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Factorizations of complete graphs into tadpoles

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ABSTRACT
A tadpole (also a *canoe paddle* or *lollipop*) is a graph that arises from a cycle and a path by gluing a terminal vertex of the path to an arbitrary vertex of the cycle. In this article, we show that all tadpoles factorize the complete graph K_{2n+1} if n is odd. We use methods similar to those used for isomorphic factorizations of complete graphs K_{2n} into spanning trees. In Section 4 of this article, we show that our methods do not work for isomorphic factorizations of K_{2n+1} into tadpoles if n is even.

KEYWORDS
Factorization; complete graphs; unicyclic graphs; tadpoles

1. Introduction

Let G be a simple graph with at most n vertices. A graph H with n vertices has a *decomposition* into subgraphs $G_0, G_1, G_2, \ldots, G_s$ if each edge of H belongs to exactly one G_i. When all subgraphs $G_i, 0 \leq i \leq s,$ are isomorphic to the graph G, we say that H has a *G-decomposition*. If G has exactly n vertices and none of them is isolated, then G is called a *factor* and the decomposition is called a *G-factorization* of H.

Graph factorizations have been extensively studied for many years. Special attention has been paid to G-factorizations. Among graphs whose G-factorizations have been sought, the most popular ones are the obvious candidates—complete graphs and complete bipartite graphs (see, e.g., [2, 11, 12]). In this article, we focus on isomorphic factorizations of complete graphs into tadpoles. *A tadpole* is a graph that arises from a cycle and a path by gluing a terminal vertex of the path to an arbitrary vertex of the cycle.

A simple arithmetic condition shows that only complete graphs with an odd number of vertices can be factorized into unicyclic graphs. A unicyclic graph on n vertices has n edges and a complete graph on n vertices has $\frac{n(n-1)}{2}$ edges. Integer n divides $\frac{n(n-1)}{2}$ if and only if n is odd. Therefore we deal with complete graphs and tadpoles of order $2n+1$ in this article. In particular, we focus on complete graphs and tadpoles with $4k+3$ vertices. We use the idea that when we remove from a unicyclic graph one suitable vertex v of the cycle then we obtain some tree T. And if we show that such a tree T allows a blended ρ-labeling [12], then we know that a tadpole of order $4k+3$ factorizes a complete graph K_{4k+3} if the two edges incident to vertex v satisfy one easy condition. We use T-factorizations of K_{4k+2} in our constructions, therefore we present here some well-known results of such factorizations.

It is a part of graph theory folklore that each graph K_{2n} can be factorized into hamiltonian paths P_{2n}. On the other hand, it is easy to observe that each K_{2n} can be also factorized into double stars; that is, two stars $K_{1,n-1}$ joined by an edge with the endvertices in the centers of both stars. The first attempt to fill the gap between these two extremal cases was Eldergill's thesis [1], where he dealt with symmetric trees. Some classes of non-symmetric trees were examined by Fronček [3, 4], Fronček and Kubesa [7], and by Kubesa [10]. A spanning tree of any diameter that factorizes K_{4k+2} was found by Fronček [7]. The result was completed for every $2n$ by Kovářová in [9]. Among the most general results there is the classification of caterpillars of diameter 4 (in series of papers Fronček [3, 4], Kubesa [10], Kovářová [9]). The classification of caterpillars of diameter 5 was proved through the years in a series of papers and finally completed in [5] by Fronček, Kovář, Kovářová and Kubesa. The paper was followed by an article of Fronček, Kovář, Kubesa, where authors classify all spanning trees with at most four vertices [6]. A T-factorization of K_{2n} for every feasible $\Delta(T)$ (the highest degree of a tree) was given by Kovář and Kubesa [8].

It was shown by Truszczynski in [13] that all tadpoles are gracefull and therefore every tadpole of order n decomposes a complete graph K_{2n+1}. However, nothing has been published about isomorphic factorizations of K_{2n+1} into tadpoles. This article partially fills the void.

2. Definitions and notation

Definition 1. A tadpole $TP(m, n-m)$ is a graph with n vertices that arises from a cycle C_m and a path of length $n - m$

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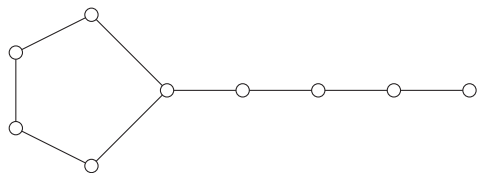

Figure 1. A tadpole $TP(5, 4)$.

by gluing a terminal vertex of the path with an arbitrary vertex of the cycle C_m (Figure 1).

Definition 2. A comet $CO(r, s, t)$ is a tree that arises from three paths of lengths r, s, t. We choose in every path exactly one terminal vertex and we glue all three paths in their chosen terminal vertices (Figure 2).

A *labeling of* G with at most $2n+1$ vertices is an injection $\lambda : V(G) \to S$, where S is often a subset of the set $\{0, 1, ..., 2n\}$. The labels of vertices u, v, denoted $\lambda(u) = i, \lambda(v) = j$, respectively, where $i, j \in S$, induce uniquely the *length* $\ell(e)$ of the edge $e = uv$ with endvertices u, v. All labelings used in this article are generalizations of labelings introduced by Rosa [11, 12].

Remark 1. In this article, we have $S = \{0_0, 1_0, ..., (n-1)_0, 0_1, 1_1, ..., (n-1)_1\}$. To simplify our notation, we often unify vertices with their respective labels. We will say "a vertex a_i" rather than "a vertex x with $\lambda(x) = a_i$". Similarly, we will say "an edge (a_i, b_j)" rather than "an edge xy, where $\lambda(x) = a_i$ and $\lambda(y) = b_j$".

The following definition was introduced in [3].

Definition 3. Let T be a tree with $2n = 4k+2$ vertices, $V(T) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2k+1$. Further, let λ be a bijection, $\lambda : V_i \to \{0_i, 1_i, 2_i, ..., (2k)_i\}, i = 0, 1$. The pure length of an edge (x_i, y_i) with $x_i, y_i \in V_i, i \in \{0, 1\}$ is defined as follows: If $\lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$, then $\ell_{ii}(x_i, y_i) = \min\{|a-b|, 2k+1-|a-b|\}$ for $i = 0, 1$. The mixed length of an edge (x_0, y_1) with $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$, is defined as $\ell_{01}(x_0, y_1) = b - a \bmod (2k+1)$ for $x_0 \in V_0, y_1 \in V_1$. We say that T has a blended ρ-labeling or just blended labeling if

$$\{\ell_{ii}(x_i, y_i) | (x_i, y_i) \in E(T)\} = \{1, 2, ..., k\}$$

for $i = 0, 1$,

$$\{\ell_{01}(x_0, y_1) | (x_0, y_1) \in E(T)\} = \{0, 1, 2, ..., 2k\}.$$

Notice that the lengths of pure and mixed edges are computed differently. Suppose we have the complete graph K_{14} with the vertex labels $0_0, 1_0, ..., 6_0, 0_1, 1_1, ...6_1$. Then both the edges $(1_0, 3_0)$ and $(1_0, 6_0)$ have the pure length 2. On the other hand, the edge $(1_0, 3_1)$ has the mixed length 2 while the edge $(1_1, 3_0)$ has the mixed length 5. Similarly, the edge $(1_0, 6_1)$ has the mixed length 5 while the edge $(6_0, 1_1)$ has the mixed length 2. We also call the pure edges (x_0, y_0) and (x_1, y_1) *pure* 00-*edges* and *pure* 11-*edges*, respectively.

Definition 4. A G-decomposition of a graph H with $2n$ vertices into subgraphs $G_0, G_1, ..., G_s$ is bicyclic if there exists an ordering $(x_0, x_1, ..., x_{n-1}, y_0, y_1, ..., y_{n-1})$ of vertices of H and isomorphisms $\phi_i : G \to G_i$, for $i = 0, 1, ..., s$, such that $\phi_i(x_j) = x_{j+i}$ and $\phi_i(y_j) = y_{j+i}$ for every $j = 0, 1, ..., n-1$, where sums $j+i$ are taken modulo n.

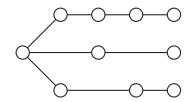

Figure 2. A comet CO(2,3,4).

In [3], it was proved that

Theorem 1. *A tree T of order $4k+2$ with a blended labeling allows a bicyclic T-factorization of K_{4k+2}.*

Let $n = 2k+1$. Notice that since our methods do not work for tadpoles with $4k+1$ vertices we deal only with tadpoles of order $2n+1 = 4k+3$.

3. A factorization of K_{4k+3} into tadpoles

Theorem 2. *Let $TP(m, n-m)$ be a tadpole and let $T = TP(m, n-m) - v$ be a tree, where $\deg(v) = 2$ and vertex v belongs to the cycle of $TP(m, n-m)$ and x, y are neighbors of v. If T bicyclicly factorizes complete graph K_{2n} into factors $T_0, T_1, ..., T_{n-1}$, where $\phi_0(x) \in \{x_0, x_1, ..., x_{n-1}\}$ and $\phi_0(y) \in \{y_0, y_1, ..., y_{n-1}\}$ (ϕ_0 is an isomorphism $\phi_0 : T \to T_0$), then tadpole $TP(m, m-n)$ factorizes complete graph K_{2n+1}.*

Proof. Let $V(K_{2n+1}) = \{v\} \cup \{x_0, x_1, ..., x_{n-1}\} \cup \{y_0, y_1, ..., y_{n-1}\}$ and $K_{2n} = K_{2n+1} - v$. Suppose that a tree $T = TP(m, n-m) - v$ bicyclicly factorizes complete graph K_{2n}, where $T_0, T_1, ..., T_{n-1}$ are factors of this factorization and $\phi_i : T \to T_i$ for $i = 0, 1, ..., n-1$ are corresponding isomorphisms.

We define $\phi_0(x) = x_i$ and $\phi_0(y) = y_j$ for $i, j \in \{0, 1, ..., (n-1)\}$.

Since $\{\phi_j(x) = x_{j+i} : j = 0, 1, ..., n-1\} = \{x_0, x_1, ..., x_{n-1}\}$ and $\{\phi_i(y) = y_{j+i} : i = 0, 1, ..., n-1\} = \{y_0, y_1, ..., y_{n-1}\}$ then the set $\{\phi_j(x)v : j = 0, 1, ..., n-1\} \cup \{\phi_i(y)v : i = 0, 1, ..., n-1\}$ is the set of all edges in K_{2n+1} incident to vertex v.

If we add to each factor $T_i, i = 0, 1, ..., n-1$ vertex v and edges $v\phi_i(x)$, $v\phi_i(y)$ we get a $TP(m, n-m)$-factorization of the complete graph K_{2n+1}. □

Lemma 3. *Tadpoles $TP(3, 4)$, $TP(4, 3)$, $TP(5, 2)$ and $TP(6, 1)$ factorize the complete graph K_7.*

Proof. The factorizations of the complete graph K_7 into tadpoles $TP(m, 7-m)$ for $m = 3, 4, 5, 6$ are in Figures 3–6. □

Lemma 4. *Every tadpole $TP(m, 4k+3-m)$ factorizes the complete graph K_{4k+3} for $m = 3k+3, 3k+4, ..., 4k+2$ and $k \geq 2$.*

Proof. First, we prove that a path P with $4k+2$ vertices admits a blended labeling. We divide a path P in three subsequent parts P_{00}, P_{01} and P_{11}, where P_{00} contains k pure 00-edges, P_{01} contains $2k+1$ mixed edges, and P_{11} contains k pure 11-edges.

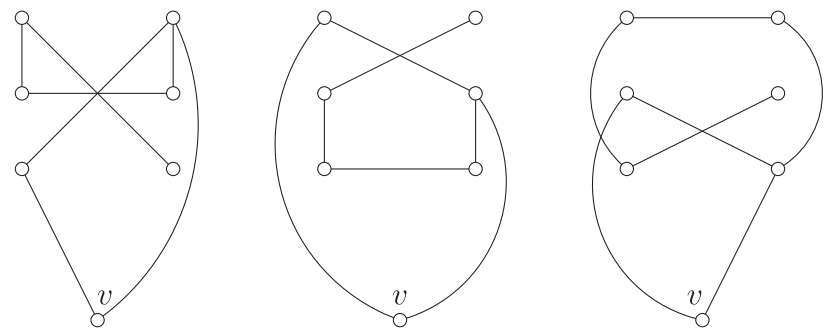

Figure 3. A *TP*(3, 4)-factorization of K_7.

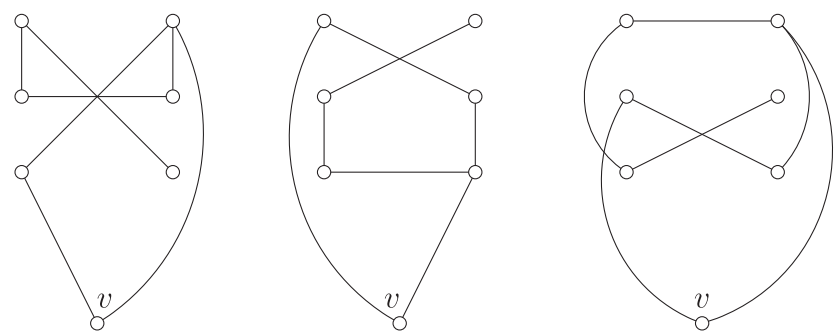

Figure 4. A *TP*(4, 3)-factorization of K_7.

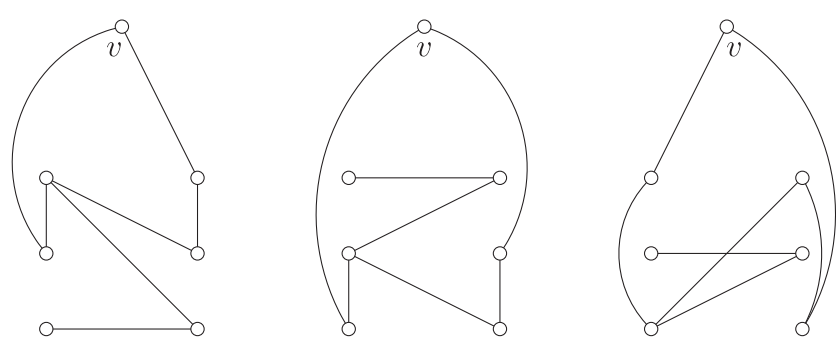

Figure 5. A *TP*(5, 2)-factorization of K_7.

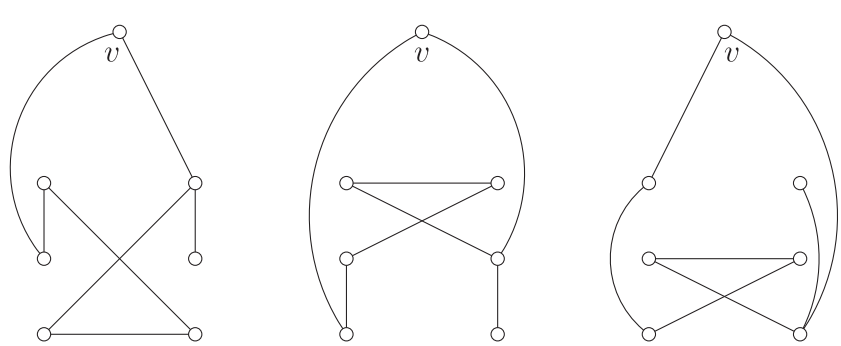

Figure 6. A *TP*(6, 1)-factorization of K_7.

- Let k be odd, $k = 2q + 1$.

$P_{00} = q_0, (q+1)_0, (q-1)_0, (q+2)_0, \ldots, (2q)_0, 0_0, (2q+1)_0,$
$P_{01} = (2q+1)_0, (4q+2)_1, (2q+2)_0, (4q+1)_1, \ldots, (3q+3)_1,$
$(3q+1)_0, (3q+2)_1, (3q+2)_0, (3q+1)_1, (3q+3)_0, \ldots,$
$(4q+1)_0, (2q+2)_1, (4q+2)_0, (2q+1)_1,$
$P_{11} = q_1, (q+1)_1, (q-1)_1, (q+2)_1, \ldots, (2q)_1, 0_1, (2q+1)_1.$

Then:
P_{00} contains pure 00-edges of lengths: $1, 2, 3, \ldots, 2q,\ 2q + 1 = k$,
P_{01} contains mixed edges of lengths: $2q + 1 = k, 2q, 2q - 1, \ldots, 2, 1,\ 0, 4q + 2 = 2k, 4q + 1 = 2k - 1, \ldots, 2q + 4,\ 2q + 3, 2q + 2 = k + 1$,
P_{11} contains pure 11-edges of lengths: $1, 2, 3, \ldots, 2q,\ 2q + 1 = k$.

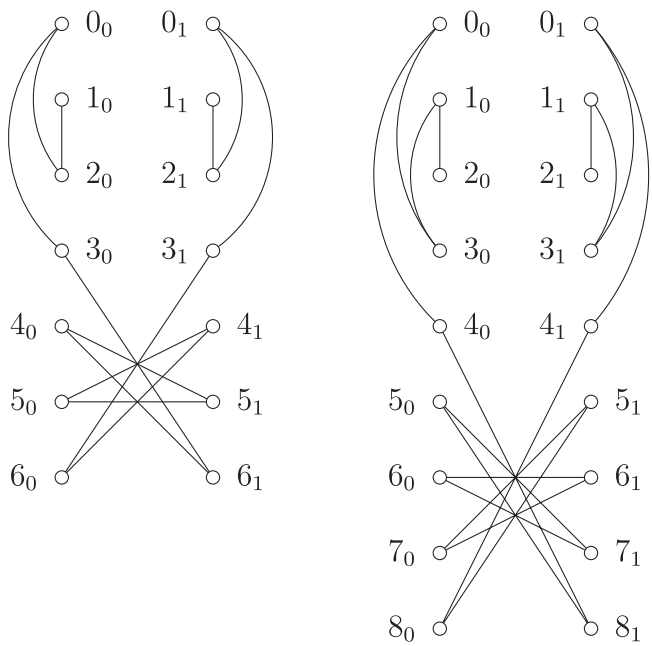

Figure 7. A basic blended labeling of a path P for $k = 2q + 1, q = 1$ and for $k = 2q, q = 2$.

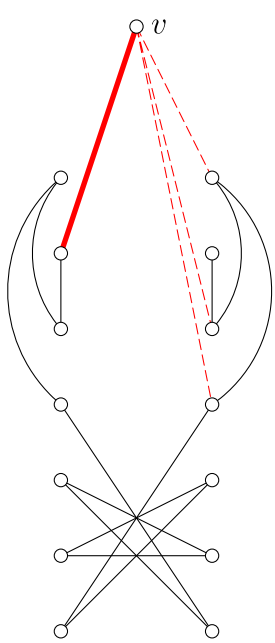

Figure 8. First factor of $TP(m, 15 - m)$-factorization of K_{15} for $m = 12, 13, 14$.

- Let k be even, $k = 2q$.

$$P_{00} = q_0, (q-1)_0, (q+1)_0, (q-2)_0, \ldots, (2q-1)_0, 0_0, (2q)_0,$$
$$P_{01} = (2q)_0, (4q)_1, (2q+1)_0, (4q-1)_1, \ldots, (3q-1)_0,$$
$$(3q+1)_1, (3q)_0, (3q)_1, (3q+1)_0, (3q-1)_1, \ldots, (4q-1)_0,$$
$$(2q+1)_1, (4q)_0, (2q)_1,$$
$$P_{11} = q_1, (q-1)_1, (q+1)_1, (q-2)_1, \ldots, (2q-1)_1, 0_1, (2q)_1.$$

Then:
P_{00} contains pure 00-edges of lengths: $1, 2, 3, \ldots, 2q - 1,\ 2q = k$,
P_{01} contains mixed edges of lengths: $2q = k, 2q - 1, 2q - 2, \ldots, 2, 1,\quad 0, 4q = 2k, 4q - 1 = 2k - 1, \ldots, 2q + \quad 3, 2q + 2,\ 2q + 1 = k + 1$,
P_{11} contains pure 11-edges of lengths: $1, 2, 3, \ldots, 2q - 1, 2q = k$ (Figure 7).

We observe that the path P admits a blended labeling for every k. We call this blended labeling a *basic blended labeling* of path P. The existence of a bicyclic factorization of complete graph K_{4k+2} into factors $P_0, P_1, \ldots, P_{2k}$, where $P_r \cong P$ for $r = 0, 1, \ldots, 2k$ follows from our discussion above. We identify terminal vertex x of a path P with a vertex $q_0 \in V_0$ ($\phi_0(x) = q_0$). Then we identify an internal vertex y of a path P with vertex $i_1 \in V_1$ ($\phi_0(y) = i_1$), where $i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-1, k\}$. If we add to each factor P_r, $r = 0, 1, \ldots, 2k$, the vertex v and edges $(q+r)_0 v, (i+r)_1 v$ (the sums are taken modulo $(2k+1)$) then we get a $TP(m, 4k+3-m)$-factorization K_{4k+3} for $m = 3k+3, 3k+4, \ldots, 4k+2$, because distances of vertices i_1, $i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-1, k\}$, and q_0 are $3k+1, 3k+2, \ldots, 4k$ (Figure 8). □

Lemma 5. *Every tadpole $TP(m, 4k+3-m)$ factorizes the complete graph K_{4k+3} for $m = 3, 4, \ldots, k+2$ and $k \geq 2$.*

Proof.

- Let k be odd, $k = 2q + 1$.
  We remove from a path P on $4k + 2$ vertices with a *basic blended labeling* pure edges $(q_0, (q+1)_0)$ and $(q_1, (q+1)_1)$ of length 1 and mixed edges $((3q+2)_0, (3q+1)_1)$ and $((3q+1)_0, (3q+2)_1)$ of lengths 1 and $2k$. Then we add pure edges $((3q+1)_0, (3q+2)_0)$ and $((3q+1)_1, (3q+2)_1)$ of length 1 and mixed edges $(q_0, (q+1)_1)$ a $((q+1)_0, q_1)$ of lengths 1 and $2k$ (Figure 9).
- Let k be even, $k = 2q$.
  We remove from a path P on $4k + 2$ vertices with a *basic blended labeling* pure edges $(q_0, (q-1)_0)$ and $(q_1, (q-1)_1)$ of length 1 and mixed edges $((3q)_0, (3q+1)_1)$ and $((3q+1)_0, (3q)_1)$ of lengths 1 and $2k$.
  Then we add pure edges $((3q)_0, (3q+1)_0)$ a $((3q)_1, (3q+1)_1)$ of length 1 and mixed edges $((q-1)_0, q_1)$ and $(q_0, (q-1)_1)$ of lengths 1 and $2k$ (Figure 9).

We get another blended labeling of path P for every $k \geq 2$. This implies the existence of bicyclic factorization of the complete graph K_{4k+2} into factors $P_0, P_1, \ldots, P_{2k}$, where $P_r \cong P$ for every $r = 0, 1, \ldots, 2k$. First we identify vertex x of path P with vertex $q_0 \in V_0$ ($\phi_0(x) = q_0$). Then we identify an internal vertex y of a path P with vertex $i_1 \in V_1$ ($\phi_0(y) = i_1$), where $i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots,\ k-1, k\}$. If we add vertex v and edges $((q+r)_0, v)$, $((i+r)_1, v)$ to each factor P_r (the sums are taken modulo $(2k+1)$) then we get a $TP(m, 4k+3-m)$-factorization of K_{4k+3} for $m = 3, 4, \ldots, k+2$, because distances of vertices $i_1, i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-1, k\}$, and q_0 are $1, 2, \ldots, k$ (Figure 10). □

Lemma 6. *Let k be odd. Then every tadpole $TP(m, 4k+3-m)$ factorizes the complete graph K_{4k+3} for $m = k+3, k+4, \ldots,\ 2k+2$.*

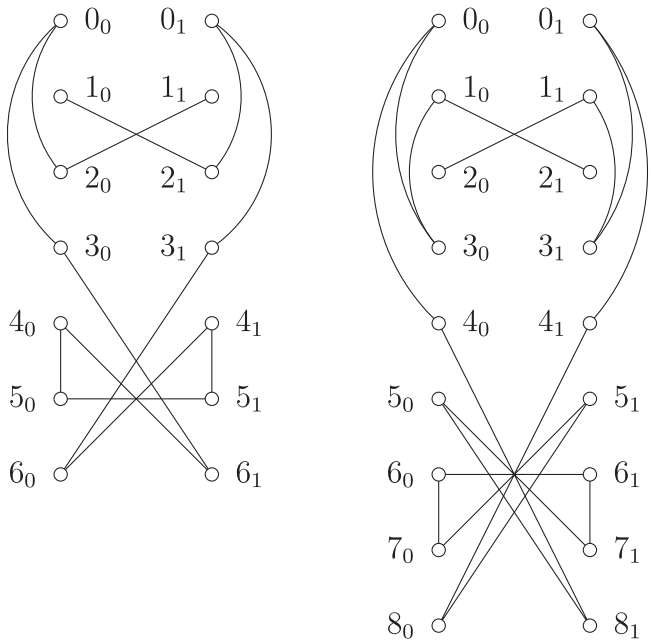

Figure 9. The other blended labeling of a path P for $k = 2q+1, q = 1$ and $k = 2q, q = 2$.

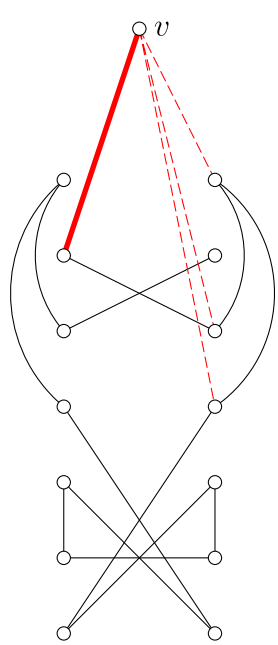

Figure 10. First factor of $TP(m, 15-m)$-factorization of K_{15} for $m = 3, 4, 5$.

Proof. Let $k = 2q+1$.

We remove from path P with a *basic blended labeling* mixed edge $((3q+2)_0, (3q+2)_1)$ of length 0 and then we add mixed edge (q_0, q_1) of length 0 (Figure 11).

With these steps we get a different blended labeling of path P. This implies the existence of a bicyclic factorization of the complete graph K_{4k+2} with factors $P_0, P_1, \ldots, P_{2k}$, where $P_r \cong P$ for every $r = 0, 1, \ldots, 2k$. First we identify vertex x of path P with vertex $(3q+2)_0 \in V_0$ $(\phi_0(x) = (3q+2)_0)$. Then we identify an internal vertex y of path P with vertex $i_1 \in V_1$ $(\phi_0(y) = i_1)$, where $i \in \{0, 1, \ldots, k-1\}$. If we add to each factor P_r vertex v and edges $(3q+2+r)_0 v, (i+r)_1 v$ (the sums are taken modulo

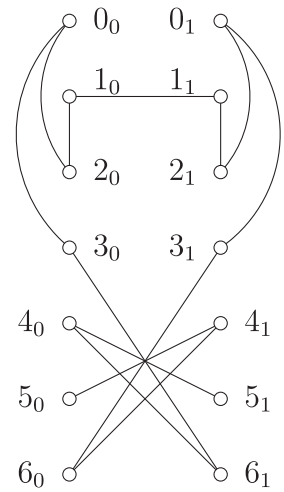

Figure 11. The other blended labeling of a path P for k odd, $k = 3$.

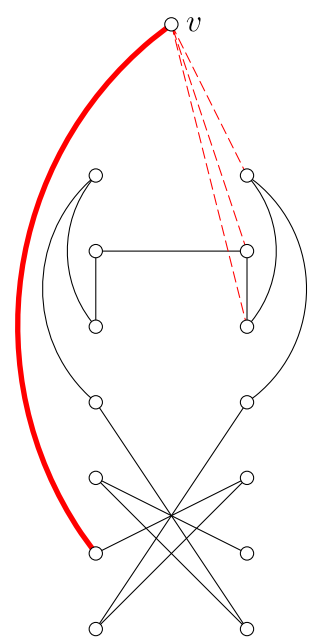

Figure 12. First factor of $TP(m, 15-m)$-factorization of K_{15} for $m = 6, 7, 8$.

$(2k+1))$, we obtain a $TP(m, 4k+3-m)$-factorization of K_{4k+3} for $m = k+3, k+4, \ldots, 2k+2$, because distances of vertices $i_1, i \in \{0, 1, \ldots, k-1\}$ and $(3q+2)_0$ are $k+1, k+2, \ldots, 2k$ (Figure 12). □

Lemma 7. *Let k be odd. Then every tadpole $TP(m, 4k+3-m)$ factorizes the complete graph K_{4k+3} for $m = 2k+4, 2k+5, \ldots, 3k+2$ and $k \geq 3$.*

Proof. Let $k = 2q+1$.

We remove from path P with a *basic blended labeling* pure edges $(q_0, (q+1)_0)$ and $(q_1, (q+1)_1)$ of length 1 and mixed edges $((3q+2)_0, (3q+1)_1), ((3q+1)_0, (3q+2)_1)$ and $((3q+2)_0, (3q+2)_1)$ of lengths $1, 2k$ and 0.

Then we add pure edges $((3q+1)_0, (3q+2)_0)$ and $((3q+1)_1, (3q+2)_1)$ of the length 1 and mixed edges $(q_0, (q+1)_1), ((q+1)_0, q_1)$ and (q_0, q_1) of lengths $1, 2k$ and 0 (Figure 13).

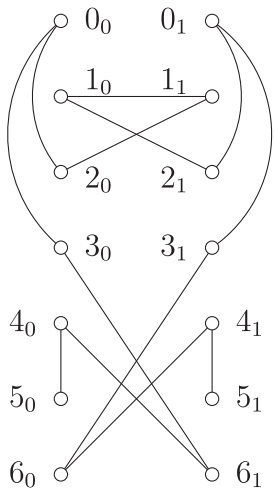

Figure 13. The other blended labeling of path P for k odd, $k = 3$.

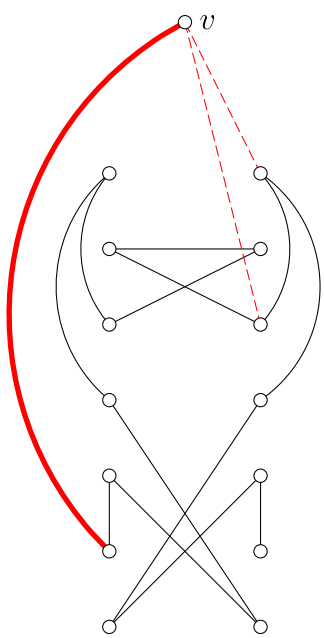

Figure 14. First factor of $TP(m, 15 - m)$-factorization of K_{15} for $m = 10, 11$.

We get another labeling of path P. This implies the existence of a bicyclic factorization of the complete graph K_{4k+2} into factors $P_0, P_1, \ldots, P_{2k}$, where $P_r \cong P$ for every $r = 0, 1, \ldots, 2k$. First we identify vertex x of a path P with vertex $(3q+2)_0 \in V_0$ $(\phi_0(x) = (3q+2)_0)$. Then we identify an internal vertex y of path P with vertex $i_1 \in V_1$ $(\phi_0(y) = i_1)$, where $i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-2, k-1\}$. If we add to each factor P_r vertex v and edges $(3q+2+r)_0 v, (i+r)_1 v$ (the sums are taken modulo $(2k+1)$) then we get a $TP(m, 4k+3-m)$-factorization of K_{4k+3} for $m = 2k+4, 2k+5, \ldots, 3k+2$, because distances of vertices $i_1, i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-2, k-1\}$ and $(3q+2)_0$ are $2k+2, 2k+3, \ldots, 3k$ (Figure 14). □

Lemma 8. *Let k be even. Then every tadpole $TP(m, 4k+3-m)$ factorizes the complete graph K_{4k+3} for $m = k+3, k+4, \ldots, 2k+1$ and $k \geq 2$.*

Proof. Let $k = 2q$.

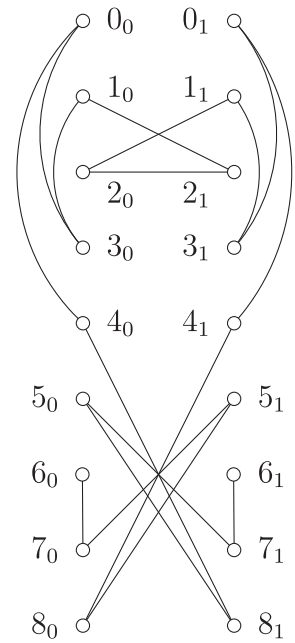

Figure 15. The other labeling of path P for k even, $k = 4$.

We remove from path P with a *basic blended labeling* pure edges $(q_0, (q-1)_0)$ and $(q_1, (q-1)_1)$ of length 1 and mixed edges $((3q)_0, (3q+1)_1), ((3q+1)_0, (3q)_1)$ and $((3q)_0, (3q)_1)$ of lengths $1, 2k$ and 0.

Then we add pure edges $((3q)_0, (3q+1)_0)$ and $((3q)_1, (3q+1)_1)$ of length 1 and mixed edges $((q-1)_0, q_1)$, $(q_0, (q-1)_1)$ and (q_0, q_1) of lengths $1, 2k$ and 0 (Figure 15).

With these steps we get a different blended labeling of path P. That implies the existence of bicyclic factorization of the complete graph K_{4k+2} with factors $P_0, P_1, \ldots, P_{2k}$, where $P_r \cong P$ for every $r = 0, 1, \ldots, 2k$. First we identify vertex x of path P with vertex $(3q)_0 \in V_0$ $(\phi_0(x) = (3q)_0)$. Then we identify an internal vertex y of path P with vertex $i_1 \in V_1$ $(\phi_0(y) = i_1)$, where $i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-2, k-1\}$. If we add to each factor P_r vertex v and edges $(3q+r)_0 v, (i+r)_1 v$ (the sums are taken modulo $(2k+1)$) then we get a $TP(m, 4k+3-m)$-factorization of K_{4k+3} for $m = k+3, k+4, \ldots, 2k+1$, because distances of vertices $i_1, i \in \{0, 1, \ldots, q-2, q-1, q+1, q+2, \ldots, k-2, k-1\}$ and $(3q)_0$ are $k+1, k+2, \ldots, 2k-1$ (Figure 16). □

Lemma 9. *Let k be even. Then every tadpole $TP(m, 4k+3-m)$ factorizes the complete graph K_{4k+3} for $m = 2k+3, 2k+4, \ldots, 3k+2$ and $k \geq 2$.*

Proof. Let $k = 2q$.

We remove from path P with a *basic blended labeling* mixed edge $((3q)_0, (3q)_1)$ of length 0 and then we add mixed edge (q_0, q_1) of length 0 (Figure 17).

We get another blended labeling of path P. This implies the existence of a bicyclic factorization of the complete graph K_{4k+2} into factors $P_0, P_1, \ldots, P_{2k}$, where $P_r \cong P$ for every $r = 0, 1, \ldots, 2k$. First we identify vertex x of path P with vertex $(3q)_0 \in V_0$ $(\phi_0(x) = (3q)_0)$. Then we identify an internal vertex y of path P with vertex $i_1 \in V_1$ $(\phi_0(y) = i_1)$, where $i \in \{0, 1, \ldots, k-1\}$. If we add to each factor P_r vertex

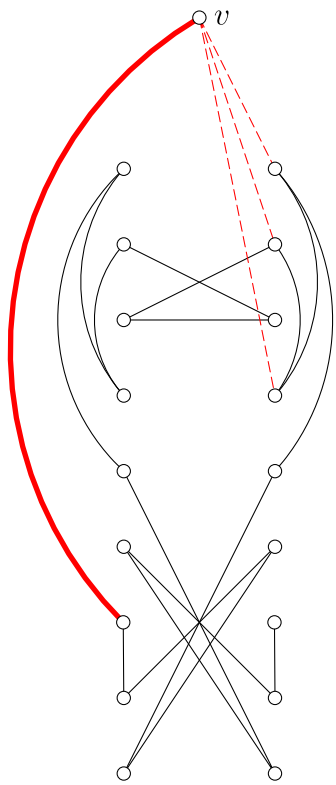

Figure 16. First factor of $TP(m, 19-m)$-factorization of K_{19} for $m = 7, 8, 9$.

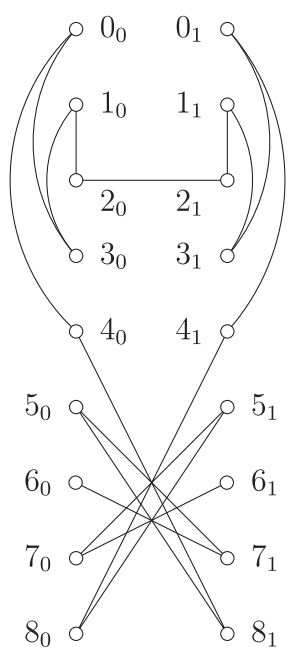

Figure 17. The order blended labeling of path P for k even, $k = 4$.

v and edges $(3q+r)_0 v, (i+r)_1 v$ (the sums are taken modulo $(2k+1)$) then we get a $TP(m, 4k+3-m)$-factorization of K_{4k+3} for $m = 2k+3, 2k+4, \ldots, 3k+2$, because distances of vertices $i_1, i \in \{0, 1, \ldots, k-1\}$ and $(3q)_0$ are $2k+1, 2k+2, \ldots, 3k$ (Figure 18). □

Note that in previous lemmas we did not prove that a tadpole $TP(2k+3, 2k)$ for k odd or $TP(2k+2, 2k+1)$ for k even, respectively, factorizes the complete graph K_{4k+3}. This we prove now.

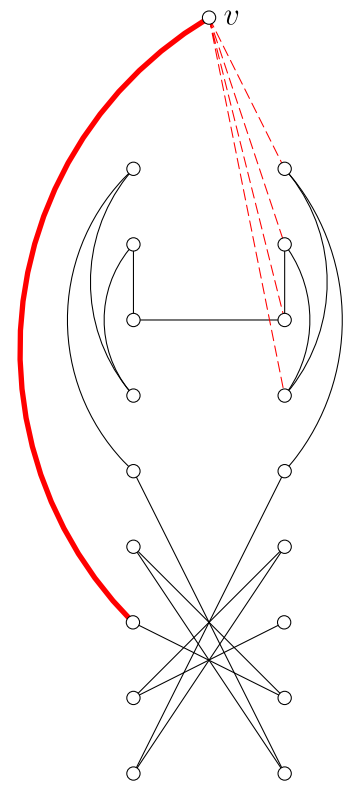

Figure 18. First factor of $TP(m, 19-m)$-factorization of K_{19} for $m = 11, 12, 13, 14$.

Lemma 10. *A tadpole $TP(2k+3, 2k)$ factorizes the complete graph K_{4k+3} for every k odd, $k \geq 3$.*

Proof. Let k be odd. We remove vertex v from the cycle in a tadpole $TP(2k+3, 2k)$ so that we obtain a comet $CO(k, k+1, 2k)$. Two neighbors of vertex v we denote x, y. We show that $CO(k, k+1, 2k)$ has a blended labeling such that a vertex $x \in V_0$ and $y \in V_1$.

Let $k \geq 3$ then comet $CO(k, k+1, 2k)$ contains:

- pure 00-edges of a path $P_{00} = 0_0, (k+1)_0, (2k)_0, (k+2)_0, (2k-1)_0, (k+3)_0, \ldots, (\frac{3k+1}{2}+1)_0, (\frac{3k+1}{2})_0$ of lengths $k, k-1, k-2, \ldots, 1$,
- a single mixed edge and pure 11-edges of a path $P_{11} = 0_0, 0_1, k_1, 1_1, (k-1)_1, 2_1, \ldots, (\frac{k+1}{2}-1)_1, (\frac{k+1}{2})_1$, where the first mixed edge has the length 0 and further pure 11-edges have lengths $k, k-1, k-2, \ldots, 1$ and
- mixed edges of a path $P_{01} = 0_0, (2k)_1, 1_0, (2k-1)_1, 2_0, \ldots, (k+1)_1, k_0$ of lengths $2k, 2k-1, 2k-2, 2k-3, \ldots, 1$.

We see that comet $CO(k, k+1, 2k)$ admits a blended labeling (Figure 19), where terminal vertices x of path P_{00} of length k and y of path P_{11} of length $k+1$, respectively, are identified with vertices $(\frac{3k+1}{2})_0$ and $(\frac{k+1}{2})_1$, respectively. Thus $x \in V_0$ and $y \in V_1$ (Figure 20).

By Theorem 2 tadpole $TP(2k+3, 2k)$ factorizes the complete graph K_{4k+3} for every k odd, $k \geq 3$ (Figure 20). □

Lemma 11. *A tadpole $TP(2k+2, 2k+1)$ factorizes the complete graph K_{4k+3} for every k even, $k \geq 2$.*

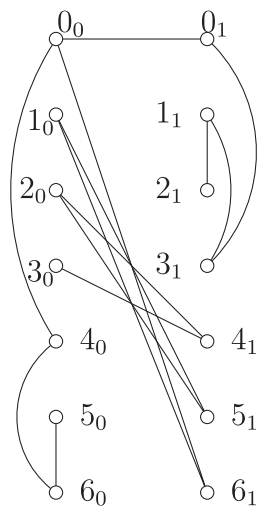

Figure 19. A blended labeling of comet $CO(k, k+1, 2k)$ for k odd, $k=3$.

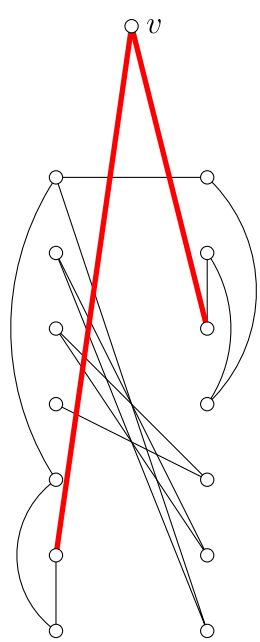

Figure 20. First factor of $TP(2k+3, 2k)$-factorization of K_{4k+3} for $k=3$.

Proof. For $k=2$ there is first factor of $TP(6, 5)$-factorization of K_{11} in Figure 21. Let k be even and $k \geq 4$. We remove vertex v from the cycle C_{2k+2} in a tadpole $TP(2k+2, 2k+1)$ so that we obtain a comet $CO(k-1, k+1, 2k+1)$. Two neighbors of vertex v we denote by x, y. We show that $CO(k-1, k+1, 2k+1)$ has a blended labeling such that a vertex $x \in V_0$ and $y \in V_1$.

Comet $CO(k-1, k+1, 2k+1)$ contains:

- a single mixed edge and pure 11-edges of a path $P_{11} = 0_0, 0_1, k_1, 1_1, (k-1)_1, 2_1, \ldots, (\frac{k+1}{2}-1)_1, (\frac{k+1}{2})_1$, where the first mixed edge has length 0 and further pure 11-edges have lengths $k, k-1, k-2, \ldots, 1$,
- mixed edges and a single pure 00-edge of a path $P_{01} = 0_0, (2k)_1, 1_0, (2k-1)_1, 2_0, \ldots, (k+1)_1, k_0, (2k)_0$, where mixed edges have lengths $2k, 2k-1, 2k-2, 2k-3, \ldots, 1$ and the last pure 00-edge has length k and

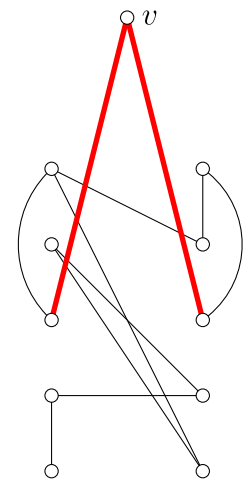

Figure 21. First factor of $TP(6, 5)$-factorization of K_{11}.

- pure 00-edges of a path $P_{00} = 0_0, (k+3q+2)_0, (2k-3q-1)_0, (k+3q+1)_0, (2k-3q-3)_0, (k+3q+3)_0, (2k-3q-2)_0, (k+3(q+1)+2)_0$, where we substitute $q = 0, 1, 2, \ldots$. We see that the first edge $(0_0, (k+3q+2)_0)$ for $q=0$ has the length $k+2-0 = (2k+1)-(k+2) = k-1$ and the rest pure 00-edges have lengths $k-6q-3, k-6q-2, k-6q-4, k-6q-6, k-6q-5$ and $2k-3q-2-(k+3(q+1)+2) = k-6q-7$.

If we substitute parameter q in lengths of pure 00-edges then we obtain

- for $q=0$ lengths of edges $k-3, k-2, k-4, k-6, k-5, k-7$,
- for $q=1$ lengths of edges $k-9, k-8, k-10, k-12, k-11, k-13$,
- for $q=2$ lengths of edges $k-15, k-14, k-16, k-18, k-17, k-19$ and so on.

Now we have to specify how to construct the last part of path P_{00} for a particular k.

- If $k \equiv 0 \bmod 6$ then the last vertex from the sequence in P_{00} is $(2k-3q-3)_0$ for $q = \frac{k-6}{6}$, where $0 \leq q \leq \frac{k-6}{6}$. Therefore the last edge from the sequence in P_{00} is $\left((k+3\frac{k-6}{6}+1)_0, (2k-3\frac{k-6}{6}-3)_0\right)$ of length $2k-3\frac{k-6}{6}-3-\left(k+3\frac{k-6}{6}+1\right) = k-k+6-4 = 2$ and then there follows a pure 00-edge $\left((2k-3\frac{k-6}{6}-3)_0, (2k-3\frac{k-6}{6}-2)_0\right)$ of length $2k-\frac{k-6}{2}-2-2k+\frac{k-6}{2}+3 = 1$.
- If $k \equiv 2 \bmod 6$ then the last vertex from the sequence in P_{00} is again $(2k-3q-3)_0$ for $q = \frac{k-8}{6}$, where $0 \leq q \leq \frac{k-8}{6}$. Therefore the last edge from the sequence in P_{00} is $\left((k+3\frac{k-8}{6}+1)_0, (2k-3\frac{k-8}{6}-3)_0\right)$ of length $2k-3\frac{k-8}{6}-3-\left(k+3\frac{k-8}{6}+1\right) = k-k+8-4 = 4$ and then there

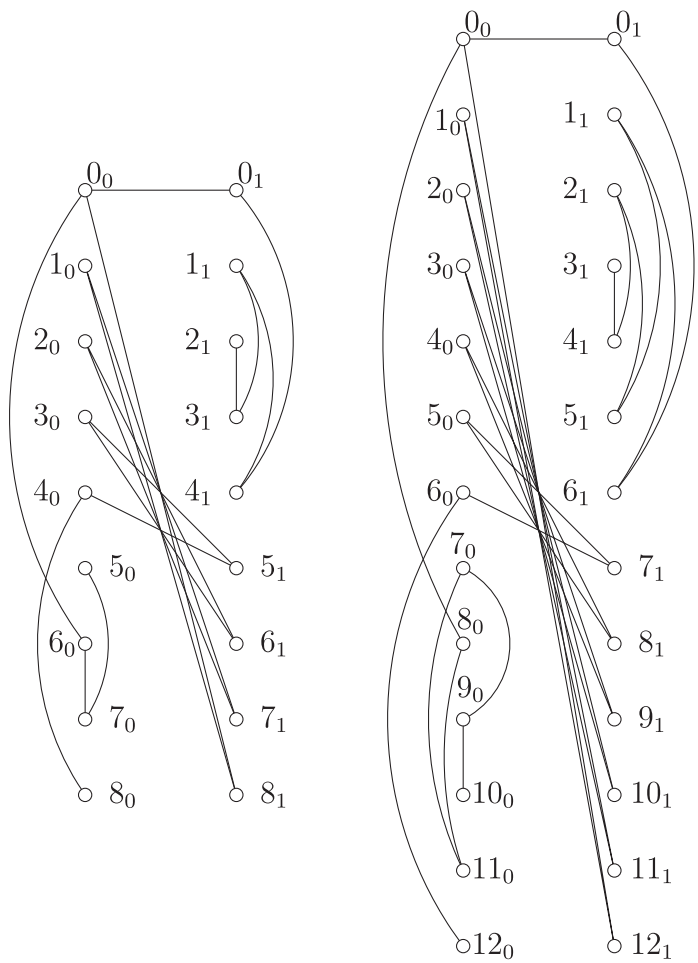

Figure 22. Blended labeling of comet $CO(k-1, k+1, 2k+1)$ for $k=4$ and $k=6$.

follow three pure 00-edge $\left((2k-3\frac{k-8}{6}-3)_0,\ (2k-3\frac{k-8}{6}-4)_0\right)$, $\left((2k-3\frac{k-8}{6}-4)_0, (2k-3\frac{k-8}{6}-2)_0\right)$ and $((2k-3\frac{k-8}{6}-2)_0, (2k-3\frac{k-8}{6}-5)_0)$ of lengths $2k-\frac{k-8}{2}-3-2k+\frac{k-8}{2}+4=1, 2k-\frac{k-8}{2}-2-2k+\frac{k-8}{2}+4=2$ and $2k-\frac{k-8}{2}-2-2k+\frac{k-8}{2}+5=3$.

- If $k \equiv 4 \bmod 6$ then the last vertex from the sequence in P_{00} is $(k+3q+1)_0$ for $q=\frac{k-4}{6}$, where $0 \le q \le \frac{k-4}{6}$. Therefore the last two edges from the sequence in P_{00} are $\left((k+3\frac{k-4}{6}+2)_0, (2k-3\frac{k-4}{6}-1)_0\right)$ and $\left((2k-3\frac{k-4}{6}-1)_0, (k+3\frac{k-6}{6}+1)_0\right)$ of lengths $2k-3\frac{k-4}{6}-1-\left(k+3\frac{k-4}{6}+2\right)=k-k+4-3=1$ and $2k-3\frac{k-4}{6}-1-\left(k+3\frac{k-4}{6}+1\right)=k-k+4-2=2$ (Figure 22).

Recall that k is even, therefore k cannot be congruent to 1, 3, 5 (mod6). Thus, we have covered above all values of k.

We see that comet $CO(k-1, k+1, 2k+1)$ admits a blended labeling (Figure 22), where terminal vertex x of a path of length $k-1$ or terminal vertex y of a path of length $k+1$, belongs to V_0 or V_1, respectively.

Therefore by Theorem 2 a tadpole $TP(2k+2, 2k+1)$ factorizes a complete graph K_{4k+3} for every k even (Figure 23). □

In previous lemmas we prove following:

Theorem 12. *Every tadpole of order $4k+3$ factorizes the complete graph K_{4k+3} for every $k \ge 1$.*

Proof. It follows from Lemmas 3–11. □

4. A factorization of K_{4k+1} into tadpoles

Theorem 13. *Tadpoles $TP(3, 2)$ and $TP(4, 1)$ do not factorize the complete graph K_5.*

Proof. A complement of tadpole $TP(3, 2)$ is tadpole $TP(4, 1)$ and vice versa. □

By providing an example for $k=4$, we illustrate the reasons why the above methods fail for tadpoles of order $4k+1$.

Again, from a tadpole of order $4k+1$ we remove one vertex v from a cycle and suppose that we obtain a path P. Then we look for a suitable labeling that guarantees a bicyclic T-factorization of K_{4k} for some tree T.

For order $4k$ there exists a similar labeling (or a sufficient condition) as for order $4k+2$, namely *swapping labeling*.

Definition 5. A graph G with $4n-1$ edges and with at most $4n$ vertices has a swapping labeling if the following is satisfied. The vertex set $V(G)=V_0 \cup V_1, V_0 \cap V_1 = \emptyset$ and $|V_0|, |V_1| \le 2n$. Let λ be an injection $\lambda : V_i \to \{0_i, 1_i, \ldots,$

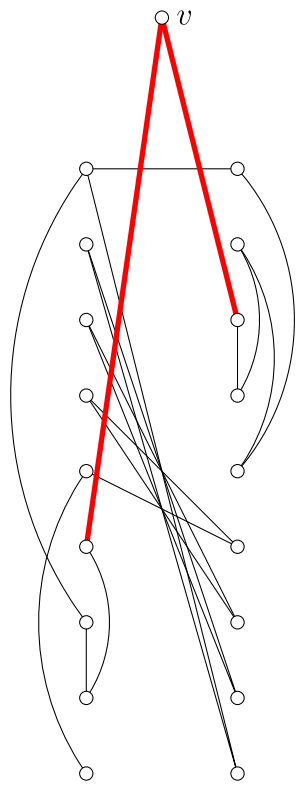

Figure 23. First factor of $TP(2k+2, 2k+1)$-factorization of K_{4k+3} for $k=4$.

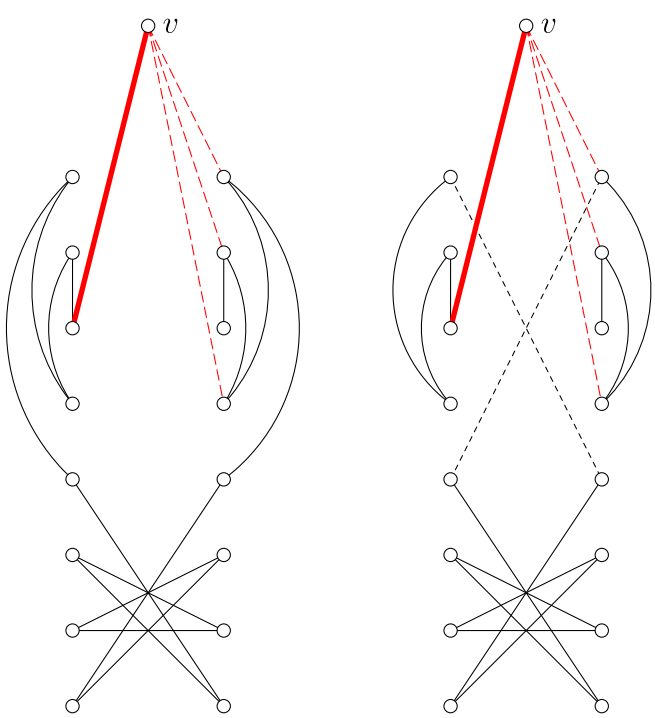

Figure 24. The basic swapping labeling of path P (in black color) of order $4n$ for first n factors and second n factors if $n=4$.

$(2n-1)_i\}$ for $i=0,\ 1$ and let the pure length l_{ii} and the mixed length l_{01} are defined as in Definition 3. Then

- $\{l_{ii}(x_i, y_i) : (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$ for $i = 0,\ 1$,
- there exists an isomorphism ϕ such that G is isomorphic to G', where $V(G') = V(G)$ and $E(G') = (E(G) \setminus \{(r_0, (r+n)_0), (s_1, (s+n)_1)\}) \cup \{(r_0, (k+n)_1), (s_1, (s+n)_0)\}$, $r, s \in \{0, 1, \ldots, 2n\}$ and the sums are taken modulo $2n$,
- $\{l_{01}(x_0, y_1) : (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n\} \setminus \{n\}$.

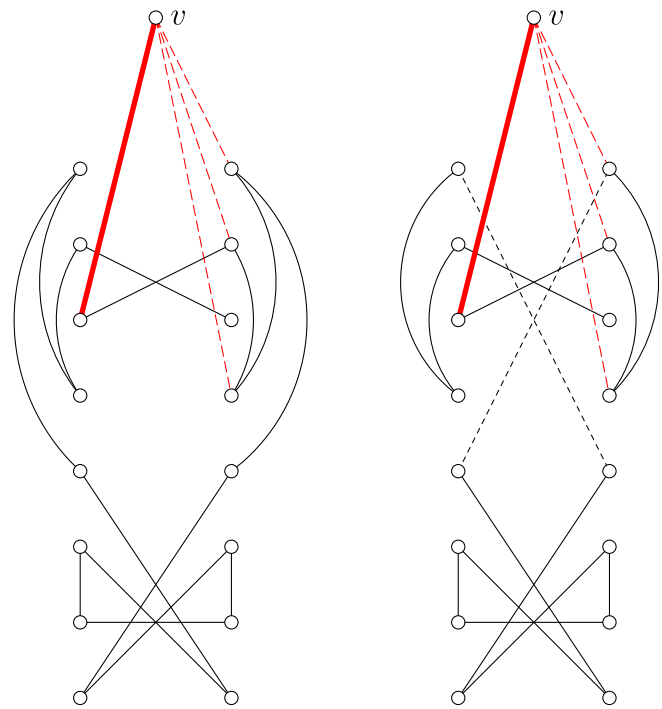

Figure 25. The other swapping labeling of path P (in black color) of order $4n$ for first n factors and second n factors if $n=4$.

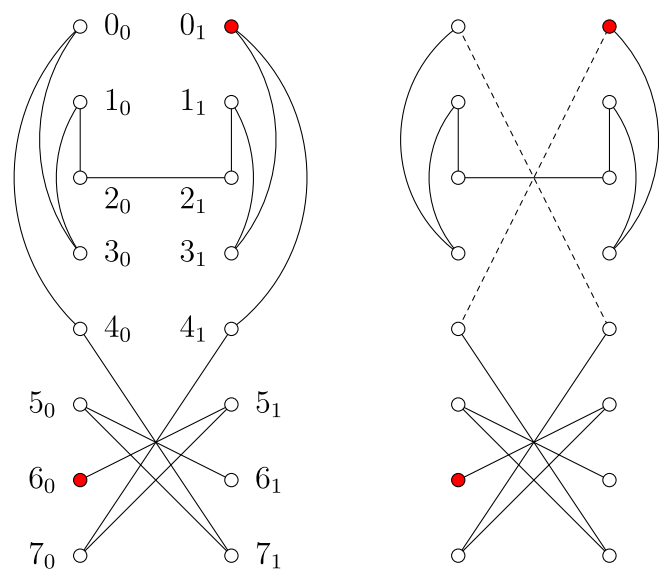

Figure 26. The other swapping labeling of path P of order $4n$, where the distance between vertices $6_0, 0_1$ was changed.

From the previous definition it follows that graph G' arises from graph G by removing pure edges of length n (the longest pure edges) from G and replacing them by mixed edges of length n which are missing in G. Thus, in the first n factors there are both pure edges of length n and in the remaining n factors there are two mixed edges of length n.

In [9], Kovářová proved.

Theorem 14. *If a graph G with $4k-1$ edges and with at most $4k$ vertices allows a swapping labeling, then there exists a bicyclic G-decomposition of a complete graph K_{4k}.*

If we use swapping labelings for paths in Figures 24 and 25 then we can prove that tadpoles $TP(m, 17-m)$ factorize K_{17} for $m=14$, 15, 16 and $m=3$, 4, 5, respectively (Figures 24 and 25). It follows from the fact that if we replace pure edges $(0_0, 4_0)$ and $(0_1, 4_1)$ by mixed edges $(0_0, 4_1)$ and $(0_1, 4_0)$ then the distance between vertices 2_0 and i_1 for $i \in \{0, 1, 3\}$ remains the same. Therefore all factors have cycles of the same length.

It can be probably generalize that all tadpoles $TP(m, 4k+1-m)$ factorize K_{4k+1} for $m = 3k+2, 3k+3, \ldots, 4k$ and $m = 3, 4, \ldots, k+1$, respectively.

While if we use a swapping labeling of path P in Figure 26, then we observe following. If we take (for example) terminal vertex 7_0 and internal vertex 0_1 of a path then they are at distance 4, but after swapping edges the same two vertices have the distance 11. Therefore, in the first 4 factors it will be cycles of length 6 and in remaining factors we have cycles of length 13. We do not obtain an isomorphic factorization.

5. Conclusion

Let k be an arbitrary positive integer. We proved that no tadpole of order $4k$ or $4k+2$ can factorize complete graph K_{4k} or K_{4k+2}, because the number of edges of such tadpoles does not divide the number of edges in the complete graph. For tadpoles of order $4k+3$ we show that all factorize a complete graph K_{4k+3}. For tadpoles of order $4k+1$ we can show that tadpoles with long (longer than $3k+1$) and short (shorter than $k+2$) cycles factorize the complete graph K_{4k+1} for $k \geq 2$. But for lengths between $k+2$ and $3k+1$ no proof is known.

In spite of the lack of the above mentioned constructions, we conjecture that such factorizations exist nevertheless.

Conjecture 1. *Every tadpole of order $2n+1$ factorizes the complete graph K_{2n+1} for each $n \geq 3$.*

Disclosure statement

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