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The modeling of the equatorial undercurrent using the Navier–Stokes equations in rotating spherical coordinates

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ABSTRACT

Using the Navier–Stokes equations in rotating spherical coordinates, we discuss a mathematical model for the equatorial current across the Pacific Ocean. Symmetry properties of the governing equations that hold in equatorial regions permit us to gain detailed insight into the structure of the ocean flow dynamics.

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1. Introduction

The equatorial ocean flows present special challenges with respect to flows at mid-latitudes since in this context neither geostrophic theory nor Ekman theory apply (see the discussions in [1,2] and [3], respectively) and, moreover, nonlinear effects are noticeable (see the discussion in [4,5]). Of all equatorial regions, that of the Pacific is the most important due to its extent – it is more than 12,000 km across the ocean, from South America to South Asia. Several recent papers are devoted to equatorial flows (see the discussions in [33-35-36]), and address the issue of underlying currents, issue that was mostly ignored in the earlier research literature (see the survey [6]). The papers [7-11] find explicit wave solutions for the governing equations in Lagrangian coordinates in the β -plane approximation, in the presence of relatively weak underlying currents. Two-dimensional flows (which ignore the meridional velocity component and use the f-plane approximation) are investigated in [1,12] (see also [13]) and inroads towards three-dimensional flows can be found in [14-17] (see [18] for a detailed description of the intricate ongoing dynamics). One issue of great current interest is the derivation of realistic models for the underlying current field. The main features of the equatorial current are that the current is mainly azimuthal and presents a strong depth-variation in the upper 500 m of the ocean (of about 4 km mean depth), while at depths larger than 500 m there is practically no motion. The most significant features of the depth-variation consist of a westward near-surface current (induced by the trade winds that blow towards the west) and a stronger eastward flow - the Equatorial Undercurrent (EUC) - that dominates the overall ocean flow, reaching speeds in excess of 1 m/s. Note the unusual feature that the EUC flows against the prevailing trade winds, as the directly wind-driven flow vanishes below 100 m. Recently, an approach was proposed to explain how such a

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This is an Open Access article distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives License (http:// creativecommons.org/licenses/by-nc-nd/4.0/), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original work is properly cited, and is not altered, transformed, or built upon in any way. current with flow-reversal might be induced by the action of the wind (see [19]). However, the model in [19] does not capture the westward Equatorial Intermediate Current (EIC), found directly below the EUC across the Pacific (see [20]). Our aim in this paper is to show that the method used in [19], to investigate the Navier–Stokes equations in rotating spherical coordinates by taking advantage of symmetry properties of the leading-order terms in the governing equations for equatorial ocean flows to gain insight into the structure of the velocity field, can be further refined to provide a model of higher accuracy. In particular, we are able to include the westward EIC as a component of the background state of the ocean.

2. Preliminaries

For large-scale ocean flows it is appropriate to consider a spherical Earth, rotating at a constant rate about its polar axis (see [21,22]). We introduce near the Equator spherical coordinates (φ, θ, r') , where r' is the radius, $\theta \in (-\pi/2, \pi/2)$ is the angle of latitude and $\varphi \in [0, 2\pi)$ is the angle of longitude. The corresponding unit vectors are $(\mathbf{e}_{\varphi}, \mathbf{e}_{\theta}, \mathbf{e}_{r})$, respectively, with \mathbf{e}_{φ} pointing from West to East, \mathbf{e}_{θ} from South to North and \mathbf{e}_{r} upwards. We denote the ocean flow velocity components by (u', v', w'). We use primes for physical (dimensional) variables, and we will remove them in the non-dimensionalisation process.

In this rotating reference system the Navier–Stokes equation and the equation of mass conservation are (see [19])

$$\begin{pmatrix} \frac{\partial}{\partial t'} + \frac{u'}{r'\cos\theta} \frac{\partial}{\partial\varphi} + \frac{v'}{r'} \frac{\partial}{\partial\theta} + w' \frac{\partial}{\partial r'} \end{pmatrix} (u', v', w') + \frac{1}{r'} \left(-u'v' \tan\theta + u'w', u'^2 \tan\theta + v'w', -u'^2 - v'^2 \right) + 2\Omega'(-v'\sin\theta + w'\cos\theta, u'\sin\theta, -u'\cos\theta) + r'\Omega'^2(0, \sin\theta\cos\theta, -\cos^2\theta) = -\frac{1}{\rho'} \left(\frac{1}{r'\cos\theta} \frac{\partial p'}{\partial\varphi}, \frac{1}{r'} \frac{\partial p'}{\partial\theta}, \frac{\partial p'}{\partial r'} \right) + (0, 0, -g') + v'_1 \left(\frac{\partial^2}{\partial r'^2} + \frac{2}{r'} \frac{\partial}{\partial r'} \right) (u', v', w') + \frac{v'_2}{r'^2} \left(\frac{1}{\cos^2\theta} \frac{\partial^2}{\partial\varphi^2} - \tan\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} \right) (u', v', w'),$$
(1)

and

$$\frac{1}{r'\cos\theta}\frac{\partial u'}{\partial\varphi} + \frac{1}{r'\cos\theta}\frac{\partial}{\partial\theta}(v'\cos\theta) + \frac{1}{r'^2}\frac{\partial}{\partial r'}(r'^2w') = 0,$$
(2)

respectively, where $p'(\varphi, \theta, r', t')$ is the pressure in the fluid, ρ' the constant density, $\Omega' \approx 7.29 \times 10^{-5}$ rad s⁻¹ is the constant rate of rotation of the Earth, and $g' = \text{constant} \approx 9.81 \text{ m s}^{-2}$ is the gravitational acceleration. The coefficients of the viscous terms are taken to be constant, with ν'_1 the vertical kinematic eddy viscosity and ν'_2 the horizontal kinematic eddy viscosity.

Let us now describe the associated boundary conditions. At the ocean's surface, $r' = R' + h'(\varphi, \theta, t')$, where $R' \approx 6378$ km is the radius of the spherical Earth, we impose the dynamic and the kinematic boundary condition:

$$p' = P'_{s}(\varphi, \theta, t') \quad \text{on} \quad r' = R' + h'(\varphi, \theta, t'), \tag{3}$$

and

$$w' = \frac{\partial h'}{\partial t'} + \frac{u'}{r'\cos\theta} \frac{\partial h'}{\partial \varphi} + \frac{v'}{r'} \frac{\partial h'}{\partial \theta} \quad \text{on} \quad r' = R' + h'(\varphi, \theta, t'), \tag{4}$$

respectively, where P'_s is the surface pressure. The wind-stress at the ocean's surface is considered to be known, and can be expressed in the form

$$\tau'_{1}(\varphi,\theta,t') = \rho' v'_{1} \frac{\partial u'}{\partial r'},$$

$$\tau'_{2}(\varphi,\theta,t') = \rho' v'_{1} \frac{\partial v'}{\partial r'},$$
(5)

the surface wind stress (τ'_1, τ'_2) being related to the vertical eddy viscosity by $\nu'_1 = \sigma'|(\tau'_1, \tau'_2)|$ on the surface, where σ' is a (dimensional) constant and $(\tau'_1, \tau'_2) = c_D \rho'_{air} U'_{wind}|U'_{wind}|$, ρ'_{air} being the density of air (about 1.2 kg/m³) and $c_D \approx 0.0013$ being a (dimensionless) drag coefficient. At the impermeable, solid, stationary bottom of the ocean, $r' = R' + d'(\varphi, \theta)$, we have the corresponding boundary condition for viscous flow:

$$u' = v' = w' = 0$$
 on $r' = R' + d'(\varphi, \theta)$, (6)

Setting r' = R' + z' and

$$p' = \rho' \left(-g'r' + \frac{1}{2}r'^2 \Omega'^2 \cos^2 \theta \right) + P'(\varphi, \theta, r', t'),$$
(7)

one can non-dimensionalise the problem by performing the change of variables

$$z' = D'z, \quad (u', v', w') = U'(u, v, kw), \quad P' = \rho' U'^2 P,$$
(8)

where $D' \approx 200$ m is the average depth of the near-surface layer to which the wind effects are confined), $U' \approx 0.1 \text{ m s}^{-1}$ is the typical speed of mid-latitude ocean currents at the surface, while the scaling factor k for the vertical velocity is less than 10^{-4} ; see the discussion in [19]. In terms of the the shallow-water parameter

$$\varepsilon = D'/R',$$

with a typical value of the order 10^{-5} , the governing equations for steady flow are transformed into the non-dimensional form

$$\left(\frac{u}{(1+\varepsilon z)\cos\theta}\frac{\partial}{\partial\varphi} + \frac{v}{1+\varepsilon z}\frac{\partial}{\partial\theta} + \frac{k}{\varepsilon}w\frac{\partial}{\partial z}\right)(u, v, kw) \\
+ \frac{1}{1+\varepsilon z}\left(-uv\tan\theta + kuw, u^{2}\tan\theta + kvw, -u^{2} - v^{2}\right) \\
+ 2\omega(-v\sin\theta + kw\cos\theta, u\sin\theta, -u\cos\theta) \\
= -\left(\frac{1}{(1+\varepsilon z)\cos\theta}\frac{\partial P}{\partial\varphi}, \frac{1}{1+\varepsilon z}\frac{\partial P}{\partial\theta}, \frac{1}{\varepsilon}\frac{\partial P}{\partial z}\right) \\
+ \frac{1}{R_{\varepsilon 1}}\left(\frac{1}{\varepsilon^{2}}\frac{\partial^{2}}{\partial z^{2}} + \frac{2}{(1+\varepsilon z)}\frac{1}{\varepsilon}\frac{\partial}{\partial z}\right)(u, v, kw) \\
+ \frac{1}{R_{\varepsilon 2}(1+\varepsilon z)^{2}}\left(\frac{1}{\cos^{2}\theta}\frac{\partial^{2}}{\partial\varphi^{2}} - \tan\theta\frac{\partial}{\partial\theta} + \frac{\partial^{2}}{\partial\theta^{2}}\right)(u, v, kw), \tag{9}$$

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and

$$\frac{1}{(1+\varepsilon z)\cos\theta}\left\{\frac{\partial u}{\partial\varphi} + \frac{\partial}{\partial\theta}(v\cos\theta)\right\} + \frac{k/\varepsilon}{(1+\varepsilon z)^2}\frac{\partial}{\partial z}\left\{(1+\varepsilon z)^2w\right\} = 0,$$
(10)

where $\omega = \Omega' R' / U' = O(1)$ and $R_{ei} = U' R' / \nu'_j$ (j = 1, 2) are the inverse Rossby number and the pair of Reynolds numbers, respectively. With (h', d') = D'(h, d), the non-dimensional boundary conditions are

$$P = \overline{P}_{s}(\varphi, \theta),$$

$$\frac{\partial u}{\partial z} = \tau_{1}(\varphi, , \theta), \quad \text{on} \quad z = h(\varphi, \theta),$$

$$\frac{\partial v}{\partial z} = \tau_{2}(\varphi, \theta),$$
(11)

$$\frac{k}{\varepsilon}w = \frac{u}{(1+\varepsilon h)\cos\theta}\frac{\partial h}{\partial\varphi} + \frac{v}{1+\varepsilon h}\frac{\partial h}{\partial\theta} \quad \text{on} \quad z = h(\varphi,\theta),$$
(12)

$$(u, v)$$
 decays rapidly below $z = h(\varphi, \theta)$. (13)

For ocean flows in the equatorial Pacific it is adequate (see the discussion in [19]) to consider

$$\omega = \mathcal{O}(1), \qquad \frac{1}{R_{e2}} = \varepsilon^2 \mu \quad \text{with} \quad \mu = \frac{\nu'_2}{\nu'_1},$$

Multiplying the third component of (9) throughout by ε , we obtain the non-dimensional Navier–Stokes system with Coriolis effects

$$\left(\frac{u}{(1+\varepsilon z)\cos\theta}\frac{\partial}{\partial\varphi} + \frac{v}{1+\varepsilon z}\frac{\partial}{\partial\theta} + \frac{k}{\varepsilon}w\frac{\partial}{\partial z}\right)(u,v,\varepsilon kw) + \frac{1}{1+\varepsilon z}\left(-uv\tan\theta + kuw,u^{2}\tan\theta + kvw, -\varepsilon u^{2} - \varepsilon v^{2}\right) + 2\omega(-v\sin\theta + kw\cos\theta, u\sin\theta, -\varepsilon u\cos\theta) = -\left(\frac{1}{(1+\varepsilon z)\cos\theta}\frac{\partial P}{\partial\varphi}, \frac{1}{1+\varepsilon z}\frac{\partial P}{\partial\theta}, \frac{\partial P}{\partial z}\right) + \left(\frac{\partial^{2}}{\partial z^{2}} + \frac{2\varepsilon}{(1+\varepsilon z)}\frac{\partial}{\partial z}\right)(u,v,\varepsilon kw) + \frac{\varepsilon^{2}\mu}{(1+\varepsilon z)^{2}}\left(\frac{1}{\cos^{2}\theta}\frac{\partial^{2}}{\partial\varphi^{2}} - \tan\theta\frac{\partial}{\partial\theta} + \frac{\partial^{2}}{\partial\theta^{2}}\right)(u,v,\varepsilon kw).$$
(14)

Correspondingly, the equation of mass conservation, (10), becomes

$$\frac{1}{(1+\varepsilon z)\cos\theta} \left\{ \frac{\partial u}{\partial\varphi} + \frac{\partial}{\partial\theta} (v\cos\theta) \right\} + \frac{k/\varepsilon}{(1+\varepsilon z)^2} \frac{\partial}{\partial z} \left\{ (1+\varepsilon z)^2 w \right\} = 0;$$
(15)

In the non-dimensional system of equations (14)–(15), the parameters ($\omega, \mu, k, \varepsilon$) are held fixed, as no approximations were performed.

The shallow-water limit corresponds to the limiting process $\varepsilon \to 0$ and $k/\varepsilon \to 0$ (see the discussion in [19]). This regime ignores the vertical velocity component and, neglecting wave perturbations

by setting h = 0, the flow dynamics being governed by the horizontal flow components u and v, subject to the nonlinear system

$$\left(\frac{u}{\cos\theta}\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta}\right)u - uv\tan\theta - 2\omega v\sin\theta = -\frac{1}{\cos\theta}\frac{\partial P}{\partial\varphi} + \frac{\partial^2 u}{\partial z^2},\tag{16}$$

$$\left(\frac{u}{\cos\theta}\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta}\right)v + u^{2}\tan\theta + 2\omega u\sin\theta = -\frac{\partial P}{\partial\theta} + \frac{\partial^{2}v}{\partial z^{2}},$$
(17)

$$\frac{\partial u}{\partial \varphi} + \frac{\partial}{\partial \theta} (v \cos \theta) = 0, \tag{18}$$

which features Coriolis terms, viscous terms and the horizontal pressure gradients, while the boundary conditions (11)–(13) simplify to

$$P = \overline{P}_{s}(\varphi, \theta),$$

$$\frac{\partial u}{\partial z} = \tau_{1}(\varphi, , \theta), \quad \text{on} \quad z = 0,$$

$$\frac{\partial v}{\partial z} = \tau_{2}(\varphi, \theta),$$
(19)

(u, v) decays rapidly below z = 0. (20)

since setting w = 0 and h = 0 makes (12) irrelevant. The system (16)–(20) captures the physical idea that the perturbation \overline{P}_s of the hydrostatic pressure and the shear stresses τ_1 and τ_2 , at the ocean's surface z = 0, are needed to produce a consistent solution for the wind-drift horizontal current (u, v). To gain insight into the structure of this system, let us note that (18) ensures (see the discussion in [23]) the existence of a stream function in spherical coordinates, $\psi(\varphi, \theta, z)$, with

$$u = -\psi_{\theta}, \quad v = \frac{1}{\cos\theta}\psi_{\varphi};$$
 (21)

see [23] for a proof of the fact that the particle paths for steady flow on the surface of a sphere are the level sets $\psi = \text{constant}$. Taking advantage of (21), we can eliminate the pressure between Equations (16) and (17) to derive (see [19]) the vorticity equation

$$\left(\psi_{\varphi}\frac{\partial}{\partial\theta} - \psi_{\theta}\frac{\partial}{\partial\varphi}\right)\left(\frac{1}{\cos^{2}\theta}\psi_{\varphi\varphi} - \psi_{\theta}\tan\theta + \psi_{\theta\theta} + 2\omega\sin\theta\right)$$
$$= \cos\theta\left(\frac{1}{\cos^{2}\theta}\psi_{\varphi\varphi} - \psi_{\theta}\tan\theta + \psi_{\theta\theta}\right)_{zz}.$$
(22)

These considerations show that (21) specifies, at leading order, the background flow, provided that the stream function $\psi(\varphi, \theta, z)$ solves the vorticity equation (22) subject to the boundary condition (19) and to the last two constraints in (20); Equations (16)–(17) then determine the associated pressure field, taking also into account the boundary condition represented by the first constraint in (19). Let us note that if we ignore the z-dependence (and thus, implicitly, consider an inviscid setting in which the wind forcing plays no role), then (22) simplifies to the ocean gyre model derived recently in [24] as a shallow-water asymptotic solution of Euler's equation in rotating spherical coordinates (with the stipulation that θ stands in [24] for the polar angle, and not for the angle of latitude) and further investigated in [25–29] in the context of the Antarctic Circumpolar Current – the largest ocean current on Earth, flowing clockwise from west to east around Antarctica (see [30,32,37]) so that, due to the lack of any landmass connecting with Antarctica, it keeps the warm ocean waters from lower latitudes away from Antarctica and thus maintains the huge ice sheets encountered near the South Pole (see the discussion in [13,31]).

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3. Main results

Finding the general solution of the equation (22) with the boundary conditions

$$\frac{\partial u}{\partial z} = \tau_1(\varphi, , \theta), \quad \text{on} \quad z = 0, \quad (23)$$

$$\frac{\partial v}{\partial z} = \tau_2(\varphi, \theta),$$

$$(u, v)$$
 decays rapidly below $z = 0.$ (24)

where the horizontal velocity field (u, v) is specified by means of (21), is not a realistic task. Instead, we try to take advantage of some special features that are typical for the equatorial region in the Pacific. We restrict our attention to an equatorial zonal band that is symmetric about the Equator, with the angle of latitude θ close to 0 and the angle of longitude φ near φ_0 , with $\varphi_0 \in (8\pi/9, 14\pi/9)$ corresponding to the region between 160 °E and 80 °W. For the twice differentiable functions $\alpha(z)$ and $\beta(z)$, we look for longitude-independent meridional velocity components and we seek a linear dependence of the azimuthal velocity component on the longitude. These features lead (see [19]) to the stream function

$$\psi(\varphi,\theta,z) = \{\varphi\alpha(z) + \beta(z)\} \ln\left[\frac{\cos\theta}{1-\sin\theta}\right] - \omega\left\{\sin\theta + \ln\left[\frac{\cos\theta}{1-\sin\theta}\right]\right\},\tag{25}$$

solution of the governing equation (22), valid for $|\varphi - \varphi_0| < \hat{\varphi}$ and $|\theta| < \hat{\theta}$ with some fixed (and small) values $\hat{\varphi} > 0$ and $\hat{\theta} > 0$. Due to (21), this corresponds to the horizontal velocity field

$$u(\varphi, \theta, z) = -\frac{\varphi\alpha(z) + \beta(z)}{\cos\theta} + \omega \frac{\sin^2\theta}{\cos\theta},$$

$$v(\varphi, \theta, z) = \frac{\alpha(z)}{\cos\theta} \ln\left[\frac{\cos\theta}{1 - \sin\theta}\right],$$
(26)

while the boundary conditions (23) are induced by a wind stress with components

$$\tau_1(\theta,\varphi) = -\frac{\varphi \alpha'(0) + \beta'(0)}{\cos \theta},$$

$$\tau_2(\theta,\varphi) = \frac{\alpha'(0)}{\cos \theta} \ln\left[\frac{\cos \theta}{1 - \sin \theta}\right].$$
 (27)

Rather than imposing (24), we pursue the considerations in [4]) and interpret the solution (26) as being valid in the near-surface ocean region above the thermocline z = -T, where we require the no-stress boundary condition

$$u_z = v_z = 0$$
 on $z = -T$. (28)

The fundamental characteristic of the trade winds in the equatorial Pacific is that, in each hemisphere, they are oriented towards the Equator, blowing westwards with a more stronger westward direction as the Equator ($\theta = 0$) is approached. This property holds for the wind stress specified in (27) if

$$\alpha'(0) < 0 < \varphi \alpha'(0) + \beta'(0)$$
(29)

for all relevant values of the longitude φ , that is, for $-\hat{\varphi} < \varphi - \varphi_0 < \hat{\varphi}$. Note that this orientation of the wind, in combination with the fact that the vanishing of the meridional component of the

Coriolis force, $\omega \sin \theta$, at the Equator prevents it from inducing a deflection from the wind direction (as is typical in Ekman theory at mid-latitudes), leads to a near-surface current that moving westward. This is ensured for the solution (26) if

$$\varphi\alpha(0) + \beta(0) > 0. \tag{30}$$

Moreover, if

$$\alpha(0) > 0, \tag{31}$$

then the meridional flow described by (26) is poleward near the surface. Note that the main features of the near-surface flow in the equatorial Pacific are a westward motion at the surface and a poleward meridional flow (see the discussion in [4]). One can see that a linear or quadratic dependence on the *z*-variable of the azimuthal velocity profile $u(\varphi, \theta, z)$ of type (26) can not accommodate the constraints (29)–(31) and (28). A solution featuring a cubic dependence was provided recently in [19], namely

$$\begin{aligned} \alpha(z) &= a \left(z^2 + 2Tz - \frac{T^2}{3} \right), \\ \beta(z) &= a \varphi_0 \left(\frac{1}{T} z^3 - 3Tz \right), \end{aligned} \qquad -T \le z \le 0,$$
 (32)

for some constant a < 0. This provides an azimuthal flow that is westward near the surface and features a faster eastward jet along the thermocline, such that at any fixed longitude $\varphi \in (\varphi_0, \frac{13}{12}\varphi_0)$, the zonal velocity u strictly decreases from a positive value at the thermocline z = -T (corresponding to an eastward flow) to a negative value at the surface z = 0 (corresponding to a westward flow), and vanishes once above z = -(T/3), having just one inflexion point, to be found between the depth levels z = -(T/3) and z = -(2T/3). The solution (32) captures the main features of the ocean current in the equatorial Pacific: a westward wind-drift overlying the stronger eastward Equatorial Undercurrent (EUC). We now show that we can refine this result, by accommodating also a weaker eastward flow beneath the EUC (see Figure 1). For this, it suffices to choose suitable quintic polynomial expressions for the azimuthal velocity profile.

Theorem 3.1: The quintic polynomial expressions

$$\begin{aligned} \alpha(z) &= \frac{144}{2125} \frac{(195T^2 - 2264)z^5}{T^5} + \frac{36}{245} \frac{(195T^2 - 2264)z^4}{T^4} - \frac{1}{2}z^2 - Tz + 1, \\ \beta(z) &= \varphi_0 \left\{ \frac{18}{2125} \frac{(2285T^2 - 24982)z^4}{T^4} + \frac{1}{4250} \frac{141180T^2 - 1487411}{T^3} z^3 + \left(\frac{3}{340} \frac{1260T^2 - 11531}{T^2} + 1 \right) z^2 + 2Tz \right\} \end{aligned}$$

accommodate an equatorial current profile that features an eastward jet at mid-depth of the near-surface layer above the thermocline z = -T, a weak eastward jet just above the thermocline and a stronger westward jet near the surface z = 0.

Proof: The corresponding azimuthal velocity $u(\varphi, \theta, z)$, obtained from (26), vanishes at z = -(T/8), z = -(2T/3) and z = -(5T/6), giving the profile depicted in the figure. Moreover, since

$$\begin{aligned} \alpha(0) &= 1 > 0, \quad \varphi_0 \alpha(0) + \beta(0) = 1 > 0, \\ \alpha'(0) &= -T < 0, \quad \varphi_0 \alpha'(0) + \beta'(0) = \phi_0 T > 0, \\ \alpha'(-T) &= 0, \quad \beta'(-T) = 0, \end{aligned}$$

all the constraints (29)–(31) and (28) are verified. The considerations preceding the statement yield now the desired result.



Figure 1. Depiction of the vertical profile of the current in the upper 200–400 m of the Pacific Ocean along the Equator: the westward wind-drift current is near the surface z = 0, below it is the eastward EUC that dominates the subsurface flows, with the weaker westward EIC found directly below the EUC, while at great depths there is practically no motion. These are the main features across the Pacific, over 12,000 km (see the data provided in [20]).

Remark 3.2: The theorem shows that it is possible to capture within our modeling framework not only the presence of the eastward EUC and of the westward wind-drift current, but also the presence of the weaker westward EIC, found directly below the EUC (see the figure).

Disclosure statement

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