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On irreducible no-hole $L(2, 1)$-coloring of Cartesian product of trees with paths

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ABSTRACT

An $L(2, 1)$-*coloring* of a graph G is a mapping $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ such that $|f(u) - f(v)| \geq 2$ for all edges uv of G, and $|f(u) - f(v)| \geq 1$ if u and v are at distance two in G. The *span of an* $L(2, 1)$-*coloring* f of G, denoted by span(f), is max $\{f(v) : v \in V(G)\}$. The *span of* G, denoted by $\lambda(G)$, is the minimum span of all possible $L(2, 1)$-colorings of G. If f is an $L(2, 1)$-coloring of a graph G with span k then an integer l is a *hole* in f if $l \in (0, k)$ and there is no vertex v in G such that $f(v) = l$. A *no-hole coloring* is defined to be an $L(2, 1)$-coloring with no hole in it. An $L(2, 1)$-coloring is said to be *irreducible* if the color of none of the vertices in the graph can be decreased and yield another $L(2, 1)$-coloring of the same graph. An *irreducible no-hole coloring* of a graph G, in short *inh-coloring* of G, is an $L(2, 1)$-coloring of G which is both irreducible and no-hole. A graph G is *inh-colorable* if there exists an inh-coloring of it. For an inh-colorable graph G the *lower inh-span* or simply *inh-span* of G, denoted by $\lambda_{inh}(G)$, is defined as $\lambda_{inh}(G) = \min\{\text{span } (f) : f \text{ is an inh-coloring of } G\}$. In this paper, we prove that the Cartesian product of trees with paths are inh-colorable.

KEYWORD
$L(2, 1)$-coloring; no-hole coloring; irreducible coloring; span of a graph; Cartesian product of graphs

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1. Introduction

The channel assignment problem is to assign frequencies to a given group of radio transmitters so that interfering transmitters are assigned frequencies with at least a minimum allowed separation. Griggs and Yeh [3] mentioned that Roberts proposed the problem of efficiently assigning radio channels to transmitters at several locations, using nonnegative integers to represent channels, so that close locations receive different channels, and channels for very close locations are at least two apart such that these channels would not interfere with each other. This problem can be modeled as a graph coloring problem where transmitters are represented by vertices, frequencies are represented by colors (non-negative integers), and based on the proximity of the transmitters and power of transmissions, edges are placed between vertices to represent possible interference. Motivated by this problem, Griggs and Yeh [3] proposed $L(2, 1)$-coloring problem of graphs. An $L(2, 1)$-*coloring* (or *labeling*) of a graph G is a mapping $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ such that $|f(u) - f(v)| \geq 2$ for all edges uv of G, and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. The *span of an* $L(2, 1)$-*coloring* f of a graph G, denoted by span(f), is equal to $\max\{f(v) : v \in V(G)\}$. The *span of a graph* G, denoted by $\lambda(G)$, is equal to $\min\{\text{span}(f) : f \text{ is an } L(2, 1)\text{-coloring of } G\}$. An $L(2, 1)$-coloring of a graph G with span equal to $\lambda(G)$ is called a *span coloring* of G.

We denote the maximum degree of a graph G by $\Delta(G)$, unless otherwise stated. Griggs and Yeh [3] found the span of paths, cycles, and obtained upper and lower bounds for the span of hypercubes. They proved that for a tree T, $\Delta(T) + 1 \leq \lambda(T) \leq \Delta(T) + 2$. Griggs and Yeh [3] also conjectured the following.

Conjecture 1.1 [3]. *For any graph G with $\Delta(G) \geq 2, \lambda(G) \leq \Delta(G)^2$.*

We refer the following proposition and lemma due to Griggs and Yeh [3] in the sequel.

Proposition 1.2 [3]. *For any graph G, $\lambda(G) \geq \Delta + 1$. Further, if $\lambda(G) = \Delta + 1$ then in any span coloring of G, the maximum degree vertices must be colored with 0 (or $\Delta + 1$) and its neighbors must be colored with $2 + i$ (or i), $i = 0, 1, \cdots, \Delta - 1$.*

Lemma 1.3 [3]. *If a graph G contains three vertices with maximum degree $\Delta(G) \geq 2$, and one of them is adjacent to the other two vertices then $\lambda(G) \geq \Delta(G) + 2$.*

For any two graphs G and H, a graph on the vertex set $V(G) \times V(H)$ is called the *Cartesian product* of G and H, denoted by $G \square H$, if vertices (x, y) and (x', y') are adjacent whenever $x = x'$ and $yy' \in E(H)$ or $xx' \in E(G)$ and $y = y'$. Georges et al. [2] obtained a relation between the span of a graph and the path covering number of its complement. In the same paper, they also found the span of the Cartesian product of complete graphs. Whittlesey et al. [19] studied the $L(2, 1)$-coloring of hypercubes and the Cartesian product of paths. Kuo and Yan [9] worked on the $L(2, 1)$-coloring of the Cartesian product of a cycle and a path. Shao and Yeh

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[18] proved that the Griggs and Yeh conjecture is true for the Cartesian product of graphs. The $L(2,1)$-coloring of the Cartesian product of cycles were studied by Jha et al. [6], and Schwarz and Troxell [17]. Then Kim et al. [8] studied the $L(2,1)$-coloring of the Cartesian product of a complete graph and a cycle.

In general, for a non-negative integer d, an $L(d,1)$-coloring of a graph G is a vertex coloring $f : V(G) \to Z^+ \cup \{0\}$ such that $|f(u)-f(v)| \geq d$ for all edges uv of G, and $|f(u)-f(v)| \geq 1$ if $d(u,v)=2$. The *span of an* $L(d,1)$*-coloring* f of a graph G is equal to $\max\{f(v) : v \in V(G)\}$. The $L(d,1)$-span of a graph G, denoted by $\lambda_1^d(G)$, is equal to the minimum span over all possible $L(d,1)$-colorings of G. For $d \geq 2$, a *k-circular* $L(d,1)$*-coloring* of a graph G is a function

$$f : V(G) \to \{0,1,\cdots,k-1\} \text{ such that } |f(u)-f(v)|_k \geq \begin{cases} d \text{ if } d(u,v)=1, \\ 1 \text{ if } d(u,v)=2, \end{cases}$$

where $|x|_k = \min\{|x|, k-|x|\}$ is the circular difference modulo k. For any positive integer k the k-circular $L(d,1)$-coloring was introduced by Heuvel et al. [4]. Karst et al. [7] studied the $L(2,1)$-coloring of the Cartesian Product of a flower and a path. In the sequel we use the following theorem by Karst et al. [7].

Theorem 1.4. [Theorem 2.1, [7]] *Let* $P_n = u_1u_2\cdots u_n$*. Let* f *be a k-circular* $L(3,1)$*-coloring of a graph* G *where* $k \geq 5$*. Let* g *be a coloring of the graph* $G\square P_n$ *defined as* $g(x,u_i) = f(x)+2i-2 \pmod{k}$*, for* $x \in V(G)$*. Then* g *is an* $L(2,1)$*-coloring of* $G\square P_n$ *with* $span(g) \leq k-1$.

For a graph G and an $L(2,1)$-coloring f of it with span k, an integer l is called a *hole* in f if $l \in (0,k)$ and there is no vertex v in G such that $f(v)=l$. An $L(2,1)$-coloring f of a graph G with no hole in it is called a *no-hole coloring* of G. Frequencies are generally used in a block. Fishburn and Roberts [1] introduced no-hole coloring of graphs. An $L(2,1)$-coloring f of a graph G is *reducible* if there exists another $L(2,1)$-coloring g of G such that $g(u) \leq f(u)$ for all vertices $u \in V(G)$ and the strict inequality holds for at least one. Otherwise f is said to be *irreducible*. An irreducible coloring with no hole in it is called *irreducible no-hole coloring* and is referred as an *inh-coloring*. A graph is called *inh-colorable* if there exists an inh-coloring of it. For an inh-colorable graph G, the *lower inh-span* or simply *inh-span* of G, denoted by $\lambda_{inh}(G)$, is defined as $\lambda_{inh}(G) = \min\{\text{span }(f) : f \text{ is an inh-coloring of } G\}$. Irreducibility assures no wastage of color.

Laskar and Villalpando [11] introduced irreducible no-hole coloring of graphs and obtained upper and lower bounds for inh-span of unicyclic graph and triangular lattices. Laskar et al. [10] proved that every tree T different from a star is inh-colorable with $\lambda_{inh}(T) = \lambda(T)$. Jacob et al. [5] studied irreducible no-hole coloring of bipartite graphs and proved that for $n,m \geq 3, \lambda_{inh}(P_n\square P_m) \leq 6$, and equality for $n,m \geq 4$; for $n,m \geq 3, \lambda_{inh}(K_n\square K_m) = mn-1$; and for $n \geq 4, m \geq 2, \lambda_{inh}(K_n\square P_m) = 2n-1$. Mandal and Panigrahi [12] solved some open problems related to irreducible no-hole coloring of graphs. The same authors also studied inh-colorability of hypercubes [13], subdivision of graphs [14], and Cartesian product of a complete graph and a cycle [15].

The following result given in [16] will be useful in the sequel.

Lemma 1.5 [16]. *Let* G *be a graph and* G_1 *be a subgraph of it. If there is an* $L(2,1)$*-coloring* f *of* G *with span* k *which induces an inh-coloring of* G_1 *with span greater than or equal to* $k-1$*, then* G *is inh-colorable and* $\lambda_{inh}(G) \leq k$.

In this paper, we prove that for $n \geq 3$, the Cartesian product of trees with paths P_n are inh-colorable and give upper bounds to the inh-span of these graphs.

2. Our results

We first investigate inh-colorability of the Cartesian product of star graphs $K_{1,m}$ with paths P_n.

Theorem 2.1. *The Cartesian product of star graphs* $K_{1,m}$ *with paths* P_n *are inh-colorable and* $\lambda_{inh}(K_{1,3}\square P_2) = 6, \lambda_{inh}(K_{1,m}\square P_2) = m+2$ *for* $m \geq 4$*. Furthermore for* $m,n \geq 3, \lambda_{inh}(K_{1,m}\square P_n) \leq m+4$ *and the equality holds for* $n \geq 5$.

Proof. Let $P_n = u_1u_2\cdots u_n$. Let for $m \geq 3$, $V(K_{1,m}) = \{v, w_1, w_2, \cdots, w_m\}$ where v is the maximum degree vertex of $K_{1,m}$.

We prove that $\lambda_{inh}(K_{1,3}\square P_2) = 6$. We give an inh-coloring f to $K_{1,3}\square P_2$ with span 6: $f(v,u_1)=0, f(w_1,u_1)=3, f(w_2,u_1)=4, f(w_3,u_1)=5$ and $f(v,u_2)=6, f(w_1,u_2)=1, f(w_2,u_2)=2, f(w_3,u_2)=3$. We can easily check that f is an inh-coloring. Thus $\lambda_{inh}(K_{1,3}\square P_2) \leq 6$. Since $\Delta(K_{1,3}\square P_2)=4, \lambda(K_{1,3}\square P_2) \geq 5$. Let $\lambda(K_{1,3}\square P_2)=5$ and g be an $L(2,1)$-coloring of $K_{1,3}\square P_2$ with span 5. From Proposition 1.2 we conclude that (v,u_1) and (v,u_2) have colors 0 and 5 since they are adjacent maximum degree vertices. We may assume that $g(v,u_1)=0$ and $g(v,u_2)=5$. Hence the vertices $(w_1,u_2),(w_2,u_2)$ and (w_3,u_2) have colors 1, 2, and 3. Then only the colors 3 and 4 are available for the vertices $(w_1,u_1),(w_2,u_1)$ and (w_3,u_1), which leads to a contradiction. Thus $\lambda_{inh}(K_{1,3}\square P_2) \geq \lambda(K_{1,3}\square P_2) \geq 6$ and so we get $\lambda_{inh}(K_{1,3}\square P_2)=6$.

For $m \geq 4$ we give an inh-coloring f_1 of $K_{1,m}\square P_2$ with span $m+2$ as follows: $f_1(v,u_1)=0, f_1(w_i,u_1)=i+1$ for $1 \leq i \leq m$ and $f_1(v,u_2)=m+2, f_1(w_1,u_2)=m, f_1(w_i,u_2)=i-1$ for $2 \leq i \leq m$. We check that f_1 is an inh-coloring of $K_{1,m}\square P_2$ for $m \geq 4$. Thus we get $\lambda_{inh}(K_{1,m}\square P_2) \leq m+2$. Since $\Delta(K_{1,m}\square P_2)=m+1, \lambda_{inh}(K_{1,m}\square P_2) \geq \lambda(K_{1,m}\square P_2) \geq m+2$. Hence $\lambda_{inh}(K_{1,m}\square P_2)=m+2$.

Now we consider the case $m,n \geq 3$. We give an $L(3,1)$-coloring f_2 of $K_{1,m}$ with span $m+2$ as follows: $f_2(v)=0, f_2(w_i)=i+2$ for $1 \leq i \leq m$. Then f_2 is a $m+5$-circular $L(3,1)$-coloring. From f_2 we construct an $L(2,1)$-coloring f_2' of $K_{1,m}\square P_n$ as $f_2'(x,u_i) = f_2(x)+2i-2 \pmod{(m+5)}$, for $x \in V(K_{1,m})$. Then span $(f_2') = m+4$. Consider the subgraph G of $K_{1,m}\square P_n$ induced on the vertex set

$\{(v,u_1),(v,u_2),(v,u_3),(w_i,u_1),(w_i,u_2),(w_{m-1},u_3),(w_m,u_3):1\le i\le m\}$. We check that f_2' induces an irreducible coloring of G with span $m+4$. We have $f_2'(v,u_1)=0, f_2'(w_m,u_3)=1, f_2'(v,u_2)=2, f_2'(w_1,u_1)=3, f_2'(v,u_3)=4$ and $f_2'(w_i,u_2)=i+4$ for $1\le i\le m$. Hence f_2' induces a no-hole coloring of G. Thus f_2' induces an inh-coloring of G with span $m+4$. Then from Lemma 1.5, for $m,n\ge 3, \lambda_{inh}(K_{1,m}\square P_n)\le m+4$. Now $\Delta(K_{1,m}\square P_n)=m+2$. If $n\ge 5$ then $K_{1,m}\square P_n$ contains a vertex of maximum degree adjacent to two other vertices of maximum degree. Then from Lemma 1.3, $\lambda(K_{1,m}\square P_n)\ge m+4$. Thus we get for $m\ge 3, n\ge 5, \lambda_{inh}(K_{1,m}\square P_n)\ge m+4$. Hence for $m\ge 3, n\ge 5$, $\lambda_{inh}(K_{1,m}\square P_n)=m+4$. □

In the rest of the paper we use greedy $L(d,1)$-coloring of a graph which is given below.

Algorithm 2.2 (Greedy coloring). Let G be a graph whose few vertices might have been colored before.

1. Order the vertices of the given graph as $u_1,u_2,\cdots,u_n$ such that all colored vertices (if any) appear at the beginning of the list.
2. Let u_i be the first uncolored vertex that appears in the list.
3. Color u_i with the smallest possible color k such that no lower indexed neighbor of u_i in the list is colored with $k-d+1, k-d+2, \cdots, k-1, k, k+1, \cdots, k+d-2$ or $k+d-1$ and no lower indexed vertex at distance two from u_i is colored with k.
4. If all the vertices of the graph have received color then stop; otherwise set $i=i+1$ and go to 3.

Theorem 2.3. *Algorithm 2.2 gives an $L(d,1)$-coloring of G if and only if the pre-colored vertices of G satisfy constraints of $L(d,1)$-coloring in the graph G.*

Now we investigate the inh-colorability of the Cartesian product of trees, different from star graphs, with paths.

Theorem 2.4. *Let T be a tree with maximum degree $\Delta\ge 3$. If T has a maximum degree vertex v such that no vertex adjacent to v is a leaf then for any path P_n, $n\ge 3, T\square P_n$ is inh-colorable and $\lambda_{inh}(T\square P_n)\le \Delta+6$.*

Proof. Let $P_n=u_1u_2\cdots u_n$. We give a $(\Delta+7)$-circular $L(3,1)$-coloring f of T as below. We first order the vertex set of T as $V(T)=\{v_1,v_2,\cdots,v_m\}$, where $v_1=v$; and $v_2,v_3,\cdots,v_{\Delta+1}$ are the neighbors of v_1; for $\Delta+2\le i\le 2\Delta+1$, v_i is a neighbor of $v_{i-\Delta}$ and for all $i>1$, v_i has exactly one neighbor in $\{v_1,\cdots,v_{i-1}\}$. This can be done since T a tree. Now we describe an $L(3,1)$-coloring f of T with span less than or equal to $\Delta+4$. We color v_1 with 1 and then the other vertices greedily following Algorithm 2.2. Since each v_i, $2\le i\le m$, is adjacent to only one v_j, $j<i$, and is distance two away from at most $\Delta-1$ vertices v_k, $k<i$, there are at most $\Delta+4$ colors that can not be used by v_i. Hence at least one color in $[0,\Delta+4]$ is available for v_i. Thus span $(f)\le\Delta+4$ and we check that f is a $(\Delta+7)$-circular $L(3,1)$-coloring. Note that, $f(v_1)=1$, $f(v_i)=i+2$ for $2\le i\le\Delta+1$, and $f(v_i)=0$ for $\Delta+2\le i\le 2\Delta+1$. From f we construct an $L(2,1)$-

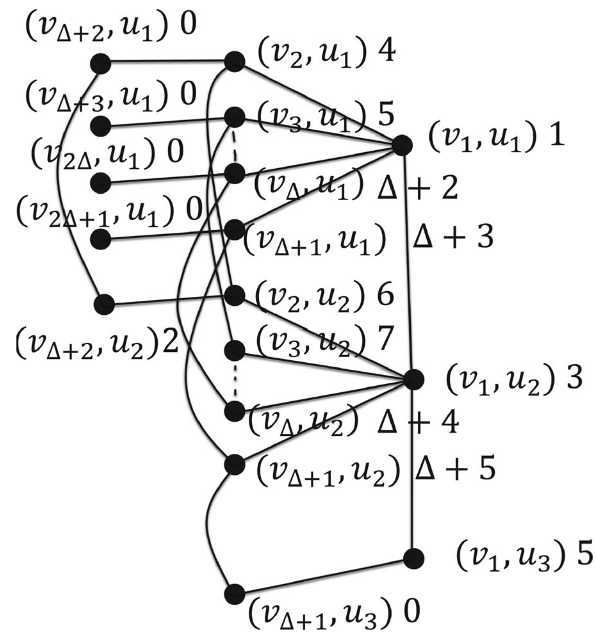

Figure 1. The coloring g of G_1.

coloring g of $T\square P_n$ as $g(x,u_i)=f(x)+2i-2 \pmod{(\Delta+7)}$, for $x\in V(T)$. Then span $(g)\le\Delta+6$.

Consider the subgraph G_1 of $T\square P_n$ induced on the vertex set $\{(v_{i_1},u_1),(v_{i_2},u_2),(v_1,u_3),(v_{\Delta+1},u_3):1\le i_1\le 2\Delta+1, 1\le i_2\le\Delta+2\}$. We prove that g induces an inh-coloring of G_1 with span $\Delta+5$. Figure 1 illustrates the coloring g of G_1. In the following we name vertices and give reasons for which their colors can not be reduced: $(v_{\Delta+1},u_3)$ and $(v_j,u_1), \Delta+2\le j\le 2\Delta+1$ because g assigns the color 0 to them; (v_1,u_1) because $g(v_1,u_1)=1$ and it is at distance two from the vertex $(v_{\Delta+2},u_1)$ colored with 0; $(v_{\Delta+2},u_2)$ because $g(v_{\Delta+2},u_2)=2$ and it is adjacent to the vertex $(v_{\Delta+2},u_1)$ colored with 0; (v_1,u_2) because $g(v_1,u_2)=3$ and it is adjacent to the vertex (v_1,u_1) colored with 1; $(v_j,u_1), 2\le j\le\Delta+1$, because $g(v_j,u_1)=j+2$ and (v_j,u_1) is adjacent to the vertex (v_1,u_1) colored with 1 and at distance two from the vertex (v_1,u_2) colored with 3 and the vertices (v_k,u_1) colored with $k+2$ for $2\le k\le j-1$; (v_1,u_3) because $g(v_1,u_3)=5$ and it is adjacent to the vertices $(v_{\Delta+1},u_3)$ and (v_1,u_2) colored with 0 and 3, respectively; $(v_j,u_2), 2\le j\le\Delta+1$ because $g(v_j,u_2)=j+4$ and (v_j,u_2) is adjacent to the vertex (v_1,u_2) colored with 3, and at distance two from the vertices $(v_{j+\Delta},u_1),(v_1,u_1)$ and (v_1,u_3) colored with 0, 1, and 5, respectively and the vertices (v_k,u_2) colored with $k+4$ for $2\le k\le j-1$. Thus g induces an irreducible coloring of G_1 with span $\Delta+5$. Since all the colors from 0 to $\Delta+5$ are used g induces an inh-coloring of G_1. Hence from Lemma 1.5, $T\square P_n$ is inh-colorable and $\lambda_{inh}(T\square P_n)\le\Delta+6$. □

Theorem 2.5. *Let T be a tree with maximum degree $\Delta\ge 3$. If T satisfies the condition that each maximum degree vertex is adjacent to a leaf then for any path P_n, $n\ge 3, T\square P_n$ is inh-colorable and $\lambda_{inh}(T\square P_n)\le\Delta+6$.*

Proof. Let $P_n=u_1u_2\cdots u_n$. Let v be a maximum degree vertex of T. We consider three cases depending on the values of Δ.

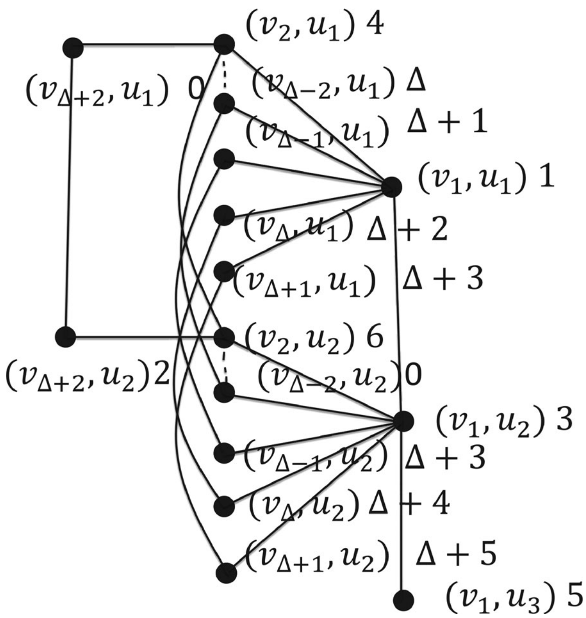

Figure 2. The coloring g_1' of G_2.

Case 1: In this case we take $\Delta \geq 5$. We give a $(\Delta+7)$-circular $L(3, 1)$-coloring f_1 of T as below. We first order $V(T)$ so that $V(T) = \{v_1, v_2, \cdots, v_m\}$, where $v_1 = v$; $v_2, v_3, \cdots, v_{\Delta+1}$ are the neighbors of v_1 such that v_2 is not a leaf; $v_{\Delta+2}$ is a vertex adjacent to v_2; $v_{\Delta-2}$ is a leaf and for all $i > 1$, v_i has exactly one neighbor in $\{v_1, \cdots, v_{i-1}\}$. This can be done since T a tree. Now we describe an $L(3, 1)$-coloring f_1 of T with span less than or equal to $\Delta+4$. We color v_1 as 1, then the other vertices greedily following Algorithm 2.2. We check that span $(f_1) \leq \Delta+4$ and f_1 is a $(\Delta+7)$-circular $L(3, 1)$-coloring. Note that, $f_1(v_1) = 1$, for $2 \leq i \leq \Delta+1, f_1(v_i) = i+2, f_1(v_{\Delta+2}) = 0$. From f_1 we construct an $L(2, 1)$-coloring g_1 of $T \Box P_n$ as $g_1(x, u_i) = f_1(x) + 2i - 2$ (mod $(\Delta+7)$), for $x \in V(T)$. Then span $(g_1) \leq \Delta+6$. Now $g_1(v_{\Delta-2}, u_2) = \Delta+2$. Since $v_{\Delta-2}$ is a leaf of T, the vertices adjacent to $(v_{\Delta-2}, u_2)$ are $(v_1, u_2), (v_{\Delta-2}, u_1)$ and $(v_{\Delta-2}, u_3)$. Since $g_1(v_1, u_2) = 3, g_1(v_{\Delta-2}, u_1) = \Delta$ and $g_1(v_{\Delta-2}, u_3) = \Delta+4$, no vertex adjacent to $(v_{\Delta-1}, u_2)$ is colored with 0 or 1. The vertices at distance two from $(v_{\Delta-2}, u_2)$ are $(v_{\Delta-2}, u_4)$, $(v_1, u_1), (v_1, u_3)$ and the vertices of the form (v_j, u_2) for $2 \leq j \leq \Delta+1$ such that $j \neq \Delta-2$. Since $g_1(v_{\Delta-2}, u_4) = \Delta+6, g_1(v_1, u_1) = 1$, $g_1(v_1, u_3) = 5$ and the colors of the vertices of the form $(v_j, u_2), 2 \leq j \leq \Delta+1, j \neq \Delta-2$ range between 6 and $\Delta+5$, no vertex at distance two from $(v_{\Delta-2}, u_2)$ is colored with 0. So we recolor $(v_{\Delta-2}, u_2)$ with color 0 and get the coloring g_1'.

Consider the subgraph G_2 of $T \Box P_n$ induced on the vertex set $\{(v_{i_1}, u_1), (v_{i_2}, u_2), (v_1, u_3), : 1 \leq i_1 \leq \Delta+2, 1 \leq i_2 \leq \Delta+2\}$. We prove that g_1' induces an inh-coloring of G_2 with span $\Delta+5$. Figure 2 illustrates the coloring g_1' of G_2.In the following we name vertices and give reasons for which their colors can not be reduced: $(v_{\Delta+2}, u_1)$ and $(v_{\Delta-2}, u_2)$ because g_1' assigns the color 0 to them; (v_1, u_1) because $g_1'(v_1, u_1) = 1$ and it is at distance two from the vertex $(v_{\Delta+2}, u_1)$ colored with 0; $(v_{\Delta+2}, u_2)$ because $g_1'(v_{\Delta+2}, u_2) = 2$ and it is adjacent to the vertex $(v_{\Delta+2}, u_1)$ colored with 0; (v_1, u_2) because $g_1'(v_1, u_2) = 3$ and it is adjacent to the vertex (v_1, u_1) colored with 1; (v_j, u_1), $2 \leq j \leq \Delta+1$, because $g_1'(v_j, u_1) = j+2$ and (v_j, u_1) is adjacent to the vertex (v_1, u_1) colored with 1 and at distance two from the vertex (v_1, u_2) colored with 3 and the vertices (v_k, u_1) colored with $k+2$ for $2 \leq k \leq j-1$; (v_1, u_3) because $g_1'(v_1, u_3) = 5$ and it is adjacent to the vertex (v_1, u_2) colored with 3 and at distance two from the vertices $(v_{\Delta-2}, u_2)$ and (v_1, u_1) colored with 0 and 1, respectively; (v_j, u_2), $2 \leq j \leq \Delta-3$, because (v_j, u_2) is adjacent to the vertex (v_1, u_2) colored with 3, and at distance two from the vertices $(v_{\Delta-2}, u_2), (v_1, u_1)$ and (v_1, u_3) colored with 0, 1, and 5, respectively and the vertices (v_k, u_2) colored with $k+4$ for $2 \leq k \leq j-1$; $(v_{\Delta-1}, u_2)$ because $g_1'(v_{\Delta-1}, u_2) = \Delta+3$ and $(v_{\Delta-1}, u_2)$ is adjacent to the vertices (v_1, u_2) and $(v_{\Delta-1}, u_1)$ colored with 3 and $\Delta+1$, respectively and at distance two from the vertices $(v_{\Delta-2}, u_2), (v_1, u_1)$ and (v_1, u_3) colored with 0, 1, and 5, respectively and the vertices (v_k, u_2) colored with $k+4$ for $2 \leq k \leq \Delta-3$; (v_Δ, u_2) because $g_1'(v_\Delta, u_2) = \Delta+4$ and (v_Δ, u_2) is adjacent to the vertices (v_1, u_2) and (v_Δ, u_1) colored with 3 and $\Delta+2$, respectively, and at distance two from the vertices $(v_{\Delta-2}, u_2), (v_1, u_1)$ and (v_1, u_3) colored with 0,1 and 5, respectively and the vertices (v_k, u_2) colored with $k+4$ for $2 \leq k \leq \Delta-3$; $(v_{\Delta+1}, u_2)$ because $g_1'(v_{\Delta+1}, u_2) = \Delta+5$ and $(v_{\Delta+1}, u_2)$ is adjacent to the vertices (v_1, u_2) and $(v_{\Delta+1}, u_1)$ colored with 3 and $\Delta+3$, respectively, and at distance two from the vertices $(v_{\Delta-2}, u_2), (v_1, u_1)$ and (v_1, u_3) colored with 0, 1, and 5, respectively and the vertices (v_k, u_2) colored with $k+4$ for $2 \leq k \leq \Delta-3$. Thus g_1' induces an irreducible coloring of G_2 with span $\Delta+5$. Since all the colors from 0 to $\Delta+5$ are used g_1' induces an inh-coloring of G_2. Hence from Lemma 1.5, $T \Box P_n$ is inh-colorable and $\lambda_{inh}(T \Box P_n) \leq \Delta+6$.

Case 2: In this case we take $\Delta = 4$. We give an 11-circular $L(3, 1)$-coloring f_2 of T as below. We first order $V(T)$ so that $V(T) = \{v_1, v_2, \cdots, v_m\}$, where $v_1 = v$; v_2, v_3, v_4, v_5 are the neighbors of v_1 such that v_2 is a leaf, v_5 is not a leaf; v_6 is a neighbor of v_5 and for all $i > 1$, v_i has exactly one neighbor in $\{v_1, \cdots, v_{i-1}\}$. This can be done since T a tree. Now we describe an $L(3, 1)$-coloring f_2 of T with span less than or equal to 8. We color v_1 as 1 and then the other vertices greedily following Algorithm 2.2. We check that span $(f_2) \leq 8$ and f_2 is a 11-circular $L(3, 1)$-coloring. Note that, $f_2(v_1) = 1$, for $2 \leq i \leq 5$, $f_2(v_i) = i+2, f_2(v_6) = 0$. From f_2 we construct an $L(2, 1)$-coloring g_2 of $T \Box P_n$ as $g_2(x, u_i) = f_2(x) + 2i - 2$ (mod 11), for $x \in V(T)$. Then span $(g_2) \leq 10$. Now $g_2(v_2, u_2) = 6$. Since v_2 is a leaf of T, the vertices adjacent to (v_2, u_2) are $(v_2, u_1), (v_2, u_3)$ and (v_1, u_2). Since $g_2(v_2, u_1) = 4, g_2(v_2, u_3) = 8$ and $g_2(v_1, u_2) = 3$, no vertex adjacent to (v_2, u_2) is colored with 0 or 1. The vertices at distance two from (v_2, u_2) are $(v_1, u_1), (v_1, u_3)$, $(v_3, u_2), (v_4, u_2), (v_5, u_2)$ and (v_2, u_4). Since $g_2(v_1, u_1) = 1, g_2(v_1, u_3) = 5, g_2(v_3, u_2) = 7, g_2(v_4, u_2) = 8, g_2(v_5, u_2) = 9$ and $g_2(v_2, u_4) = 10$, no vertex at distance two from (v_2, u_2) is colored with 0. So we recolor the vertex (v_2, u_2) with color 0 and get the coloring g_2'.

Consider the subgraph G_3 of $T \Box P_n$ induced on the vertex set $\{(v_{i_1}, u_1), (v_{i_2}, u_2), (v_1, u_3), (v_5, u_3) : 1 \leq i_1 \leq 6, 1 \leq i_2 \leq 6\}$. We prove that g_2' induces an inh-coloring of G_3

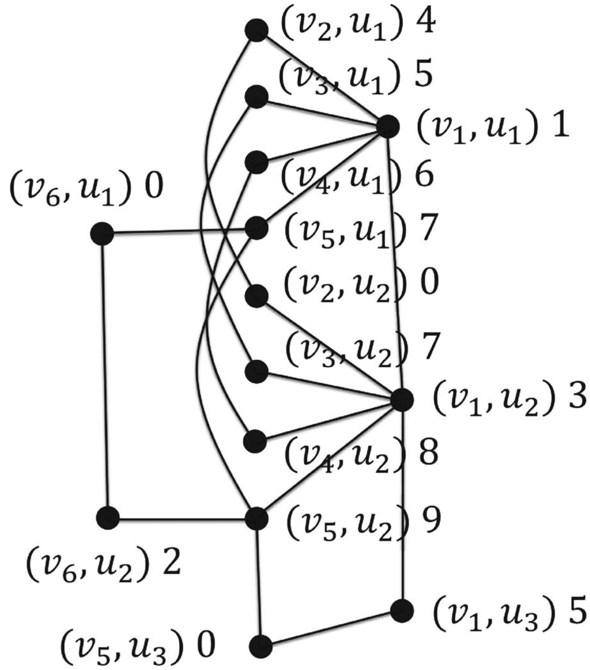

Figure 3. The coloring of g_2' of G_3.

with span 9. Figure 3 illustrates the coloring g_2' of G_3. In the following we name vertices and give reasons for which their colors can not be reduced: $(v_6, u_1), (v_2, u_2)$ and (v_5, u_3) because g_2' assigns the color 0 to them; (v_1, u_1) because $g_2'(v_1, u_1) = 1$ and it is at distance two from the vertex (v_6, u_1) colored with 0; (v_6, u_2) because $g_2'(v_6, u_2) = 2$ and it is adjacent to the vertex (v_6, u_1) colored with 0; (v_1, u_2) because $g_2'(v_1, u_2) = 3$ and it is adjacent to the vertex (v_1, u_1) colored with 1; $(v_j, u_1), 2 \leq j \leq 5$, because $g_2'(v_j, u_1) = j + 2$ and (v_j, u_1) is adjacent to the vertex (v_1, u_1) colored with 1 and at distance two from the vertex (v_1, u_2) colored with 3 and the vertices (v_k, u_1) colored with $k + 2$ for $2 \leq k \leq j - 1$; (v_1, u_3) because $g_2'(v_1, u_3) = 5$ and it is adjacent to the vertices (v_5, u_3) and (v_1, u_2) colored with 0 and 3, respectively; (v_3, u_2) because $g_2'(v_3, u_2) = 7$ and it is adjacent to the vertices (v_1, u_2) and (v_3, u_1) colored with 3 and 5, respectively and at distance two from the vertices (v_2, u_2) and (v_1, u_1) colored with 0 and 1, respectively; (v_4, u_2) because $g_2'(v_4, u_2) = 8$ and it is adjacent to the vertices (v_1, u_2) and (v_4, u_1) colored with 3 and 6, respectively and at distance two from the vertices (v_2, u_2) and (v_1, u_1) colored with 0 and 1, respectively; (v_5, u_2) because $g_2'(v_5, u_2) = 9$ and it is adjacent to the vertices (v_1, u_2) and (v_5, u_1) colored with 3 and 7, respectively and at distance two from the vertices $(v_2, u_2), (v_1, u_1)$ and (v_1, u_3) colored with 0, 1 and 5, respectively. Thus g_2' induces an irreducible coloring of G_3 with span 9. Since all the colors from 0 to 9 are used g_2' induces an inh-coloring of G_3. Hence from Lemma 1.5, $T \square P_n$ is inh-colorable and $\lambda_{inh}(T \square P_n) \leq 10$.

Case 3: In this case we take $\Delta = 3$. We consider two subcases depending on the values of n.

Subcase (i): In this subcase we take $n \geq 4$. We give a 10-circular $L(3, 1)$-coloring f_3 of T as below. We first order $V(T)$ so that $V(T) = \{v_1, v_2, \cdots, v_m\}$, where $v_1 = v$; v_2, v_3, v_4 are the neighbors of v_1 and for all $i > 1$, v_i has exactly one neighbor in $\{v_1, \cdots, v_{i-1}\}$. This can be done since T a tree. Now we describe an $L(3, 1)$-coloring f_3 of T with span less than or equal to 7. We color v_1 as 0 and then the other vertices greedily following Algorithm 2.2. We check that span $(f_3) \leq 7$ and f_3 is a 10-circular $L(3, 1)$-coloring. Note that, $f_3(v_1) = 0$, for $2 \leq i \leq 4, f_3(v_i) = i + 1$. From f_3 we construct an $L(2, 1)$-coloring g_3 of $T \square P_n$ as $g_3(x, u_i) = f_3(x) + 2i - 2 \pmod{10}$, for $x \in V(T)$. Then span $(g_3) \leq 9$.

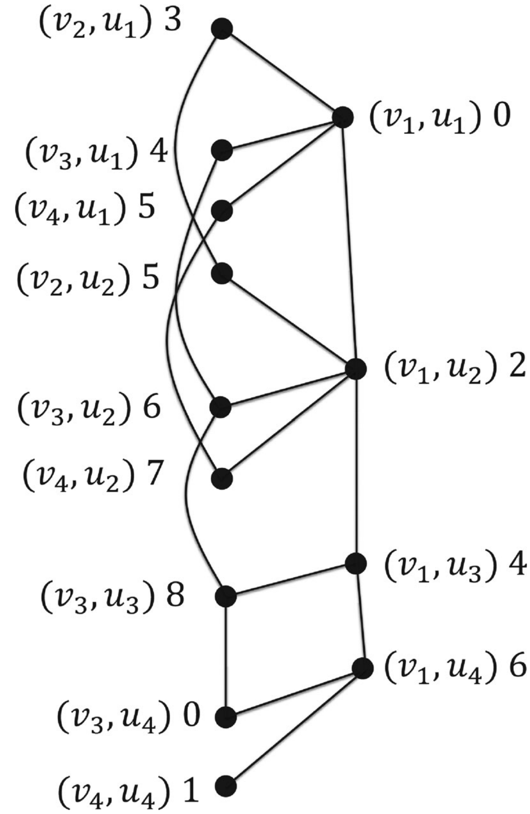

Figure 4. The coloring g_3 of G_4.

Consider the subgraph G_4 of $T \square P_n$ induced on the vertex set $\{(v_{i_1}, u_1), (v_{i_2}, u_2), (v_1, u_3), (v_3, u_3), (v_1, u_4), (v_3, u_4), (v_4, u_4) : 1 \leq i_1 \leq 4, 1 \leq i_2 \leq 4\}$. We prove that g_3 induces an inh-coloring of G_4 with span 8. Figure 4 illustrates the coloring g_3 of G_4. In the following we name vertices and give reasons for which their colors can not be reduced: (v_1, u_1) and (v_3, u_4) because g_3 assigns the color 0 to them; (v_4, u_4) because $g_3(v_4, u_4) = 1$ and it is at distance two from the vertex (v_3, u_4) colored with 0; (v_1, u_2) because $g_3(v_1, u_2) = 2$ and it is adjacent to the vertex (v_1, u_1) colored with 0; $(v_j, u_1), 2 \leq j \leq 4$, because $g_3(v_j, u_1) = j + 1$ and (v_j, u_1) is adjacent to the vertex (v_1, u_1) colored with 0 and at distance two from the vertex (v_1, u_2) colored with 2 and the vertices (v_k, u_1) colored with $k + 1$ for $2 \leq k \leq j - 1$; (v_1, u_3) because $g_3(v_1, u_3) = 4$ and it is adjacent to the vertex (v_1, u_2) colored with 2 and at distance two from the vertex (v_1, u_1) colored with 0; $(v_j, u_2), 2 \leq j \leq 4$, because $g_3(v_j, u_2) = j + 3$ and (v_j, u_2) is adjacent to the vertex (v_1, u_2) colored with 2 and at distance two from the vertices (v_1, u_1) and (v_1, u_3) colored with 0 and 4, respectively and the vertices (v_k, u_2) colored with $k + 3$ for $2 \leq k \leq j - 1$; (v_1, u_4) because $g_3(v_1, u_4) = 6$ and it is adjacent to the

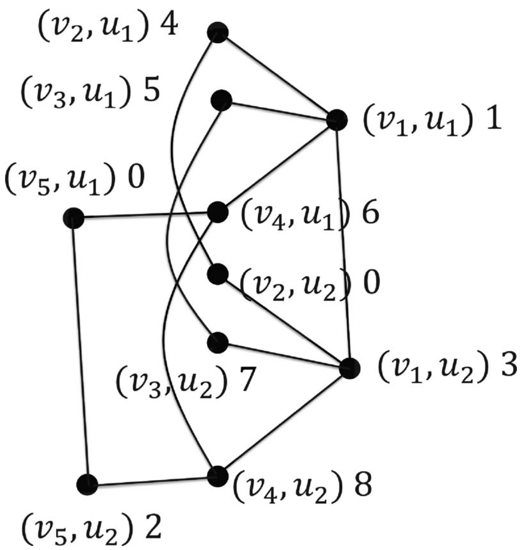

Figure 5. The coloring g'_4 of G_5.

vertices (v_1,u_3) and (v_4,u_4) colored with 4 and 1, respectively; (v_3,u_3) because $g_3(v_3,u_3)=8$ and it is adjacent to the vertices $(v_3,u_4),(v_1,u_3)$ and (v_3,u_2) colored 0, 4, and 6, respectively and at distance two from the vertex (v_1,u_2) colored with 2. Thus g_3 induces an irreducible coloring of G_4 with span 8. Since all the colors from 0 to 8 are used g_3 induces an inh-coloring of G_4. Hence from Lemma 1.5, $T\square P_n$ is inh-colorable and $\lambda_{inh}(T\square P_n)\leq 9$.

Subcase (ii): Here we take $n=3$. We give a 10-circular $L(3, 1)$-coloring f_4 of T as below. We first order $V(T)$ so that $V(T)=\{v_1,v_2,\cdots,v_m\}$, where $v_1=v$; v_2,v_3,v_4 are the neighbors of v_1 such that v_2 is a leaf, v_4 is not a leaf; v_5 is a neighbor of v_4 and for all $i>1$, v_i has exactly one neighbor in $\{v_1,\cdots,v_{i-1}\}$. This can be done since T a tree. Now we describe an $L(3, 1)$-coloring f_4 of T with span less than or equal to 7. We color v_1 as 1, then the other vertices greedily following Algorithm 2.2. Then span $(f_4)\leq 7$ and thus f_4 is a 10-circular $L(3, 1)$-coloring. Note that, $f_4(v_1)=1$, for $2\leq i\leq 4, f_4(v_i)=i+2$ and $f_4(v_5)=0$. From f_4 we construct an $L(2, 1)$-coloring g_4 of $T\square P_3$ as $g_4(x,u_i)=f_4(x)+2i-2$ (mod 10), for $x\in V(T)$. Then span $(g_4)\leq 9$. Now $g_4(v_2,u_2)=6$. Since v_2 is a leaf of T, the vertices adjacent to (v_2,u_2) are $(v_2,u_1),(v_2,u_3)$ and (v_1,u_2), where $g_4(v_2,u_1)=4, g_4(v_2,u_3)=8$ and $g_4(v_1,u_2)=3$. Hence no vertex adjacent to (v_2,u_2) is colored with 0 or 1. The vertices at distance two from (v_2,u_2) are $(v_1,u_1),(v_1,u_3),(v_3,u_2)$ and (v_4,u_2). Since $g_4(v_1,u_1)=1, g_4(v_1,u_3)=5, g_4(v_3,u_2)=7$ and $g_4(v_4,u_2)=8$, no vertex at distance two from (v_2,u_2) is colored 0. We recolor the vertex (v_2,u_2) with color 0 and get the coloring g'_4.

Consider the subgraph G_5 of $T\square P_3$ induced on the vertex set $\{(v_{i_1},u_1),(v_{i_2},u_2), : 1\leq i_1\leq 5, 1\leq i_2\leq 5\}$. We prove that g'_4 induces an inh-coloring of G_5 with span 8. Figure 5 illustrates the coloring g'_4 of G_5. In the following we name vertices and give reasons for which their color can not be reduced: (v_5,u_1) and (v_2,u_2) because g'_4 assigns the color 0 to them; (v_1,u_1) because $g'_4(v_1,u_1)=1$ and it is at distance two from the vertex (v_5,u_1) colored with 0; (v_5,u_2) because $g'_4(v_5,u_2)=2$ and it is adjacent to the vertex (v_5,u_1) colored with 0; (v_1,u_2) because $g'_4(v_1,u_2)=3$ and it is adjacent to the vertex (v_1,u_1) colored with 1; (v_j,u_1), $2\leq j\leq 4$, because $g'_4(v_j,u_1)=j+2$ and (v_j,u_1) is adjacent to the vertex (v_1,u_1) colored with 1 and at distance two from the vertex (v_1,u_2) colored with 3 and the vertices (v_k,u_1) colored with $k+2$ for $2\leq k\leq j-1$; (v_3,u_2) because $g'_4(v_3,u_2)=7$ and it is adjacent to the vertices (v_1,u_2) and (v_3,u_1) colored with 3 and 5, respectively and at distance two from the vertices (v_2,u_2) and (v_1,u_1) colored with 0 and 1, respectively; (v_4,u_2) because $g'_4(v_4,u_2)=8$ and it is adjacent to the vertices (v_1,u_2) and (v_4,u_1) colored with 3 and 6, respectively, and at distance two from the vertices (v_2,u_2) and (v_1,u_1) colored with 0 and 1, respectively. Thus g'_4 induces an irreducible coloring of G_5 with span 8. Since all the colors from 0 to 8 are used, g'_4 induces an inh-coloring of G_5. Hence from Lemma 1.5, $T\square P_3$ is inh-colorable and $\lambda_{inh}(T\square P_3)\leq 9$. □

3. Concluding remarks

In this paper we have proved that the Cartesian product of trees with paths P_n, $n\geq 3$, are inh-colorable. We have given upper bounds to the inh-span of these graphs. In Theorem 2.1 we have found the exact value of $\lambda_{inh}(K_{1,m}\square P_n)$ when $n=2$ or $n\geq 5$. So the following problems remain open.

1. What is the exact value of inh-span of $K_{1,m}\square P_3$ and $K_{1,m}\square P_4$ for $m\geq 3$?
2. Are the Cartesian products of trees different from star graphs with P_2 inh-colorable?
3. What is the exact value of the inh-span of $T\square P_n$ where $n\geq 3$ and T is any tree different from a star?

Declarations of interest

No potential conflict of interest was reported by the author(s).

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