AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2020.02.003

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ABSTRACT

For two positive numbers d and m, an m-$L(1, d)$-labeling of a graph G is a mapping $f: V(G) \to [0, m]$ such that $|f(u) - f(v)| \geq 1$ if $d(u, v) = 1$, and $|f(u) - f(v)| \geq d$ if $d(u, v) = 2$. The *span* of f is the maximum difference among the numbers assigned by f. The *L(1, d)-labeling number* of G, denoted by $\lambda_{1,d}(G)$, is the minimum span over all $L(1, d)$-labelings of G. In this paper, we determine the $L(1, 2)$-labeling numbers of square of cycles C_n for all values n.

KEYWORDS

$L(1; d)$-labeling number; code assignment; direct product

1. Introduction

For two positive numbers d and m, an m-$L(1, d)$-labeling of a graph G is a mapping $f: V(G) \to [0, m]$ such that $|f(u) - f(v)| \geq 1$ if $d(u, v) = 1$, and $|f(u) - f(v)| \geq d$ if $d(u, v) = 2$. The *span* of f is the maximum difference among the numbers assigned by f. The *L(1, d)-labeling number* of G, denoted by $\lambda_{1,d}(G)$, is the minimum span over all $L(1, d)$-labelings of G.

The wireless computer network is called *Packet Radio Network* (PRN) if this network communicates by radio frequencies. In PRN, there exist two major types of collisions (or interferences), one is *Direct collision*, which caused by the transmission of adjacent stations (computers); another is *Hidden terminal collision*, which caused by distance-two stations that transmit to the same receiving station at the same time.

Suppose direct interference can be ignored since it is weak in a wireless computer network. Bertossi and Bonuccelli [1] introduced an optimal code assignment to avoid the hidden terminal interference. This code assignment problem is equivalent to the $L(0, 1)$-labeling problem if codes are corresponded to labels.

In general, the direct interference cannot be ignored. That is, it needs to avoid hidden terminal collision as well as direct collision. In order to avoid direct interference, any two adjacent stations are required to be assigned different codes, then any two distance-two stations need to be assigned at least d apart codes in order to avoid hidden terminal interference and direct interference, here $d \geq 1$. Then, Jin and Yeh [6] abstracted this code assignment problem to $L(1, d)$-labeling problem with $d \geq 1$.

$L(d, 1)$-labeling numbers of graphs for $d \geq 1$ have been studied in many articles. Please referred to the surveys [2, 16].

By now, many researchers have paid attention to the $L(1, d)$-labeling numbers of graphs for $d \geq 1$ and introduced some results. For example, Niu [8] obtained $L(1, d)$-labeling numbers of paths and cycles. Griggs and Jin [4] studied $L(1, d)$-labeling numbers of lattices (grids). Furthermore, Jayasree and Nicholas [5] introduced $L(1, 2)$-labeling numbers of certain generalizes Petersen graphs and n-star. In [3], the authors introduced the $L(j, k)$-labeling numbers of trees and stars with maximum degree. Lam, Lin and Wu [7] worked on $L(j, k)$-labeling numbers of product of completed graphs. Recently, Shiu and Wu determined $L(1, d)$-labeling numbers of square of paths in [10]. Moreover, the authors also introduced some $L(1, d)$-labeling numbers of graphs in [9, 11–15].

The *kth power* G^k of an undirected graph G is a graph with the same set of vertices and an edge between two vertices when their distance in G is at most k. G^2 is called the *square* of G.

Two labels are *t-separated* if the difference between them is at least t. Similarly, a set of labels is *t-separated* if the distance of any two labels from this set is not less than t.

For any $a \in \mathbb{R}$, $[a]_m \in [0, m)$ denotes the remainder of a upon division of m.

Lemma 1.1. *Let d be a positive number with $1 \leq d$. Suppose G is a graph and H is an induced subgraph of G. Then $\lambda_{1,d}(G) \geq \lambda_{1,d}(H)$.*

Note that Lemma 1.1 is not true if H is not an induced subgraph. For example, graph $K_{1,3}$ is a subgraph but not an induced subgraph of graph K_4 and it is easy to verify that $\lambda_{1,2}(K_{1,3}) = 4 > 3 = \lambda_{1,2}(K_4)$.

Lemma 1.2. [10] *Let d be a positive number with $1 \leq d < 3$. Then $\lambda_{1,d}(P_6^2) \geq \min\{5, 3 + d\}$.*

2. $L(1, 2)$-labeling numbers of square of cycles

In this section, the cycle C_n^2 is represented by $v_0 v_1 \cdots v_{n-1} v_0$. The number of vertices in set S is denoted by $|S|$.

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Theorem 2.1. *Let n be a positive integer. For $3 \le n \le 9$, $\lambda_{1,2}(C_n^2) = n-1$.*

Proof. For $3 \le n \le 9$, any two vertices of C_n^2 are adjacent or distance two apart, this means that labels of vertices in $V(C_n^2)$ should be different from each other. Then $\lambda_{1,2}(C_n^2) \ge n-1$ for $3 \le n \le 9$.

On the other hand, define an $L(1, 2)$-labeling for C_n^2 as follows.

$$f(v_i) = i \text{ for } 0 \le i \le n-1.$$

It is easy to verify that f is an $(n-1)$-$L(1, 2)$-labeling of C_n^2 for $3 \le n \le 9$. Then $\lambda_{1,2}(C_n^2) \le n-1$ for $3 \le n \le 9$. □

Theorem 2.2. *Let n be a positive integer. For $n \ge 12$ and $n \equiv 0 \pmod 6$, $\lambda_{1,2}(C_n^2) = 5$.*

Proof. For $n=6$, any two vertices of C_6^2 are adjacent or distance two apart, then $\lambda_{1,2}(C_6^2) \ge 5$.
For $n \ge 12$, P_6^2 is an induced subgraph of C_n^2, by Lemma 1.1 and 1.2, then $\lambda_{1,2}(C_n^2) \ge \lambda_{1,2}(P_6^2) = 5$.

On the other hand, given an $L(1, 2)$-labeling of C_n^2 for $n \equiv 0 \pmod 6$ as follows.

$$f(v_i) = [i]_6 \text{ where } 0 \le i \le n-1.$$

It is not difficult to verify that f is a 5-$L(1, 2)$-labeling of C_n^2 for $n \equiv 0 \pmod 6$ and $n \ge 12$.
Hence, $\lambda_{1,2}(C_n^2) = 5$ for $n \equiv 0 \pmod 6$ and $n \ge 12$. □

For $n \ge 10$, it is easy to verify that the vertices of C_n^2 have the following property.

Property 1. For $0 \le i \le n-1$, vertex v_i is adjacent to vertices v_s and at distance two from u_t, where $s = [i \pm 1]_n$, $[i \pm 2]_n$, $t = [i \pm 3]_n, [i \pm 4]_n$, then $|f(v_i) - f(v_s)| \ge 1$ and $|f(v_i) - f(v_t)| \ge 2$.

Theorem 2.3. $\lambda_{1,2}(C_{10}^2) = 8$.

Proof. Let f be an $L(1, 2)$-labeling of C_{10}^2.

Claim 1. *If v_i, v_j are two different vertices of graph C_{10}^2 and $f(v_i) = f(v_j)$, then $j = [i+5]_{10}$ and $|f(v_k) - f(v_i)| \ge 2$ for $i \ne j \ne k$ and $i, j, k \in \mathbb{Z}_{10}$.*

Proof of Claim 1. Suppose the claim does not hold. That is, if v_i, v_j are two different vertices of graph C_{10}^2 and $f(v_i) = f(v_j)$, then we have the following two *cases*.

1. If $j \ne [i+5]_{10}$, then $j = [i+p]_{10}$, where $p \in \{-1, 1, -2, 2, -3, 3, -4, 4\}$. By Property 1, v_i, v_j are adjacent or distance two apart, this means that $|f(v_i) - f(v_j)| \ge 1$, it contradicts to $f(v_i) = f(v_j)$.
2. If $j = [i+5]_{10}$ and $|f(v_k) - f(v_i)| < 2$ for some $k \ne i, j$, this means that v_k should be adjacent to v_i or at least distance three apart from v_i. In graph C_{10}^2, only vertex $v_{[i+5]_{10}}$ is at distance three from v_i. Since $j = [i+5]_{10}$ and $k \ne j$, then v_k should be adjacent to v_i. By the property of C_{10}^2, v_k should be at distance two from vertex v_j, then $|f(v_k) - f(v_j)| \ge 2$. It is a contradiction since $f(v_i) = f(v_j)$.

Hence, the claim is proved. □

Suppose m pairs of vertices have the same labels. This implies that $(10-2m)$ vertices have different labels.

- If $0 \le m \le 4$, by Claim 1, $\lambda_{1,2}(C_{10}^2) \ge 2(m-1) + (10-2m) + 1 = 9$.
- If $m = 5$, by Claim 1, $\lambda_{1,2}(C_{10}^2) \ge 2(5-1) = 8$.

Then, $\lambda_{1,2}(C_{10}^2) \ge 8$.

On the other hand, define an 8-$L(1, 2)$-labeling f for C_{10}^2 as following shows.

$$f(v_0) = f(v_5) = 0,\ f(v_1) = f(v_6) = 2,\ f(v_2) = f(v_7) = 4,\ f(v_3) = f(v_8) = 6,\ f(v_4) = f(v_9) = 8.$$

It is easy to verify that f satisfies the constraints of $L(1, 2)$-labeling. Hence $\lambda_{1,2}(C_{10}^2) = 8$. □

Theorem 2.4. $\lambda_{1,2}(C_{11}^2) = 6$.

Proof. Let f be an $L(1, 2)$-labeling of C_{11}^2. By Property 1, $f(v_0)$ is different from $f(v_i)$ for $i = 1, 2, 3, 4, 7, 8, 9, 10$. Then we have the following cases.

- If all vertices have the different labels, then $\lambda_{1,2}(C_{11}^2) \ge 10$.
- If there exist two vertices which have the same label, by the symmetry of cycle and Property 1, without loss of generality, let $f(v_0) = f(v_5) = a$. Since vertices v_1, v_4, v_8 are at distance two from each other, v_0 is at distance two from v_4, v_8, and v_5 is at distance two from v_1, v_8, then set $\{a, f(v_1), f(v_4), f(v_8)\}$ are 2-separated. Then $\lambda_{1,2}(C_{11}^2) \ge 6$.

On the other hand, given an $L(1, 2)$-labeling f for C_{11}^2 as follows.

$$f(v_0) = f(v_6) = 0,\ f(v_1) = 1,\ f(v_2) = f(v_7) = 2,$$
$$f(v_3) = f(v_9) = 4,\ f(v_4) = 5,\ f(v_5) = f(v_{10}) = 6,\ f(v_8) = 3.$$

It is easy to verify that f is a 6-$L(1, 2)$-labeling of C_{11}^2. Hence $\lambda_{1,2}(C_{11}^2) = 6$. □

Let f be an $L(1, 2)$-labeling of G and $v_i \in V(G)$ for $0 \le i \le n-1$. Then $(f(v_0), f(v_1), \cdots, f(v_{n-1}))$ is called *label sequence*, denoted by $S_f(G)$. The set of labels $\{f(v_i) | v_i \in V(G), i \in \mathbb{Z}_n\}$ is denoted by $f(G)$.

Lemma 2.5. *Let n be a positive integer. For $n \ge 13$ and $n \ne 0 \pmod 6$, $\lambda_{1,2}(C_n^2) \ge 6$.*

Proof. Suppose $\lambda_{1,2}(C_n^2) < 6$. Since P_6^2 is an induced subgraph of C_n^2 for $n \ge 13$, then $\lambda_{1,2}(C_n^2) \ge \lambda_{1,2}(P_6^2) = 5$ by Lemma 1.1 and 1.2. Suppose $\lambda_{1,2}(C_n^2) = \lambda$ if $n \ge 13$ and $n \ne 0 \pmod 6$. Let f be a λ-$L(1, 2)$-labeling of C_n^2.

If a vertex is labeled by 0, the vertex is called *0-vertex*. After removing 0-vertices from cycle C_n, the resulting graph is a disjoint union of paths. More precisely, suppose the number of 0-vertices is m. For convenience, let the cycle $C_n = u_1P_1u_2\cdots u_mP_mu_1$ and $f(u_1) = f(u_2) = \cdots = f(u_m) = 0$, where P_i includes at least one vertex.

Note that each vertex in P_i cannot be labeled by 0.

Claim 2. *For $1 \le i \le m$, P_i must include five vertices.*

Proof of Claim 2. We consider the vertices $u_iP_iu_{i+1}$ for an arbitrary $i \in \{1, 2, 3, \cdots, m\}$. Without loss of generality, let $u_iP_iu_{i+1} = v_0v_1v_2\cdots v_pv_{p+1}$, where $f(v_0) = f(u_i) = f(u_{i+1}) = f(v_{p+1}) = 0$. By Property 1, $p \ge 4$.

1. If $p = 4$, then $f(v_0) = f(v_5) = 0$, by Property 1, $f(v_j) \in [2,6)$, where $j = 1, 2, 3, 4$. According to the symmetry of cycle, without loss of generality, let $f(v_2) \in [2,3) \cup [3,4)$.
   - If $f(v_2) \in [2,3)$, then $f(v_1), f(v_4) \in [3,6)$ since v_1 and v_4 are at distance two from one vertex which is labeled by 0 and adjacent to v_2. Moreover, v_1 and v_4 are distance two apart mutually, then $f(v_1), f(v_4) \in [3,4) \cup [5,6)$. Since v_3 is adjacent to v_1 and v_4, $f(v_3) \in [4,5)$. Since v_6 is at distance two from v_2, v_3 and adjacent to v_5, basing on the above conclusion, $f(v_6) \ge 6$, where index 6 should be taken in modulo n. It contradicts the hypothesis.
   - If $f(v_2) \in [3,4)$, then $f(v_1), f(v_4) \in [2,3) \cup [4,6)$ since v_1 and v_4 are at distance two from one vertex which is labeled by 0 and adjacent to v_2. Moreover, v_1 and v_4 are distance two apart mutually, then one of $f(v_1)$ and $f(v_4)$ lies in $[2,3)$ and $f(v_3) \in [4,6)$. If $f(v_1) \in [2,3)$, by Property 1, we have $f(v_{n-1}) \in [1,2)$ since the length of $[2,3)$ is less than 1. Similarly, by Property 1, $f(v_{n-2}) \in [5,6)$. This forces $f(v_{n-3}) \in [4,5)$ and hence $f(v_{n-4}) \in [3,4)$. Since v_{n-5} is distance two apart from v_{n-1} and v_{n-2}, according to the above conclusion and Property 1, $f(v_{n-5}) \in [3,4)$. It contradicts $f(v_{n-4}) \in [3,4)$ since the length of $[3,4)$ is less than 1 and vertex v_{n-5} is adjacent to vertex v_{n-4}.

     Thus, $f(v_4) \in [2,3)$. Since v_6 is at distance two from v_2, v_3 and $f(v_5) = 0$, $f(v_6) \in [1,2)$. Recall that $f(v_4) \in [2,3)$ and $f(v_3) \in [4,6)$. Since v_7 is at distance two from v_3, v_4, by Property 1, $f(v_7) \in [0,1)$. It contradicts $f(v_5) = 0$ since the length of $[3,4)$ is less than 1 and $d(v_5, v_7) = 1$.
2. If $p \ge 6$, then $f(v_i) \in [1,6)$ for $i \in \{1,2,3,4,5,6\}$. Since vertices $v_1, v_2, \cdots, v_6$ induce a subgraph P_6^2 of C_n^2. By Lemma 1.2, $\lambda_{1,2}(P_6^2) \ge 5$, it is a contradiction since the length of $[1,6)$ is less than 5.

Hence, $p = 5$.That is, the claim is proved.

Since $n \not\equiv 0 \pmod 6$, there exists at least one part P_s, which cannot include 5 vertices, for some $s \in \{1, 2, \cdots, m\}$. It contradicts to Claim 2. Hence, $\lambda_{1,2}(C_n^2) \ge 6$ for $n \not\equiv 0 \pmod 6$. □

Theorem 2.6. $\lambda_{1,2}(C_n^2) = 7$ *for* $n = 15, 16$.

Proof. Suppose $\lambda_{1,2}(C_n^2) = \lambda < 7$. By Lemma 2.5, $\lambda_{1,2}(C_n^2) \ge 6$ for $n = 15, 16$. Let f be a λ-$L(1, 2)$-labeling for graph C_n^2, where $n = 15, 16$. It means that $f(C_n^2) \subseteq [0,7)$ for $n = 15, 16$. For convenience, let $[0,7) = \cup_{i=0}^{6} I_i$ and the subinterval $I_i = [i, i+1)$. It implies that n vertices are divided into seven parts $p_0, p_1, \cdots, p_6$, the labels of vertices in p_i lie in I_i, where $i \in \mathbb{Z}_7$ and $n = 15, 16$. This forces that

$$\text{at least one part contains atleast 3 vertices.} \tag{2.1}$$

By Property 1, any two vertices in set $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ are labeled by 1-separated labels, where $i \in \mathbb{Z}_n$ and the indices should be taken in modulo n. For convenience, let *a-vertex* represent the vertex whose label lies in I_a.

1. If $n = 15$, then

   $$p_i \text{ contains at most three vertices of graph } C_{15}^2, \text{ where } i \in \mathbb{Z}_7. \tag{2.2}$$

   According to conclusion (2.1) and (2.2), at least one part of $p_0, p_1, \cdots, p_6$ contains 3 and only 3 vertices.Let $p_i = \{u_1, u_2, u_3\}$ for some $i \in \mathbb{Z}_7$. Then the i-vertices divide graph C_{15}^2 into three parts. For convenience, let $C_{15}^2 = u_1P_1u_2P_2u_3P_3u_1$. By Property 1, $|P_s| \ge 4$, where $s = 1, 2, 3$. Since $|P_1| + |P_2| + |P_3| = 12$, then $|P_1| = |P_2| = |P_3| = 4$. According to Property 1, $f(P_s) \subseteq [0, i-1) \cup [i+2, 7)$ for $s = 1, 2, 3$ if the intervals exist. Then the labels of 12 vertices lie in four or five subintervals. That is, we can obtain the following conclusion.

   $$\text{If } p_i \text{ contains 3 vertices, then no vertex lies in } p_{i\pm1} \text{ if the parts exist, where } i \in \mathbb{Z}_7. \tag{2.3}$$

   - If $i \ne 0, 6$, then 12 vertices lie in four parts, say p_a, p_b, p_c, p_d. By (2.2), each of p_a, p_b, p_c, p_d contains 3 vertices. According to (2.3), the differences among i, a, b, c, d are at least 2, it is impossible since $i, a, b, c, d \in \mathbb{Z}_7$.
   - If $i = 0$ or 6, then 12 vertices lie in five parts, say p_a, p_b, p_c, p_d, p_e. By (2.2), at least two of p_a, p_b, p_c, p_d, p_e contain 3 vertices. Without loss of generality, let p_a, p_b contain 3 vertices. by (2.3), at least 3 parts contain no vertices. This implies that four parts contains 15 vertices. It forces that at least one part contains at least 4 vertices, it contradicts the conclusion (2.2).
2. If $n = 16$, by Property 1, p_i contains at most four vertices of graph C_{16}^2, where $i \in \mathbb{Z}_7$. Suppose p_i contains four vertices of graph C_{16}^2 for some $i \in \mathbb{Z}_7$. Let $p_i = \{u_1, u_2, u_3, u_4\}$. Then the i-vertices divide graph C_{16}^2 into four parts. For convenience, let $C_{16}^2 = u_1P_1u_2P_2u_3P_3u_4P_4u_1$. By Property 1, $|P_s| \ge 4$ for $s = 1,2,3,4$. It is a contradiction since $|P_1| + |P_2| + |P_3| + |P_4| + 4 \ge 20$ and C_{16}^2 contains only 16 vertices. Thus,

   $$p_i \text{ contains at most three vertices of graph } C_{16}^2, \text{ where } i \in \mathbb{Z}_7 \tag{2.4}$$

   According to conclusion (2.1) and (2.4), at least two parts of $p_0, p_1, \cdots, p_6$ contain 3 and only 3 vertices. Let p_i, p_j contain 3 vertices for some $i, j \in Z_7$ and

$i \neq j$.Without loss of generality, let $p_i = \{v_0, v_5, v_{10}\}$ for some $i \in \mathbb{Z}_7$. For convenience, let $C_{16}^2 = v_0P_1v_5P_2v_{10}P_3v_0$ and $|P_3| = 5$ and $|P_1| = |P_2| = 4$.

- If $i \neq 0, 6$, then $f(P_1), f(P_2) \subseteq [0, i-1) \cup [i+2, 7)$ if the intervals exist. It means that no entry of P_1, P_2 lies in p_{i-1}, p_i, p_{i+1}. This forces that $f(v_1)$, $f(v_2)$, $f(v_3)$ and $f(v_4)$ lie in four different intervals respectively. Let $f(v_1) \in I_a$, $f(v_2) \in I_b$, $f(v_3) \in I_c$, and $f(v_4) \in I_d$, where $\{a, b, c, d\} = \mathbb{Z}_7 \setminus \{i-1, i, i+1\}$. Since $d(v_1, v_4) = 2$, $|a-d| \geq 2$. Similarly, $f(v_6)$, $f(v_7)$, $f(v_8)$, $f(v_9) \in I_a \cup I_b \cup I_c \cup I_d$. Since v_6 is at distance two from v_2 and v_3 and adjacent to v_4, $f(v_6) \notin I_b \cup I_c \cup I_d$, that is, $f(v_6) \in I_a$. This implies that $|a-b| \geq 2$ and $|a-c| \geq 2$. Similar to the above discussion, we can obtain that $f(v_7) \in I_b$ and $f(v_8) \in I_c$. Since v_7 is distance two apart from v_3,v_4 and $d(v_8, v_4) = 2$, $|b-c| \geq 2$, $|b-d| \geq 2$ and $|c-d| \geq 2$. According to the above conclusion, any two numbers of i, a, b, c, d are 2-separated, it is impossible since $i, a, b, c, d \in \mathbb{Z}_7$.
- If $i = 0$ or 6, according to the above case, $j \in \{0, 6\}$. Without loss of generality, let $i = 0, j = 6$. Then, $|p_0| = |p_6| = 3$. It means that $p_0 = \{v_0, v_5, v_{10}\}$ and each part of P_1, P_2 and P_3 contains one vertex which lies in p_6. That is,$f(v_1), f(v_2), f(v_3) \in [2, 7)$. According to the symmetry of the graph, without loss of generality, let one of $f(v_{11})$, $f(v_{12})$ and $f(v_{13})$ lie in I_6.
  - (a) If $f(v_{11}) \in I_6$, by the Property 1, $f(v_6) \in I_6$ and then $f(v_1) \in I_6$. This forces that $f(v_3)$, $f(v_4)$, $f(v_7)$ lie in I_2, I_3, I_4 separately. It is impossible since v_7 is at distance two from vertices v_3 and v_4 and $f(v_7)$ should be 2-separated from $f(v_3)$ and $f(v_4)$ but the length of $I_2 \cup I_3 = [2, 4)$ is less than 2.
  - (b) If $f(v_{12}) \in I_6$, then $f(v_6) \in I_6$ or $f(v_7) \in I_6$.
    - Suppose $f(v_6) \in I_6$. Then $f(v_1) \in I_6$. It forces that $f(v_2), f(v_3), f(v_4) \in I_2 \cup I_3 \cup I_4$. According to the above conclusion, we have that $f(v_7) \in I_5$ and then $f(v_{15}) \in I_1$. That is, $f(v_2) \in I_2$, $f(v_3) \in I_3$, $f(v_4) \in I_4$. It forces $f(v_{11}) \in I_3$. Since vertex v_{14} is at distance two from vertices v_1, v_2, v_{11} and $f(v_1), f(v_2), f(v_{11}) \in I_3 \cup I_4 \cup I_6, f(v_{14}) \in I_0 \cup I_1$. It is impossible since v_{14} is adjacent to vertices v_0, v_{15} whose labels lie in I_0 and I_1.
    - Suppose $f(v_7) \in I_6$. According to the Property 1, $f(v_2) \in I_6$ or $f(v_1) \in I_6$. If $f(v_2) \in I_6$, then $f(v_3)$, $f(v_4)$, $f(v_6)$, $f(v_8)$, $f(v_9) \in I_2 \cup I_3 \cup I_4$. Since v_6 is at distance two from v_3 and v_9, $f(v_3)$, $f(v_6)$, $f(v_9) \in I_2 \cup I_4$. That is, $f(v_4), f(v_8) \in I_3$. It is a contradiction since $d(v_4, v_8) = 2$ and the length of I_3 is less than 1. Thus, $f(v_1) \in I_6$. Basing on the above conclusion, we have $f(v_3), f(v_4), f(v_8), f(v_9)$, $f(v_{13}) \in I_2 \cup I_3 \cup I_4$. Since $d(v_4, v_8) = d(v_9, v_{13}) = 2$, $f(v_4), f(v_8)$ lie in I_2 and I_4 separately, so do $f(v_9), f(v_{13})$. It means that $f(v_8)$, $f(v_9) \in I_2 \cup I_4$.
      Recall that $f(v_7) \in I_6$ and $f(v_{10}) \in I_0$, this forces $f(v_{11}) \in I_1$. It implies that $f(v_9) \in I_2$, $f(v_8) \in I_4$, and then $f(v_{13}) \in I_4$, $f(v_4) \in I_2$. Since v_{14} is at distance two from v_{11} and v_1, $f(v_{14}) \in I_3$. Similarly, we obtain that $f(v_{15}) \in I_5$. Since v_{15} is distance two apart from v_2 and v_3 which are adjacent to v_4, $f(v_2), f(v_3) \in I_3$. It is impossible since the length of I_3 is less than 1.
  - (c) If $f(v_{13}) \in I_6$, then $f(v_8) \in I_6$ or $f(v_7) \in I_6$.
    - Suppose $f(v_7) \in I_6$. Then $f(v_2) \in I_6$. It forces that $f(v_1), f(v_3), f(v_4), f(v_6) \in I_2 \cup I_3 \cup I_4$. Since $d(v_1, v_4) = 2$, $f(v_1), f(v_4) \in I_2 \cup I_4$. This forces that $f(v_3) \in I_3$. Since v_3 is distance two apart from v_6, $f(v_6) \notin I_2 \cup I_3 \cup I_4$. It is a contradiction since $f(v_6) \in I_2 \cup I_3 \cup I_4$.
    - Suppose $f(v_8) \in I_6$. By the symmetry of the graph and the above case, $f(v_3) \notin I_6$, that is, $f(v_2) \in I_6$. It causes $f(v_1)$, $f(v_4)$, $f(v_6)$, $f(v_9) \in I_2 \cup I_3 \cup I_4$. Since $d(v_1, v_4) = 2$, then $f(v_1)$, $f(v_4)$ lie in I_2 and I_4 separately. Similarly, $f(v_6)$, $f(v_9)$ also lie in I_2 and I_4 separately. Since v_4, v_6 are two adjacent vertices, $f(v_4)$ and $f(v_6)$ are 1-separated. That is, $f(v_4)$ and $f(v_6)$ lie in I_2 and I_4 separately and so do $f(v_9)$ and $f(v_1)$. By the symmetry of the graph, without loss of generality, let $f(v_9) \in I_2$ and $f(v_1) \in I_4$. Since v_{12} is at distance two from v_8, v_9 and adjacent to v_{10}, $f(v_{12}) \in I_4$. Similar to the above discussion, $f(v_{14}) \in I_2$. Moreover, $d(v_{11}, v_{14}) = d(v_{11}, v_8) = 2$ and $d(v_{11}, v_{10}) = d(v_{11}, v_{12}) = 1$, $f(v_{11}) \notin I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_6$. It is a contradiction.

Thus, $\lambda_{1,2}(C_n^2) \geq 7$ for $n = 15$, 16.

On the other hand, given two $L(1, 2)$-labelings f_1 and f_2 for C_{15}^2, C_{16}^2, respectively, as follows.

$$S_{f_1}(C_{15}^2) = (0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, 7);$$
$$S_{f_2}(C_{16}^2) = (0, 1, 2, 3, 4, 5, 6, 7, 0, 1, 2, 3, 4, 5, 6, 7).$$

It is easy to verify that f_1, f_2 are two 7-$L(1, 2)$-labelings of C_{15}^2, C_{16}^2, respectively.
Thus, $\lambda_{1,2}(C_n^2) \leq 7$ for $n = 15$, 16.

Hence, $\lambda_{1,2}(C_n^2) = 7$ for $n = 15$, 16.

Theorem 2.7. *$\lambda_{1,2}(C_n^2) = 6$ for $n \geq 13, n \neq 15, 16$ and $n \not\equiv 0 \pmod 6$.*

Proof. By Lemma 2.5, $\lambda_{1,2}(C_n^2) \geq 6$ for $n \geq 13$ and $n \not\equiv 0 \pmod 6$.

On the other hand, define several $L(1, 2)$-labelings for C_n^2 as follows.

1. For $n \equiv 1 \pmod 6$, given an $L(1, 2)$-labelings f_1 for C_n^2 as follows.
   $f_1(v_i) = [i]_6$ for $0 \leq i \leq n-2$;

$f_1(v_{n-1}) = 6.$

2. For $n \equiv 2 \pmod 6$, given an $L(1, 2)$-labelings f_2 for C_n^2 as follows.

   $$f_2(v_i) = [i]_6 \text{ for } 0 \le i \le n-15;$$

   $$f_2(v_i) = [i-n+14]_7 \text{ for } n-14 \le i \le n-1.$$

3. For $n \equiv 3 \pmod 6$, given an $L(1, 2)$-labelings f_3 for C_n^2 as follows.
   $f_3(v_i) = [i]_6$ for $0 \le i \le n-22$;
   $f_3(v_i) = [i-n+21]_7$ for $n-21 \le i \le n-1$.
4. For $n \equiv 4 \pmod 6$, given an $L(1, 2)$-labelings f_4 for C_n^2 as follows.
   $f_4(v_i) = [i]_6$ for $0 \le i \le n-29$;
   $f_4(v_i) = [i-n+28]_7$ for $n-28 \le i \le n-1$.
5. For $n \equiv 5 \pmod 6$, given an $L(1, 2)$-labelings f_5 for C_n^2 as follows.
   $f_5(v_i) = [i]_6$ for $0 \le i \le n-36$;
   $f_5(v_i) = [i-n+35]_7$ for $n-35 \le i \le n-1$.
6. For $n=17$, given an $L(1, 2)$-labeling f_6 for C_{17}^2 as follows.

   $$S_{f_6}(C_{17}^2) = (0,1,2,4,5,6,0,2,3,4,5,0,1,2,3,4,6).$$

7. When $n=22$, 23, 29, given an $L(1, 2)$-labelings f_7, f_8 and f_9 for C_{22}^2, C_{23}^2 and C_{29}^2, respectively, as follows.

   $$S_{f_7}(C_{22}^2) = (0,1,2,4,5,6,0,2,3,4,6,0,1,2,4,5,6,0,2,3,4,6);$$

   $$S_{f_8}(C_{23}^2) = (0,1,2,3,4,5,6,0,2,3,4,6,0,1,2,4,5,6,0,2,3,4,6);$$

   $$S_{f_9}(C_{29}^2) = (0,1,2,3,4,5,6,0,1,2,4,5,6,0,2,3,4,6,0,1,2,4,5,6,0,2,3,4,6).$$

It is not difficult to verify that f_i satisfies the constraints of 6-$L(1, 2)$-labeling, where $i = 1, 2, \cdots, 9$, here we omit the detail. Then $\lambda_{1,2}(C_n^2) \le 6$ for $n \ge 13$, $n \ne 15, 16$ and $n \ne 0 \pmod 6$.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was completed with the support of National Natural Science Foundation of China10.13039/501100001809 (No.11601391).

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