{2}-1$ cycles of length $2(k-1)$ or a 2-base or in a disconnected dacard. □

It is clear that all the $2m$ dacards of $B(n,n,mP_k)$ obtained by deleting middle vertices at equal distance i from the nearest hub vertex are mutually isomorphic; let M_i denote such a dacard. Then i can be 1, 2,…, or $\lceil \frac{k}{2} \rceil - 1$. These dacards will be used in proving the next main result.

Theorem 4. *For* $k>4$,

$$adrn(B(n,n,mP_k)) = \begin{cases} 5 & \text{if } (n,m)=(1,2) \\ max\{m,n\}+2 & \text{if } k=5 \text{ and } (n,m)\neq(1,2) \\ n+2 & \text{otherwise} \end{cases}$$

Proof. The dadeck of $G=B(n,n,mP_k)$ consists of $2n(1,L)$, $m(k-2)(2,M_i)$ and $2(n+m,K)$. Let D denote any dacard of G.
Lower bound: The graph G shares $n+1$ leaf dacards with the graph $B(n+1,n-1,mP_k)$. Hence $adrn(G)\geq n+2$. …… (Eq-1)

For $(n,m)=(1,2)$, the graph G shares two leaf dacards and two middle dacards with a graph obtained from L by joining x to a 2-vertex at distance $k-4$ from the base. Hence $adrn(G)\geq 5$.

For $k=5$ and $(n,m)\neq(1,2)$, G shares a leaf dacard and m middle dacards with a graph obtained from M_1 by joining x to two 1-vertices at distance $2(k-2)$. Hence from (Eq-1), $adrn(G)\geq max\{m,n\}+2$.
Upper bound: We proceed by eleven cases and we prove that the collection of dacards considered under each case determines G uniquely.

Case 1. L and any dacard D other than L and M_1.

The dacard L forces G to be connected and to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in K, join x to the n isolated vertices and to m vertices of degree 1 at distance $k-2$ from the unique $(n+m)$-vertex or in middle dacard (other than M_1) join x to two 1-vertices at distance $2(k-2)$ and the extension then obtained is G.

Case 2. L, M_1 and D when $(n,m)\neq(1,2)$ and $k\neq 5$.

The dacard L forces G to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in M_1, to get an extension non-isomorphic to G, join x with a 1-neighbour of the $(n+m-1)$-base and to a vertex at distance $2(k-2)$. The resulting graph has exactly one dacard isomorphic to M_1 (obtained by removing x), no middle dacards (since the removal of any other 2-vertex results in a dacard with two bases of degree at least 3 at distance less than $k-1$ or a vertex of degree at least 3 with $n-1$ neighbours of degree 1), exactly one dacard L (obtained by removing the 1-neighbour of the 3-base adjacent to x) as the removal of any other 1-vertex results in a dacard with two bases at distance less than $k-1$ and has no dacards H (since the removal of any $(n+m)$-vertex results in a dacard with no $(n+m)$-base).
Claim 3. αL, βM_1, $\alpha,\beta\geq 1$, $\alpha+\beta=m+1$ and D when $(n,m)\neq(1,2)$ and $k=5$.

The dacard L forces every extension to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in the dacard M_1, to get an extension non-isomorphic to G, vertex x must be joined to a 1-neighbour of the $(n+m-1)$-base and to some vertex at distance 6. If x is joined to a 1-neighbour of an $(n+m)$-base, the the resulting extension H (Figure 3) has exactly m dacards isomorphic to M_1 (obtained by removing each 2-neighbour at distance 2 from the $(n+m)$-base lying on a cycle), has no more middle dacards (since the removal of any other 2-vertex results in a dacard with a base of degree at least 3 having $n-1$ neighbours of degree 1), exactly one leaf dacard (obtained by removing 1-neighbour of the 2-base) as the removal of any leaf results in a dacard having a 2-base and has no hub dacard (since the removal of the unique $(n+m)$-vertex results in a dacard with two nontrivial components). If x is joined to a 2-base, then the resulting extension has exactly one dacard isomorphic to each L (obtained by removing the 1-neighbour of the 3-base adjacent to x) and M_1 (obtained by removing x), has no

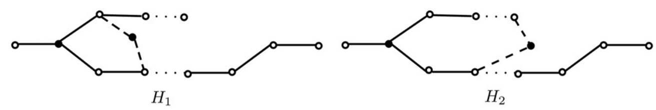

Figure 4. The extensions H_1 and H_2.

more leaf or middle dacards (since the removal of any leaf or 2-vertex results in a dacard with three vertices of degree at least 3 or two adjacent vertices each of degree at least 3) and has no dacards K (since the removal of an $(n+m)$-vertex results in a dacard having no $(n+m)$-vertex).

Case 4. αL, βM_1, $\alpha,\beta \geq 1$, $\alpha+\beta = 4$ and one D when $(n,m) = (1,2)$.

The dacard L forces every extension to have a cycle of length $2(k-1)$. Hence the only two extensions non-isomorphic to G of M_1 are obtained by annexing a vertex and joining it to the 2-base at minimum distance from the 3-base or the 1-neighbour of the 3-base (when $k=5$) and to the 1-vertex at maximum distance from the 3-base. The first extension has exactly two dacards isomorphic to L (obtained by removing the two leaves) and has exactly two dacards isomorphic to M_1 (obtained by removing the 2-neighbours of the bases that are not common neighbours). The second extension has exactly one dacard isomorphic to L (obtained by removing the unique leaf) and has exactly two dacards isomorphic to M_1 (obtained by removing the 2-vertices at distance 2 from the unique 3-vertex). Both of these two extensions have no more dacards M_1 (since the removal of any other 2-vertex results in a dacard having two 3-bases or exactly two leaves or having two leaves at distance 2 or 3 from a 3-vertex or in a disconnected dacard or a base with two 1-neighbours), no more middle dacards M_i $\left(2 \leq i \leq \lceil \frac{k}{2} \rceil\right)$ (since the removal of 2-vertex results in a dacard with two 3-bases at distance either less than $k-1$ or greater than $k-1$ or with at most one 3-base) and no hub dacard (since the removal of any 3-vertex results in a dacard with no 3-base or having a 1-vertex at distance greater than $k-2$ from a 3-base).

Case 5. $M_{\lceil \frac{k}{2} \rceil -1}$ and K when $k=5$.

As in Case 1.1 of Theorem 2, the dacards $M_{\lceil \frac{k}{2} \rceil -1}$ and K force G to have every base with at most n neighbours of degree one. Hence in $M_{\lceil \frac{k}{2} \rceil -1}$, the newly added vertex x must be adjacent to a 1-neighbour of each base with $n+1$ neighbours of degree 1 and the resulting extension is isomorphic to G.

Case 6. αM_1, βK, $\alpha,\beta \geq 1$, $\alpha+\beta = 4$ and one D when $k=5$.

Case 6.1. $(n,m) = (1,2)$.

The dacard M_1 forces every extension to be connected and hence every extension of K is obtained by joining x to the isolated vertex in K. If the other neighbours of x are a 2-base and the unique 1-neighbour of the 3-base, then the resulting extension non-isomorphic to G has exactly two middle dacards M_1 (obtained by removing the two 2-neighbours adjacent to the 3-bases, exactly two hub dacards K (obtained by removing the two 3-bases) and no other hub dacards (since the removal of any other 3-vertex results in a dacard with no isolated vertex) and has no leaf dacards (since the removal of any leaf results in a dacard with a cycle of length less than $2(k-1)$)). If x is joined to the 1-neighbour of a 2-base and to the other 2-base or to the 1-neighbour of a 2-base and the 1-neighbour of the 3-base, then the resulting extension non-isomorphic to G has exactly one dacard K (since the removal of any 3-vertex other than x results in a dacard having a base with two 1-neighbours), exactly one dacard M_1 (obtained by removing a 2-vertex adjacent to two 3-bases or a 2-vertex lying on a cycle at distance 3 from the 3-base). The above extensions have no more middle dacards and the remaining extensions non-isomorphic to G have no middle dacards (since the removal of any other 2-vertex results in a disconnected dacard or in a dacard having a 1-vertex at distance $k-2$ from the nearest 3-base or 4-base or having no 3-base).

Case 6.2. $(n,m) \neq (1,2)$.

The dacards K and M_1 force G to be connected and to have two vertices of degree at least $n+m$. Hence in every extension non-isomorphic to G of M_1, one neighbour of the newly added vertex must be the unique $(n+m-1)$-vertex. If the other neighbour is the 2-base, then the resulting graph has exactly one dacard M_1 (obtained by removing x) and no more middle dacards (since the removal of any other 2-vertex results in a dacard having two adjacent bases of degree at least 3), exactly one dacard K (obtained by removing an $(n+m)$-vertex adjacent to the 3-base) as the removal of any other $(n+m)$-vertex results in a dacard having a 1-vertex at distance $k-3$ from the nearest $(n+m)$-base). The remaining extensions other than G have no dacard K (since the removal of any $(n+m)$-vertex results in a dacard having two nontrivial components having a 1-vertex at distance 2 from an $(n+m)$-base or an $(n+m+1)$-base or a cycle of length less than $2k-2$).

Case 7. M_i, $1 \leq i \leq \lceil \frac{k}{2} \rceil - 1$, K and D when $k \neq 5$.

Case 7.1. M_1, K and D.

The dacards K and M_1 force G to be connected and to have two vertices of degree at least $n+m$. Hence in every extension non-isomorphic to G of M_1, one neighbour of the newly added vertex must be the unique $(n+m-1)$-vertex. The extensions H_1, H_2 (Figure 4), H_3 (Figure 5) when $(n,m) = (1,2)$ and H_4 when $(n,m) \neq (1,2)$ (Figure 5) have exactly one dacard M_1 (obtained by removing x), no more middle dacards (since the removal of any other 2-vertex results in a dacard having two 3-vertices at distance less than $k-1$ or no 3-base or two nontrivial components) and exactly one hub dacard (obtained by removing the unique 3-base in H_1 and H_2, removing the 3-base adjacent to a 3-vertex in H_3 and removing the $(n+m)$-base adjacent to the 3-vertex in H_4). The above extensions H_1, H_2, H_3 and H_4 have no more dacards K and the remaining extensions other than G have no dacards K (since the removal of any other $(n+m)$-vertex results in a dacard having two nontrivial

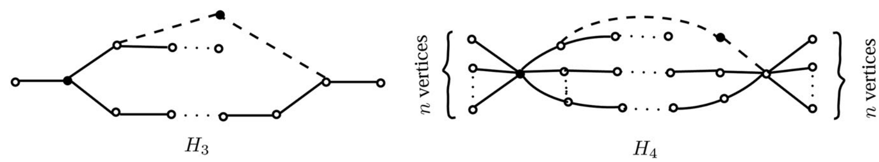

Figure 5. The extensions H_4 and H_5.

components or no $(n+m)$-base or an $(n+m)$-base with $n+1$ neighbours of degree 1 or an $(n+m)$-vertex or having a 1-vertex at distance 2 or less than $k-2$ (>1) or an $(n+m+1)$-vertex).

Case 7.2. M_i $\left(2 \le i \le \lceil \frac{k}{2} \rceil - 1\right)$, K and D.

Let the middle dacard have 1-vertices at distance $l_1 > 1$ and $l_2 \ge 1$ such that $l_1 \ge l_2$ from the nearest $(n+m)$-base. The extension non-isomorphic to G of M_i (obtained by joining x to a 1-vertex at distance l_2 (or l_1) from the nearest $(n+m)$-base and to the 2-neighbour of the other $(n+m)$-base which is at distance $l_1 - 1$ (or $l_2 - 1$ if $l_2 > 1$) from the other 1-vertex) has exactly one middle dacard (obtained by removing x) as the removal of any other 2-vertex results in a dacard having two bases of degree at least 3 at distance less than k – 1, no dacard L as the removal of any leaf results in a dacard having a cycle of length less than $2(k-1)$ and exactly one hub dacard (obtained by removing the $(n+m)$-neighbour of a 3-base adjacent to x). The above extensions have no more hub dacard and the remaining extensions non-isomorphic to G has no hub dacard (since the removal of any $(n+m)$-vertex results in a dacard having two nontrivial components or a 1-vertex at distance less than $k-1$ (>1) from the nearest $(n+m)$-base or at most $n+m-1$ isolated vertices or an $(n+m+1)$-vertex.

Case 8. $M_{\lceil \frac{k}{2} \rceil -1}$ and $M_{\lceil \frac{k}{2} \rceil -2}$ when k is even ($k \ne 6$ for $(n,m) = (1,2)$).

Every extension non-isomorphic to G of $M_{\lceil \frac{k}{2} \rceil -1}$ does not have the dacard $M_{\lceil \frac{k}{2} \rceil -2}$ (since the removal of any 2-vertex other than x results in a dacard having an $(n+m+1)$-vertex or a 1-vertex at distance $\frac{k}{2} - 1$ or $\frac{k}{2} - 2$ $(k \ge 8)$ from the nearest $(n+m)$-base or a cycle of length less than $2(k-1)$ or exactly one $(n+m)$-base (for $k \ge 8$) or results in a disconnected dacard).

Case 9. $M_{\lceil \frac{k}{2} \rceil -1}$ and '$M_{\lceil \frac{k}{2} \rceil -2}$ or $M_{\lceil \frac{k}{2} \rceil -3}$' when k is odd ($k \ne 7$ for $(n,m) = (1,2)$) but only the first two dacards determine G when $k=7$ and $(n,m) = (1,2)$.

Every extension non-isomorphic to G of $M_{\lceil \frac{k}{2} \rceil -1}$ does not have the second dacard $M_{\lceil \frac{k}{2} \rceil -2}$, since the removal of any 2-vertex (other than x) results in a dacard having a base with $n+1$ or $n+2$ neighbours of degree 1 (when $k=5$) or a 1-vertex at distance $\lceil \frac{k}{2} \rceil - 2$ from the nearest $(n+m)$-vertex or having an $(n+m+1)$-vertex or a cycle of length less than $2(k-1)$ or exactly one $(n+m)$-base (for $k \ge 9$) or results in a disconnected dacard.

Case 10. M_i, M_j, $i \ne j$, $1 \le i < j \le \lceil \frac{k}{2} \rceil - 1$ and D.

We exclude Cases 8 and 9 that come under Case 10.

Case 10.1 M_i has exactly one vertex of maximum degree.

Let the other middle dacard, say M_j, have 1-neighbours at distance $l_1 > 1$ and $l_2 \ge 1$ such that $l_1 \ge l_2$, respectively from the nearest $(n+m)$-base.

Case 10.1.1. $m = 2$.

The extension ($\ncong G$) of the dacard M_i (obtained by joining x to the 2-base at a maximum distance from the 3-base (when $n=1$) or to the $(n+m-1)$-base (when $n>1$) and to a 2-vertex at distance $l_1 - 2$ (if $l_1 > 3$) or $l_2 - 2$ (if $l_2 > 3$) from the 1-neighbour of another 2-base) has exactly one dacard M_i (obtained by removing x) and exactly one dacard M_j (obtained by removing a 2-vertex adjacent to a 3-neighbour of x which is at minimum distance from an $(n+m)$-vertex non adjacent to x) and the extension obtained by joining x to the 1-neighbour of the 2-base which is at minimum distance from the 3-base and to a 2-vertex at distance l_1 (if $l_1 > 1$) or l_2 (if $l_2 > 1$) from the 1-neighbour of another 2-base) have exactly one dacard M_i (obtained by removing x) and have exactly one dacard M_j (obtained by removing a central vertex of a path connecting two vertices of degree at least 3, disjoint from a path containing x where the central vertex is at minimum distance from x). The above extensions have no more dacards M_j (since the removal of any other 2-vertex results in a dacard having a cycle or two vertices of degree at least 3 at distance less than $k-1$, no more dacards M_i (since the removal of any other 2-vertex results in a dacard having two bases of degree at least 3 or a cycle or no $(n+m)$-base or having a 1-vertex at distance less than $k-3$ (>1) from a base of degree at least 3) and have no leaf dacards (since the removal of any leaf results in a dacard having a cycle of length less than $2(k-1)$) and have no hub dacards (since the removal of any $(n+m)$-vertex results in a dacard with no $(n+m)$-base). All other extensions non-isomorphic to G have either two isomorphic dacards M_i or exactly one dacard M_i (since the removal of any other 2-vertex results in a dacard having a cycle or at least $2n+2$ vertices of degree 1 or a base with $n+1$ neighbours of degree 1 or a 1-vertex at distance less than $k-3$ (>1) from a base of degree at least 3 or having no $(n+m)$-base.

Case 10.1.2. $m > 2$.

The extension non-isomorphic to G of the dacard M_i, obtained by joining x to a $(n+m)$-base to a 2-vertex at distance $l_1 - 2$ (if $l_1 > 2$) or $l_2 - 2$ (if $l_2 > 2$) from 1-neighbour of another 2-base, has exactly two middle dacards (obtained by removing x and by removing a 2-neighbour of a 3-base adjacent to x) and has no more dacards (because of the similar reasons described in Case 10.1).

Case 10.2. M_i has two vertices of maximum degree.

Proof is similar to Case 10.1.

Case 11. $(n+2)L$ (this case arise when $n > 1$) or $3M_i$ $(1 \leq i \leq \lceil \frac{k}{2} \rceil - 1)$.

Proof is similar to Case 4 of Theorem 2.

From all the preceding cases, we conclude that

$$adrn(B(n,n,mP_k)) = \begin{cases} 5 & \text{if } (n,m) = (1,2) \text{ (Cases } 1,\ 3,\ \ 5-11) \\ max\{m,n\}+2 & \text{if } k=5 \text{ and } (n,m) \neq (1,2) \\ & \text{(Cases } 1,\ 4-6,\ 9-11) \\ n+2 & \text{otherwise} \quad \text{(Cases } 1,\ 4-6,\ 9-11) \end{cases}$$

□

From the full deck of a graph G, it is easy to compute the degrees of the deleted vertices; hence the RC is equivalent to the reconstruction of graphs from their dadecks. From the partial deck of a graph, the degree of the deleted vertex is hard to compute, and the dacard (degree associated card) provides more information. Hence graphs may be reconstructible using fewer dacards than cards.

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