
Taylor & Francis
Taylor & Francis Group

AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

# On graphoidal length of a tree in terms of its diameter

Purnima Gupta, Megha Agarwal & Rajesh Singh

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2019.12.012



# On graphoidal length of a tree in terms of its diameter

Purnima Gupta[a], Megha Agarwal[b], and Rajesh Singh[c]

[a]Department of Mathematics, Sri Venkateswara College, University of Delhi, Delhi, India; [b]Department of Mathematics, Ramanujan College, University of Delhi, Delhi, India; [c]Department of Mathematics, Ramanujan College, University of Delhi, Delhi, India

**ABSTRACT**

A *graphoidal cover* of a graph $G$ is a set $\Psi$ of non-trivial paths (which are not necessarily open) in $G$ such that every vertex of $G$ is an internal vertex of at most one path in $\Psi$ and every edge of G is in exactly one path in $\Psi$. We denote the set of all graphoidal covers of graph $G$ by $\mathcal{G}_G$. The graphoidal length $gl(G)$ of a graph $G$ is defined as

$$gl(G) = \max_{\Psi \in \mathcal{G}_G}\{\min_{P \in \Psi} l(P)\}.$$

In this paper, we obtain bounds for the graphoidal length of a tree in terms of its diameter. We prove that if $G$ is any tree (excepts paths) of diameter $d$, then graphoidal length $gl(G)$ is less than equal to $\lfloor 2d/3 \rfloor$. Further, we characterize trees attaining the upper bound. Also, the trees for which $gl(G) = k$ where $\lfloor d/2 \rfloor < k < \lfloor 2d/3 \rfloor$ are characterized.

**KEYWORDS**
Graphoidal cover; graphoidally covered graph; graphoidal length; graphoidal covering number

**2010 MATHEMATICS SUBJECT CLASSIFICATION**
05C70; 20D60

## 1. Introduction

For standard terminology and notation in graph theory, as also for pictorial representations of graphs, we refer the standard text-books such as Harary [10], West [15] or Chartrand [8]. Terms and notations, not specifically defined or described in these text-books, will be separately defined as and when found necessary. Throughout we consider only nontrivial, connected, finite, undirected graphs without loops and multiple edges.

For any graph $G = (V, E)$, $V$ (or $V(G)$) denotes the set of vertices of $G$, $E$ (or $E(G)$) denotes the set of edges of $G$ and $e$ (or $e(G)$) denotes the number of vertices of degree 1 (pendant vertices) in $G$. The *distance* $d(u, v)$ between two vertices $u$ and $v$ is the length of shortest path joining them. Further, the distance $d(u, S)$ of the vertex $u$ from the set $S \subseteq V$ is given by $d(u, S) = \min\{d(u, v) : \ v \in S\}$. The *diameter* of $G$ is given by $diam(G) = \max\{d(u, v) : \ u, v \in V\}$.

**Definition 1.1.** A graphoidal cover of a graph $G$ is a set $\Psi$ of non-trivial paths (which are not necessarily open) in $G$ such that:

**GC1:** Every vertex of $G$ is an internal vertex of at most one path in $\Psi$;
**GC2:** Every edge of $G$ is in exactly one path in $\Psi$.

The set of all graphoidal covers of a graph $G$ is denoted by $\mathcal{G}_G$ and for a given $\Psi \in \mathcal{G}_G$, the ordered pair $(G, \Psi)$ is called a graphoidally covered graph. For any graph $G$, the set $E$ of edges (consisting of paths of length one) is a graphoidal cover of $G$ and is referred to as the trivial graphoidal cover of $G$.

There are three types of vertices in a graphoidally covered graph $(G, \Psi)$ (see Figure 1):

1. **$\Psi$-Exterior Vertex or Black Vertex:** A vertex which is not an internal vertex of any $\Psi$-edge. In the diagrammatic representation of $(G, \Psi)$, it is shown as a small filled circle.
2. **$\Psi$-Interior Vertex or White Vertex:** A vertex which is not an end-vertex of any $\Psi$-edge. In the diagrammatic representation of $(G, \Psi)$, it is shown as a small unfilled circle.
3. **$\Psi$-Composite Vertex:** A vertex which is neither an exterior vertex nor an interior vertex of $(G, \Psi)$, which means it is an internal vertex to a $\Psi$-edge and also an end-vertex to at least one other $\Psi$-edge. In the diagrammatic representation, this vertex is represented as an unfilled circle with as many small tangents to the circle as the number of $\Psi$-edges for which this vertex is the end-vertex.

In Figure 2, we give diagrammatic representation of a graphoidally covered graph $(G, \Psi)$ with $\Psi = \{(a, b, e), (a, c, f), (a, d, g), ((c, d))\}$. The vertices $a, e, f, g$ are all black vertices ($\Psi$-exterior vertices), $b$ is a white vertex ($\Psi$-interior vertices) and $c$, $d$ are $\Psi$-composite vertices.

Acharya and Sampathkumar [3] in 1987 introduced the concept of graphoidal covers as a close variant of another emerging discrete structure called *semigraphs* [13]. Many interesting notions like graphoidal covering number [3], graphoidal labeling [12], graphoidal signed graphs [14], etc

**CONTACT** Purnima Gupta purnimachandni1@gmail.com Department of Mathematics, Sri Venkateswara College, University of Delhi, Delhi 110021, India.

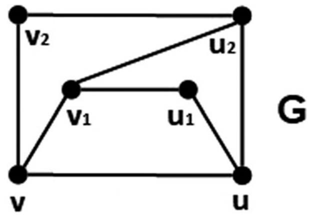


**Figure 1.** Each of the collections $\Psi_1 = \{(u, u_1, v_1, v), (u, u_2, v_2, v), (u, v), (v_1, u_2)\}$ and $\Psi_2 = \{(u, u_1, v_1, v, u), (v_1, u_2, v_2, v), (u, u_2)\}$ of paths in $G$ is a graphoidal cover for the graph $G$.

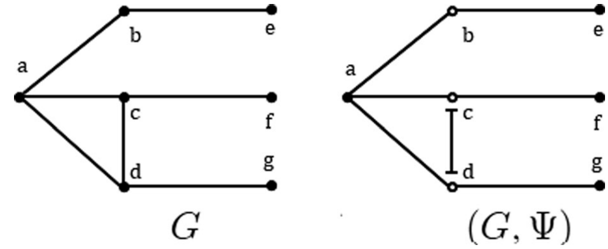


**Figure 2.** Diagrammatic representation of a graphoidal cover $\Psi$ of graph $G$.

were introduced and are being studied extensively. In particular, notion of graphoidal covering number of a graph has attracted many researchers and numerous work is present in literature on graphoidal covering number [5–7, 11]. Later on in 1999, Acharya and Gupta [9] extended the notion of graphoidal covers to infinite graphs and introduced the notion of domination in the discrete structure, called *graphoidally covered graph* [2,9]. A detailed treatment of graphoidal covers and graphoidally covered graphs is given in [9, 1].

**Definition 1.2.** [3] The graphoidal covering number of $G$, denoted by $\eta(G)$, is the minimum cardinality of a graphoidal cover of $G$.

$$\eta(G) = min\{|\Psi| \ : \Psi \in \mathcal{G}_G\}$$

A graphoidal cover of $G$ with cardinality $\eta(G)$ is referred to as an $\eta$-graphoidal cover.

In [4], a new graph invariant $gl(G)$, called graphoidal length of a graph $G$, based on graphoidal covers was introduced and some bounds were found for this parameter. Therein, an upper bound for graphoidal length of a tree was obtained in terms of its edges and pendant vertices. In this paper we establish another upper bound for graphoidal length of a tree in terms of its diameter. It will be proved that these two upper bounds for graphoidal length of a tree are independent.

**Definition 1.3. (Length of a Graphoidal Cover)** Given a non-trivial graph $G$, associate a function $\Theta$ with as follows:

$$\Theta \ : \ \mathcal{G}_G \to \mathbb{Z}^+$$
$$\Theta(\Psi) = \min\{l(P) \ : \ P \in \Psi\}.$$

The positive integer $\Theta(\Psi)$ associated with $\Psi$ is called the **length of the graphoidal cover** $\Psi$ and is denoted by $gl_\Psi(G)$.

**Definition 1.4. (Graphoidal Length of a Graph)** The maximum of the range of $\Theta$ is called the **graphoidal length of graph** $G$ and is denoted by $gl(G)$. Thus,

$$\begin{aligned} gl(G) &= \max\{\Theta(\Psi) \ : \ \Psi \in \mathcal{G}_G\} \\ &= \max\{gl_\Psi(G) \ : \ \Psi \in \mathcal{G}_G\} \end{aligned}$$

Obviously, for any graphoidal cover $\Psi$ of a given graph $G$, $gl(G) \geq gl_\Psi(G)$. Since $gl_E(G) = 1$ for the trivial graphoidal $\Psi = E$ of $G$, therefore $gl(G) \geq 1$. A graphoidal cover $\Psi$ of graph $G$ with $gl_\Psi(G) = gl(G)$ is called a $gl$-graphoidal cover (or $gl(G)$-graphoidal cover) of $G$.

In Figure 3, we illustrate the notion of graphoidal length by depicting four different types of graphoidal covers of the complete graph $K_4$ namely, $E, \Psi_1, \Psi_2, \Psi_3$ where $E$ is the edge set of $G$, $\Psi_1 = \{bd, bca, badc\}, \Psi_2 = \{bd, ac, badcb\}$ and $\Psi_3 = \{badc, acbd\}$.

In $\Psi = E$, each path is an open path of length one and hence $gl_E(G) = 1$. In $\Psi_1$ there are three open paths of length 1, 2 and 3 and hence $gl_{\Psi_1}(G) = 1$. In $\Psi_2$, there are two open paths of length one and one closed path of length 4, therefore $gl_{\Psi_2}(G) = 1$. In $\Psi_3$ there are two paths each of length 3, therefore $gl_{\Psi_3}(G) = 3$. Hence $gl(K_4) \geq 3$. Since $G$ has a vertex of degree 3, every graphoidal cover of $G$ has at least two paths. As $|E(K_4)| = 6$, therefore $gl_\Psi(K_4) \leq 3$ for every graphoidal cover $\Psi$. Consequently, $gl(K_4) \leq 3$. It follows that $gl(K_4) = 3$.

Observe that for any graph $G$ of order $n$ and size $m$, $gl(G) \leq \min\{n, m\}$. Moreover $gl(G) = \min\{n, m\}$ if and only if $G$ is either a path or a cycle. It is interesting to note that graphoidal length of a path or cycle coincides with its usual length. Further, if $G$ is a graph of size $m$ and maximum degree $\Delta \geq 3$, then every graphoidal cover $\Psi$ of $G$ contains at least one path of length at most $\lfloor m/2 \rfloor$.

In [4], the natural connection that graphoidal length of a graph exhibits with graphoidal covering number was exposed in the form of bound for $gl(G)$ in terms of $\eta(G)$.

**Theorem 1.5.** [4] *For any graph G*

$$1 \leq gl(G) \leq \left\lfloor \frac{|E(G)|}{\eta(G)} \right\rfloor.$$

Pakkiam and Arumugum [11] established the following interesting and useful results which give exact value of graphoidal covering number in case of trees.

**Theorem 1.6.** [11] *For any tree $T$ with $e$ pendant vertices, $\eta(G) = e - 1$.*

From Theorems 1.5 and 1.6, we have the following upper bound for graphoidal length of a tree in terms of its size and pendant vertices.

**Theorem 1.7.** *For any tree $T$ with $e$ pendant vertices,*

$$1 \leq gl(T) \leq \left\lfloor \frac{|E(T)|}{e-1} \right\rfloor.$$

## 2. Bounds on graphoidal length of trees

Clearly, graphoidal length of a path $P_{d+1}$ of diameter $d$ is $d$. For a tree $G$ ($\ncong P_{d+1}$) of diameter $d$, the graphoidal length

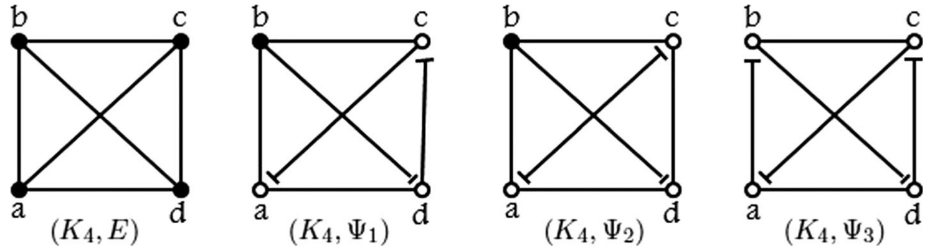


**Figure 3.** Different graphoidal covers of complete graph $K_4$.

$gl(G) < d$. This motivates us to find an upper bound for graphoidal length of a tree in terms of its diameter. In fact we show that if $G$ is any tree (excepts paths) of diameter $d$, then graphoidal length $gl(G)$ is less than equal to $\lfloor 2d/3 \rfloor$. Further, we characterize trees attaining this bound. We also characterize trees for which graphoidal length $gl(G) = k$ where $\lfloor d/2 \rfloor < k < \lfloor 2d/3 \rfloor$.

**Theorem 2.1.** *Let $G$ $(\ncong P_{d+1})$ be a tree of diameter $d$. Then*

$$gl(G) \leq \left\lfloor \frac{2d}{3} \right\rfloor.$$

*Proof.* Suppose there exists a tree $G$ such that $gl(G) > \lfloor 2d/3 \rfloor = k$ (*say*). Let $\Psi$ be a graphoidal cover of $G$ with minimum cardinality such that $gl_\Psi(G) > k$. Since $G \ncong P_{d+1}, |\Psi| \geq 2$. Thus we can choose two $\Psi$-edges $P$, $Q$ such that $V(P) \cap V(Q) \neq \emptyset$. Let $V(P) \cap V(Q) = \{v\}$. Then $v$ is end vertex of at least one $\Psi$-edge $P$ or $Q$. If $v$ is an end vertex of both $P$ and $Q$, then $\Phi = \Psi \cup \{P \cup Q\} \setminus \{P, Q\}$ is a graphoidal cover of $G$ such that $|\Phi| = |\Psi| - 1$ and $gl_\Psi(G) > k$, a contradiction. Hence $v$ is an end vertex to exactly one $\Psi$-edge $P$ or $Q$. Without loss in generality we assume that $v$ is an end vertex of $P$. Let $P = (v_0 v_1 \ldots v_p)$ and $Q = (u_0 u_1 \ldots u_q)$ with $v = v_p = u_j$ for some $j$ $(0 < j \leq \lfloor q/2 \rfloor)$.

Consider the path $R = (v_0 v_1 \ldots v_p = u_j u_{j+1} \ldots u_q)$ of length $p + (q - j)$. Clearly,

$$\begin{aligned} d \geq l(R) = p + (q-j) &\geq p + \left\lceil \frac{q}{2} \right\rceil \\ &> p + \frac{d}{3} \quad [\text{since } q = l(Q) > k = \lfloor 2d/3 \rfloor] \\ &\geq \left\lfloor \frac{2d}{3} \right\rfloor + 1 + \frac{d}{3} = d, \text{a contradiction.} \end{aligned}$$

Hence for any given tree $G$, $gl(G) \leq \lfloor 2d/3 \rfloor$. □

Thus we have found an upper bound for graphoidal length of a tree in terms of its diameter. Naturally one is tempted to ask "Is the bound sharp?" . The bound is indeed sharp. The star $K_{1,3}$ is an example of extremal graph for the upper bound.

Since the bound is sharp, we characterize extremal graphs for this bound. In this direction, we first prove the following lemma.

**Lemma 2.2.** *Let $G$ $(\ncong P_{d+1}$ or $K_{1,n}$, $n \geq 3)$ be a tree with diameter $d$ and $gl(G) = k$ for some $k$ such that $\lfloor d/2 \rfloor < k \leq \lfloor 2d/3 \rfloor$. If $\Psi$ is a graphoidal cover of $G$ such that $gl_\Psi(G) = k$, then $|\Psi| = 2$.*

*Proof.* Since $G \ncong P_{d+1}, |\Psi| \geq 2$. Now suppose, if possible, $|\Psi| > 2$. Then we can choose $\Psi$-edges $P$, $Q$, $R$ such that $V(P) \cap V(Q) \neq \emptyset$ and $V(Q) \cap V(R) \neq \emptyset$. Let $v \in V(P) \cap V(Q)$ and $u \in V(Q) \cap V(R)$. If $v$ is an end vertex for both $P$ and $Q$, then the path $P \cup Q$ has length $p + q$ and $d \geq p + q \geq 2k$, a contradiction to the fact that $k > \lfloor d/2 \rfloor$. Thus $v$ is an end vertex of exactly one of $P$ or $Q$. Similarly, $u$ is an end vertex of exactly one of $Q$ or $R$. Also, on the similar lines it can be proved that $v \neq u$. Let $P = (v_0 v_1 \ldots v_p), Q = (u_0 u_1 \ldots u_q)$ and $R = (w_0 w_1 \ldots w_r)$. Consider the following possibilities:

**Case I.** $v$ is an end vertex of $P$ and $u$ is an end vertex of $R$. Let $v = v_p = u_i$ for some $i$ $(1 \leq i \leq q)$ and $u = w_0 = u_j$ for some $j$ $(1 \leq j \leq q)$. Without loss of generality we may assume that $i < j$. Then the path $S = (v_0 v_1 \ldots v_p = v = u_i u_{i+1} \ldots u_{j-1} u_j = u = w_0 w_1 \ldots w_r)$ in $G$ has length $p + (j - i) + r$. By definition of diameter,

$$d \geq l(S) = p + (j - i) + r \geq k + (j - i) + k > d,$$

a contradiction.

**Case II.** $v$ is an end vertex of $P$ and $u$ is an end vertex of $Q$. Let $v = v_p = u_i$ for some $i$ $(1 \leq i \leq \lfloor q/2 \rfloor)$ and $u = u_q = w_j$ for some $j$ $(1 \leq j \leq \lfloor r/2 \rfloor)$. Then the path $S = (v_0 v_1 \ldots v_p = v = u_i u_{i+1} \ldots u_q = u = w_j w_{j+1} \ldots w_r)$ in $G$ has length $p + (q - i) + (r - j)$. By definition of diameter,

$$\begin{aligned} d \geq l(S) &= p + (q - i) + (r - j) \\ &= k + (q - \lfloor q/2 \rfloor) + (r - \lfloor r/2 \rfloor) \\ &= k + \lceil q/2 \rceil + \lceil r/2 \rceil \\ &\geq k + k/2 + k/2 \\ &> d, \end{aligned}$$

a contradiction.

**Case III.** $v$ is an end vertex of $Q$ and $u$ is an end vertex of $R$. Similar to Case II, by interchanging the roles of $P$ and $R$, we will arrive at a contradiction.

**Case IV.** $v$ is an end vertex of $Q$ and $u$ is an end vertex of $Q$. Let $v = v_i = u_0$ for some $i$ $(1 \leq i \leq \lfloor p/2 \rfloor)$ and $u = u_q = w_j$ for some $j$ $(1 \leq j \leq \lfloor r/2 \rfloor)$. Then the path $S = (v_p v_{p-1} \ldots v_i = v = u_0 u_1 \ldots u_q = u = w_j w_{j+1} \ldots w_r)$ in $G$ has length $(p - i) + q + (r - j)$. By definition of diameter,

$$d \geq l(S) = (p - i) + q + (r - j) > d \; [\text{similar to Case II}],$$

a contradiction.

Thus in all the cases we arrive at a contradiction. Hence our assumption is wrong and $|\Psi| = 2$. □

**Remark 2.3.** If $G(\ncong K_{1,n})$ is a tree with diameter $d$ such that $\lfloor d/2 \rfloor < gl(G) \le \lfloor 2d/3 \rfloor$, then every $gl$-graphoidal cover of $G$ is an $\eta$-graphoidal cover.

We now characterize extremal graphs for the upper bound obtained in Theorem 2.1. In fact we show that for trees with diameter not equal to 4, the only possible graphs that can be extremal for the upper bound are stars $K_{1,n}$ $(n \ge 3)$ and T-graphs $\Gamma_{m,n,k}$.

**Definition 2.4.** *A **T-graph**, denoted by $\Gamma_{m,n,k}$, is a graph* obtained by identifying a pendant vertex of each path $P_{m+1}, P_{n+1}$ and $P_{k+1}$.

**Theorem 2.5.** *Let $G \ncong P_{d+1}$ be a tree of diameter $d$ $(\neq 4)$. Then $gl(G) = \lfloor 2d/3 \rfloor$ if and only if one of the following holds:*

(1) $G \cong K_{1,n}$ $(n \ge 3)$
(2) $G \cong \Gamma_{m,n,k}$ *where* $\min\{m, n+k\} = \lfloor 2d/3 \rfloor$.

*Proof.* Suppose $gl(G) = \lfloor 2d/3 \rfloor$. If $G \cong K_{1,n}$ $(n \ge 3)$, then we are through. Let $G \ncong K_{1,n}$ $(n \ge 3)$ and $\Psi$ be a graphoidal cover of $G$ such that $gl_\Psi(G) = \lfloor 2d/3 \rfloor$. Since $d \neq 4, \lfloor 2d/3 \rfloor > \lfloor d/2 \rfloor$. Hence from Lemma 2.2, it follows that $|\Psi| = 2$.

Let $\Psi = \{P, Q\}$ and $V(P) \cap V(Q) = \{v\}$ for some vertex $v$. Clearly, $v$ is an end vertex to exactly one of $P$ and $Q$. Without loss of generality, assume that $v$ is an end vertex of $P$. Let $P = (v_0 v_1 \ldots v_m)$ and $Q = (u_0 u_1 \ldots u_q)$, where $v = v_m = u_n$ $(\lceil q/2 \rceil \le n < q)$. Then setting $k = q - n$ we get that $G \cong \Gamma_{m,n,k}$ where $\min\{m, n+k\} = \lfloor 2d/3 \rfloor$.

***Conversely,*** if $G \cong K_{1,n}$ $(n \ge 3)$, then diameter $d$ of $G$ is 2. Clearly, in this case, $gl(G) = 1 = \lfloor 2d/3 \rfloor$ and we are through. Let $G \cong \Gamma_{m,n,k}$ for some $m$, $n$, $k$ such that $\min\{m, n+k\} = \lfloor 2d/3 \rfloor$. Then $\Psi = \{P, Q\}$, where $P = P_{m+1}$ and $Q = P_{n+1} \cup P_{k+1}$, is a graphoidal cover of $G$ such that $gl_\Psi(G) = \lfloor 2d/3 \rfloor$. Hence it follows that $gl(G) = \lfloor 2d/3 \rfloor$. □

Thus we have completely characterized extremal trees with diameter $d \neq 4$ for the upper bound. What happens in case the diameter of the tree is 4? The next theorem gives a necessary and sufficient condition for a tree of diameter 4 to attain the upper bound.

**Theorem 2.6.** *Let $G$ $(\ncong P_{d+1})$ be a tree of diameter $d = 4$. Then $gl(G) = \lfloor 2d/3 \rfloor$ if and only if $|N^*(u)| \le 2$ for each $u \in V$, where $N^*(u) = \{v \in N(u) : d(v) = 1$ or $v$ is a support to two pendant vertices$\}$.*

*Proof.* ***Necessity:*** Let $G$ be a tree such that $gl(G) = \lfloor 2d/3 \rfloor = 2$. Suppose there exists $w \in V$ such that $|N^*(w)| > 2$. Let $u_1, u_2, u_3$ be vertices in $N^*(w)$.

Consider any graphoidal cover $\Psi \in \mathcal{G}_G$ such that $gl_\Psi(G) = 2$. Then every $\Psi$-edge has length at least 2. Now if for any $i$ $(1 \le i \le 3)$, $u_i$ is supporting two pendant vertices say $u_{i_1}$ and $u_{i_2}$. Then $R_i = (u_{i_1} u_i u_{i_2})$ must be a $\Psi$-edge. For otherwise either $u_i u_{i_1}$ or $u_i u_{i_1}$ is a $\Psi$-edge, a contradiction to the fact that $gl_\Psi(G) = 2$. Thus $u_i$ must be an end vertex for the $\Psi$-edge containing $u_{iw}$. Now let $P$ be the $\Psi$-edge containing the edge $u_1 w$ with $u_1$ as its end vertex. Then since $gl_\Psi(G) = 2$, $w$ must be an internal vertex of $P$. Clearly $P = (u_1 w u_2)$, for otherwise $w u_2$ will be a $\Psi$-edge, a contradiction. But then $u_3 w$ is a $\Psi$-edge, again a contradiction. Hence lemma holds and we are through.

***Sufficiency:***
Suppose the hypothesis holds. Let $w$ be the center of $G$. We shall show that $gl_\Psi(G) = 2$ for some graphoidal cover $\Psi$ of $G$. Clearly, due to the hypothesis, $d(u) \le 3$ for each $u \in N(w)$. Now partition the neighborhood $N(w)$ of $w$ as follows

$$N(w) = N_1(w) \cup N_2(w) \cup N_3(w)$$

where $N_i(w) = \{u \in N(w) : d(u) = i\}$ for $i = 1, 2, 3$. For each $u$ in $N_2(w)$, let $z_u$ denote the pendant vertex with support $u$. If $N_3(w) \neq \emptyset$ then for each $u$ in $N_3(w)$, let $x_u$, $y_u$ denote pendant vertices with support $u$. Also, let $P = \{(x_u u y_u) : u \in N_3(w)\}$. Obviously, by definition $N^*(w) = N_1(w) \cup N_3(w)$.

Now since $|N^*(w)| \le 2$, following three cases arise:

**Case 1:** $N^*(w) = \emptyset$
Then $\Psi = \{(w u z_u) : u \in N_2(w)\}$ is a graphoidal cover of $G$ of length 2.

**Case 2:** $N^*(w) = \{v_1\}$
In this case $N_2(w) \neq \emptyset$ and therefore we choose a vertex $v$ in $N_2(w)$. Then $\Psi = P \cup \{(w u z_u) : u \in N_2(w) - \{v\}\} \cup \{(z_v v w v_1)\}$ is a graphoidal cover of $G$ of length 2. Note that $P$ may possibly be empty in case $v_1 \in N_1(w)$.

**Case 3:** $N^*(w) = \{v_1, v_2\}$
Then $\Psi = P \cup \{(w u z_u) : u \in N_2(w)\} \cup \{(v_1 w v_2)\}$ is a graphoidal cover of $G$ of length 2. Again in this case, $P$ may possibly be empty if $v_1, v_2 \in N_1(w)$.

Thus in all the cases above we have constructed a graphoidal cover $\Psi$ such that $gl_\Psi(G) = 2$. Consequently, from Theorem 2.1, $gl(G) = 2 = \lfloor 2d/3 \rfloor$. □

After characterizing trees $G$ of diameter $d$ for which graphoidal length $gl(G) = \lfloor 2d/3 \rfloor$, we now make an attempt to look at the structure of trees for which graphoidal length $gl(G) < \lfloor 2d/3 \rfloor$. In our next theorem we partially answer this question by characterizing trees $G$ of diameter $d$ having graphoidal length $gl(G) = k$, where $\lfloor d/2 \rfloor < k < \lfloor 2d/3 \rfloor$.

**Theorem 2.7.** *Let $G$ be tree of diameter $d$ and $k$ be an integer such that $\lfloor d/2 \rfloor < k < \lfloor 2d/3 \rfloor$. Then $gl(G) = k$ if and only if one of the following condition hold:*

(1) $G \cong \Gamma_{m,n,d-m}$ *for some* $m$, $n$ $(m \ge n)$ *such that* $k - n = m < d - k$;
(2) $G \cong \Gamma_{k,m,d-k}$ *for some* $m$ *such that* $2k - d \le m \le d - k$.

*Proof.* Let $gl(G) = k$, where $\lfloor d/2 \rfloor < k < \lfloor 2d/3 \rfloor$. The condition $\lfloor 2d/3 \rfloor > \lfloor d/2 \rfloor + 1$ ensures that $d \ge 11$. Therefore $G \ncong K_{1,n}$ $(n \ge 3)$. Also, since $gl(G) < d$, therefore $G \ncong P_{d+1}$.

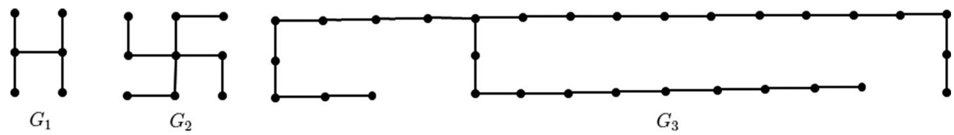


**Figure 4.** Example of trees.

Let $\Psi$ be a graphoidal cover of $G$ such that $gl_\Psi(G) = k$. Then from Lemma 2.2, $|\Psi| = 2$. Let $\Psi = \{P, Q\}$ and $V(P) \cap V(Q) = \{v\}$ for some vertex $v$. Again, $v$ cannot be an end vertex to both $P$ and $Q$. Without loss of generality, assume that $v$ is an end vertex of $P$. Let $P = (v_0 v_1 ... v_a)$ and $Q = (u_0 u_1 ... u_b)$, where $v = v_a = u_c$ for some $c$ such that $\lceil b/2 \rceil \le c \le b$. Then $G \cong \Gamma_{a,c,b-c}$. Now we have two possibilities:

**Case I.** $a \ge b$.
Then $b = k$ and $d = a + c$. Set $m = c, n = b - c$, then $a = d - m, k - n = m < d - k$. Hence $G \cong \Gamma_{m,n,d-m}$ where $m \ge n$ and $k - n = m < d - k$.

**Case II.** $b > a$.
Then $a = k$ and $d = \max\{a + c, b\}$. But since $a = k > \lfloor d/2 \rfloor \ge \lfloor b/2 \rfloor \ge b - c$, it follows that $d = a + c$. Set $m = b - c$, then $2k - d \le m \le d - k$. Hence $G \cong \Gamma_{k,m,d-k}$ where $m$ is such that $2k - d \le m \le d - k$.

***Conversely***, if $G \cong \Gamma_{m,n,d-m}$ *for some* $m$, $n$ $(m \ge n)$ *such that* $k - n = m < d - k$. Then $\Psi = \{P, Q\}$, where $P = P_{d-m+1}$ and $Q = P_{m+1} \cup P_{n+1}$, is a graphoidal cover of $G$ such that $gl_\Psi(G) = k$. Now if $G \cong \Gamma_{k,m,d-k}$ for some $m$ such that $2k - d \le m \le d - k$. Then $\Psi = \{P, Q\}$, where $P = P_{k+1}$ and $Q = P_{m+1} \cup P_{d-k+1}$, is a graphoidal cover of $G$ such that $gl_\Psi(G) = k$. Thus in either case there exists a graphoidal cover $\Psi$ of $G$ such that $gl_\Psi(G) = k$. It follows that $gl(G) = k$. Hence the theorem. □

## 3. Concluding remarks

In this paper, we focused on the problem of finding graphoidal length of a tree in terms of its diameter. We observed that for a tree of diameter $d$, $gl(G) = d$ if and only if $G \cong P_{d+1}$. We proved that for trees $G$ (other than paths) with diameter $d$, $gl(G) \le \lfloor 2d/3 \rfloor$ and that the extremal trees for this upper bound are homeomorphic to $K_{1,3}$. Further, we characterized trees $G$ of diameter $d$ having graphoidal length $gl(G) = k$, where $\lfloor d/2 \rfloor < k \le \lfloor 2d/3 \rfloor$. Thus it remains to characterize trees $G$ with $gl(G) = k$ where $k \le \lfloor d/2 \rfloor$.

**Problem 1.** *Characterize trees G of diameter d having graphoidal length* $gl(G) = k$ *such that* $k \le \lfloor d/2 \rfloor$.

In particular, one can first look to tackle the following problem.

**Problem 2.** *Characterize trees G of diameter d with* $gl(G) = \lfloor d/2 \rfloor$.

It is interesting to note that the two upper bounds for graphoidal length of a tree obtained in Theorems 1.7 and 2.1 are unrelated and independent of each other. The trees $G_1$, $G_2$ and $G_3$ in Figure 4 exhibit this fact.

$$\left\lfloor \frac{|E(G_1)|}{e-1} \right\rfloor < \left\lfloor \frac{2d(G_1)}{3} \right\rfloor, \quad \left\lfloor \frac{|E(G_2)|}{e-1} \right\rfloor = \left\lfloor \frac{2d(G_2)}{3} \right\rfloor, \quad \left\lfloor \frac{|E(G_3)|}{e-1} \right\rfloor > \left\lfloor \frac{2d(G_3)}{3} \right\rfloor.$$

**Problem 3.** *Characterize trees G of diameter d such that*

1. $\left\lfloor \frac{|E(G)|}{e-1} \right\rfloor < \left\lfloor \frac{2d}{3} \right\rfloor$
2. $\left\lfloor \frac{|E(G)|}{e-1} \right\rfloor = \left\lfloor \frac{2d}{3} \right\rfloor$
3. $\left\lfloor \frac{|E(G)|}{e-1} \right\rfloor > \left\lfloor \frac{2d}{3} \right\rfloor$.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

## References


[1] Arumugam, S., Devadas Acharya, B. and E. Sampathkumar, (1997). Graphoidal covers of a graph: a creative review In *Graph Theory and Its Applications (Tirunelveli, 1996)*, pages 1–28. Tata McGraw-Hill, New Delhi.

[2] Acharya, B. D, Gupta, P. (2007). Further results on domination in graphoidally covered graphs. *AKCE Int. J. Graphs Comb.* 4(2):127–138.

[3] Acharya, B. D, Sampathkumar, E. (1987). Graphoidal covers and graphoidal covering number of a graph. *Indian J. Pure Appl. Math.* 18(10):882–890.

[4] Arumugam, S., Gupta, P, Singh, R. (2016). Bounds on graphoidal length of a graph. *Electron. Notes Discret. Math.* 53:113–122.

[5] Arumugam, S, Pakkiam, C. (1994). Graphoidal bipartite graphs. *Graphs Combin.* 10(2–4):305–310.

[6] Arumugam, S., Rajasingh, I, Pushpam, P. R. L. (2001). Graphs whose acyclic graphoidal covering number is one less than its maximum degree. *Discrete Math.* 240(1-3):231–237.

[7] Arumugam, S, Suresh Suseela, J. (1998). Acyclic graphoidal covers and path partitions in a graph. *Discrete Math.* 190(1–3):67–77.

[8] Chartrand, G, Lesniak, L. (2005). *Graphs & Digraphs*, 4th ed. Boca Raton, FL: Chapman & Hall/CRC

[9] Devadas Acharya, B, Gupta, P. (1999). Domination in graphoidal covers of a graph. *Discret. Math.* 206(1–3):3–33. Combinatorics and number theory (Tiruchirappalli, 1996).

[10] Harary, F. (1969). *Graph Theory.*Reading, MA: Addison-Wesley Publishing Co.

[11] Pakkiam, C, Arumugam, S. (1989). On the graphoidal covering number of a graph. *Indian J. Pure Appl. Math.* 20(4):330–333.

[12] Sahul Hamid, I, Anitha, A. (2011). On label graphoidal covering number—I. *Discrete Math. Algorithm. Appl.* 3(1):1–33.

[13] Sampathkumar, E. (2000). Semigraphs and their applications. *Report on the DST Project.*

[14] Siva Kota Reddy, P, Misra, U. K. (2014). Graphoidal signed graphs. *Proc. Jangjeon Math. Soc.* 17(1):41–50.

[15] West, D. B. (1996). *Introduction to Graph Theory.*Upper Saddle River, NJ: Prentice Hall, Inc.