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Inverse source problem related to one-dimensional Saint-Venant equation

Hiroshi Takase

Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, Japan

ABSTRACT
The one-dimensional Saint-Venant equation describes unsteady water flow in channels and is derived from the one-dimensional Euler equation by imposing several physical assumptions. In this paper, we consider the linearized and simplified equation in the one-dimensional case featuring a mixed derivative term and prove the global Lipschitz stability of the inverse source problem via the global Carleman estimate.

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1. Introduction and main result

We consider

$$\begin{aligned} &Au := (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)u = H(x,t) && \text{in } Q_+ := (-\ell,\ell)\times(0,T),\\ &u(\cdot,0) = \partial_t u(\cdot,0) = 0 && \text{on } (-\ell,\ell),\\ &u(-\ell,\cdot) = u(\ell,\cdot) = 0 && \text{on } (0,T), \end{aligned} \tag{1}$$

where $a > 0$, $\ell > 0$ and $T > 0$ are positive constants and H is the source term. The differential operator A has the form of a one-dimensional wave operator plus the mixed derivative term $a\partial_x\partial_t$. This term appears when we linearize the one-dimensional Saint-Venant equation, which is the equation introduced by Saint-Venant in [1] to describe unsteady water flow in channels. The one-dimensional Saint-Venant equation comprises continuity and momentum equations. Their formulations and physical meanings are written in Cunge et al. [2]. Even though one-dimensional flow does not exist in nature, mathematically speaking, it is important to consider the simplified equation and observe its properties.

We consider the uniqueness and stability for the inverse source problem to determine H on $(-\ell,\ell)$ from the boundary observation data of the solution to (1). The argument is based on the Carleman estimate and the Bukhgeim–Klibanov method in [3]. The Carleman estimate was first introduced in Carleman [4] to prove the unique continuation property for the elliptic operator whose coefficients are not necessarily analytic. Using the Carleman estimate, Bukhugeim and Klibanov proved global uniqueness results for multidimensional coefficient inverse problems. This methodology is widely applicable to various partial differential equations provided that we can prove the Carleman estimate for the considered equations. For a hyperbolic equation, Imanuvilov and Yamamoto [5] considered

CONTACT Hiroshi Takase htakase@ms.u-tokyo.ac.jp

the global Lipschitz stability for wave equations through interior observations. Baudouin et al. [6] proved the global Carleman estimate for the wave equation and considered its applications to controllability, inverse problems, and reconstructions. Bellassoued and Yamamoto [7] considered the inverse source and coefficient problems for the wave equation on a compact Riemannian manifold with a boundary.

In proving the Carleman estimate for the operator A, the main difficulties lie in the existence of the mixed derivative term $a\partial_x\partial_t$. There are also difficulties when we apply the Carleman estimate to the extended solution to (1). In the usual case of the wave equation, an evenly extended solution with respect to time t satisfies the wave equation as well. However, considering (1), the evenly extended solution no longer satisfies the equation. We therefore need to consider a different extension.

To prove the global Lipschitz stability for inverse source problems of the hyperbolic partial differential equation, the observation time should be given for the distant wave to reach the boundary owing to the finite propagation speed. We define constants to describe this situation mathematically. Let $x_0 \in [-\ell, \ell]^c$ be a given point and

$$T_0 := \frac{1}{\sqrt{\rho}} \max_{-\ell \le x \le \ell} |x - x_0|,$$

where $\rho := ((\sqrt{a^2+4} - a)/2)^2$ is the square of the wave speed.

Theorem 1.1: *Assume $H(x,t) = f(x)R(x,t)$, where*

$$f \in L^2(-\ell,\ell), \quad R \in H^1(0,T; L^\infty(-\ell,\ell)),$$
$$f(x) = f(-x), \quad R(x,0) = R(-x,0), \quad \text{a.e. } x \in (-\ell,\ell),$$

and

$$\exists m_0 > 0 \quad \text{s.t. } |R(x,0)| \ge m_0 > 0 \text{ a.e. } x \in (-\ell,\ell).$$

Let $T > T_0$. We assume there exists a unique solution u to (1) in the class

$$u \in \bigcap_{k=0}^{2} H^{3-k}(0,T; H^k(-\ell,\ell)).$$

There then exists a constant $C > 0$ that is independent of u and f such that

$$\|f\|_{L^2(-\ell,\ell)} \le C\left(\|\partial_x\partial_t u(\ell,\cdot)\|_{L^2(0,T)} + \|\partial_x\partial_t u(-\ell,\cdot)\|_{L^2(0,T)}\right).$$

2. Proof of Theorem 1.1

2.1. Preliminary

To prove Theorem 1.1 using the Bukhugeim–Klibanov method, we need to prove the Carleman estimate for the operator A in extended domain $Q_\pm$. We consider

$$A = \partial_t^2 - \partial_x^2 + a\partial_t\partial_x, \quad Q_\pm := (-\ell,\ell) \times (-T,T),$$

where $a > 0$ is a constant. The next proposition is the global Carleman estimate, whose proof is postponed to the Appendix section.

Proposition 2.1: *Choose*

$$x_0 \notin [-\ell,\ell],$$

and β such that

$$0 < \beta < \rho,$$

and set

$$\psi(x,t) := |x-x_0|^2 - \beta t^2, \quad \varphi(x,t) := \mathrm{e}^{\gamma\psi(x,t)},$$
$$\sigma(x,t) := s\gamma\varphi(x,t), \quad (x,t) \in \overline{Q_\pm},$$

where $\gamma > 0$ *and* $s > 0$ *is some parameters. There then exists a constant* $\gamma_* > 0$ *such that for all* $\gamma \geq \gamma_*$, *the following holds.*

There exist constants $s_* = s_*(\gamma)$ *and* $C = C(s_*)$ *such that*

$$\begin{aligned}
&\int_{Q_\pm} \mathrm{e}^{2s\varphi}(\sigma|\partial_x u|^2 + \sigma|\partial_t u|^2 + \sigma^3|u|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq C\int_{Q_\pm} \mathrm{e}^{2s\varphi}|Au|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_{-T}^{\mathrm{T}} \left(\mathrm{e}^{2s\varphi(\ell,t)}\sigma(\ell,t)|\partial_x u(\ell,t)|^2 + \mathrm{e}^{2s\varphi(-\ell,t)}\sigma(-\ell,t)|\partial_x u(-\ell,t)|^2\right)\,\mathrm{d}t,
\end{aligned}$$

for all $s > s_*$ *and* $u \in \bigcap_{k=0}^2 H^{2-k}(-T,T;H^k(-\ell,\ell))$ *such that* $u(\cdot,\pm T) = \partial_t u(\cdot,\pm T) = 0$ *on* $(-\ell,\ell)$ *and* $u(\pm\ell,\cdot) = 0$ *on* $(-T,T)$.

Proving our main theorem requires several energy estimates as follows.

Lemma 2.2: *Assume* $f \in L^2(-\ell,\ell)$, $R \in L^2(-T,T;L^\infty(-\ell,\ell))$ *and* $u_1 \in L^2(-\ell,\ell)$. *Let* $u \in \bigcap_{k=0}^2 H^{2-k}(-T,T;H^k(-\ell,\ell))$ *be a solution to*

$$\begin{array}{ll}
Au = (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)u = f(x)R(x,t) & \text{in } Q_\pm, \\
u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = u_1 & \text{on } (-\ell,\ell), \\
u(-\ell,\cdot) = u(\ell,\cdot) = 0 & \text{on } (-T,T).
\end{array}$$

There then exists a constant C such that

$$\|\partial_t u(\cdot,t)\|_{L^2(-\ell,\ell)}^2 + \|\partial_x u(\cdot,t)\|_{L^2(-\ell,\ell)}^2 \leq C(\|f\|_{L^2(-\ell,\ell)}^2 + \|u_1\|_{L^2(-\ell,\ell)}^2), \quad \text{a.e. } t \in (-T,T)$$

holds.

Proof: Set $E(t) := \|\partial_t u(\cdot,t)\|_{L^2(-\ell,\ell)}^2 + \|\partial_x u(\cdot,t)\|_{L^2(-\ell,\ell)}^2$. Multiplying the first equation by $\partial_t u$ and integrating over $(-\ell,\ell)$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq E(t) + \|fR(\cdot,t)\|_{L^2(-\ell,\ell)}^2$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t}E(t)) \leq \mathrm{e}^{-t}\|fR(\cdot,t)\|_{L^2(-\ell,\ell)}^2.$$

Furthermore, integration over $(0,t)$ yields

$$E(t) \leq C(\|R\|_{L^2(-T,T;L^\infty(-\ell,\ell))}^2\|f\|_{L^2(-\ell,\ell)}^2 + \|u_1\|_{L^2(-\ell,\ell)}^2).$$

■

Lemma 2.3: *Assume* $H \in L^2(-T,T;L^2(-\ell,\ell))$. *Let* $z \in \bigcap_{k=0}^{2} H^{2-k}(-T,T;H^k(-\ell,\ell))$ *be a solution to*

$$\begin{aligned} &Az = H(x,t) && \text{in } Q_{\pm}, \\ &z(\cdot,0) = 0, \quad \partial_t z(\cdot,0) = z_1 && \text{on } (-\ell,\ell), \\ &z(-\ell,\cdot) = z(\ell,\cdot) = 0 && \text{on } (-T,T), \end{aligned}$$

and

$$z(\cdot,\pm T) = \partial_t z(\cdot,\pm T) = 0 \quad \text{on } (-\ell,\ell).$$

There then exists a constant $C > 0$ *such that*

$$\|z_1\|^2_{L^2(-\ell,\ell)} \le C\|H\partial_t z\|_{L^1(Q_{\pm})}$$

holds.

Proof: Multiplying the first equation by $2\partial_t z$ and integrating over $Q_+ = (-\ell,\ell) \times (0,T)$, we have

$$\int_{Q_+} \partial_t|\partial_t z|^2 + \int_{Q_+} \partial_t|\partial_x z|^2 + a\int_{Q_+} \partial_x|\partial_t z|^2 = 2\int_{Q_+} H\partial_t z.$$

Hence, we get

$$\|z_1\|^2_{L^2(-\ell,\ell)} \le C\|H\partial_t z\|_{L^1(Q_+)}.$$

The proof for $Q_- := (-\ell,\ell) \times (-T,0)$ is similar. ■

2.2. Proof of the main theorem

Proof of Theorem 1.1: We use the weight function $\psi(x,t) = |x-x_0|^2 - \beta t^2$ for $x_0 \in [-\ell,\ell]^c$. We assume $T > T_0$, i.e. $\rho T^2 > \rho T_0^2 = \max_{-\ell \le x \le \ell} |x-x_0|^2$, and there thus exists $0 < \beta < \rho$ such that

$$\max_{-\ell \le x \le \ell} |x-x_0|^2 < \beta T^2 (< \rho T^2).$$

There then exists $0 < \epsilon (< \frac{T}{2})$ such that

$$\max_{-\ell \le x \le \ell} |x-x_0|^2 < \beta(T-2\epsilon)^2.$$

Thus, for all $x \in [-\ell,\ell]$ and $t \in [-T,-T+2\epsilon] \cap [T-2\epsilon,T]$, we have

$$\varphi(x,t) = e^{\gamma\psi(x,t)} < 1.$$

Let u be the solution in the class

$$u \in \bigcap_{k=0}^{2} H^{3-k}(0,T;H^k(-\ell,\ell))$$

and take the extension of u to $(-T,T)$:

$$u(x,t) = \begin{cases} u(x,t) & \text{in } Q_+ := (-\ell,\ell) \times (0,T), \\ u(-x,-t) & \text{in } Q_- := (-\ell,\ell) \times (-T,0). \end{cases}$$

We also extend R:

$$R(x,t) = \begin{cases} R(x,t) & \text{in } Q_+, \\ R(-x,-t) & \text{in } Q_-. \end{cases}$$

We can then prove $u \in \bigcap_{k=0}^{2} H^{3-k}(-T,T;H^k(-\ell,\ell))$ and $R \in H^1(-T,T;L^\infty(-\ell,\ell))$. Indeed, we assume $u(\cdot,0) = \partial_t u(\cdot,0) = 0$ and we thus have

$$\partial_t u(x,t) = \begin{cases} \partial_t u(x,t) & \text{in } Q_+, \\ -\partial_t u(-x,-t) & \text{in } Q_-, \end{cases}$$

and

$$\partial_t^2 u(x,t) = \begin{cases} \partial_t^2 u(x,t) & \text{in } Q_+, \\ \partial_t^2 u(-x,-t) & \text{in } Q_-. \end{cases}$$

Furthermore, we assume the symmetry of the source term f and $R(\cdot,0)$ and thus get

$$\partial_t^3 u(x,t) = \begin{cases} \partial_t^3 u(x,t) & \text{in } Q_+, \\ -\partial_t^3 u(-x,-t) & \text{in } Q_-. \end{cases}$$

Considering R, we have

$$\partial_t R(x,t) = \begin{cases} \partial_t R(x,t) & \text{in } Q_+, \\ -\partial_t R(-x,-t) & \text{in } Q_-, \end{cases}$$

owing to the symmetry of $R(\cdot,0)$. This extended u satisfies

$$\begin{array}{ll} Au = (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)u = f(x)R(x,t) & \text{in } Q_\pm, \\ u(\cdot,0) = \partial_t u(\cdot,0) = 0 & \text{on } (-\ell,\ell), \\ u(-\ell,\cdot) = u(\ell,\cdot) = 0 & \text{on } (-T,T). \end{array}$$

From Lemma 2.2, we have

$$\|\partial_t u(\cdot,t)\|^2_{L^2(-\ell,\ell)} + \|\partial_x u(\cdot,t)\|^2_{L^2(-\ell,\ell)} \le C\|f\|^2_{L^2(-\ell,\ell)}, \quad \text{a.e. } t \in (-T,T). \tag{2}$$

Let $v := \partial_t u$. v satisfies

$$\begin{array}{ll} Av = (\partial_t^2 - \partial_x^2 + a\partial_x\partial_t)v = f(x)\partial_t R(x,t) & \text{in } Q_\pm, \\ v(\cdot,0) = 0,\ \partial_t v(\cdot,0) = f(\cdot)R(\cdot,0) & \text{on } (-\ell,\ell), \\ v(-\ell,\cdot) = v(\ell,\cdot) = 0 & \text{in } (-T,T). \end{array}$$

Also from Lemma 2.2, we have

$$\|\partial_t v(\cdot,t)\|^2_{L^2(\ell,\ell)} + \|\partial_x v(\cdot,t)\|^2_{L^2(-\ell,\ell)} \le C\|f\|^2_{L^2(-\ell,\ell)}, \quad \text{a.e. } t \in (-T,T). \tag{3}$$

We define a cut-off function η satisfying $0 \le \eta(t) \le 1$:

$$\eta(t) := \begin{cases} 1, & |t| \le T - 2\epsilon, \\ 0, & |t| \ge T - \epsilon. \end{cases}$$

We set $w := \eta\partial_t u = \eta v$. w satisfies

$$\begin{array}{ll} Aw = \eta f\partial_t R + 2\partial_t\eta\partial_t v + a\partial_t\eta\partial_x v + \partial_t^2\eta v & \text{in } Q_\pm, \\ w(\cdot,0) = 0,\ \partial_t w(\cdot,0) = f(\cdot)R(\cdot,0) & \text{on } (-\ell,\ell), \\ w(-\ell,\cdot) = w(\ell,\cdot) = 0 & \text{on } (-T,T). \end{array}$$

Moreover,

$$w(\cdot,\pm T) = \partial_t w(\cdot,\pm T) = 0, \quad \text{on } (-\ell,\ell)$$

holds. We can therefore apply Proposition 2.1 to w to obtain

$$\begin{aligned}\int_{Q_\pm} & \mathrm{e}^{2s\varphi}(s|\partial_x w|^2 + s|\partial_t w|^2 + s^3|w|^2) \\ & \le C\int_{Q_\pm} \mathrm{e}^{2s\varphi}|f\partial_t R|^2 + C\int_{Q_\pm} \mathrm{e}^{2s\varphi}(|\partial_t\eta\partial_t v|^2 + |\partial_t\eta\partial_x v|^2 + |\partial_t^2\eta v|^2) \\ & \quad + Cs\,\mathrm{e}^{Cs}\mathcal{D}^2\end{aligned}$$

for all $s \ge s_*$, where

$$\begin{aligned}\mathcal{D}^2 &:= \int_{-T}^{\mathrm{T}} \left(|\partial_x w(\ell,t)|^2 + |\partial_x w(-\ell,t)|^2\right)\,\mathrm{d}t \\ &\le \int_{-T}^{\mathrm{T}} \left(|\partial_x\partial_t u(\ell,t)|^2 + |\partial_x\partial_t u(-\ell,t)|^2\right)\,\mathrm{d}t.\end{aligned}$$

We here consider sufficiently large $\gamma > \gamma_*$ as a fixed constant. Considering the second, third, and fourth terms on the right-hand side and using (2) and (3), we get

$$\int_{Q_\pm} \mathrm{e}^{2s\varphi}(|\partial_t\eta\partial_t v|^2 + |\partial_t\eta\partial_x v|^2 + |\partial_t^2\eta v|^2) \le C\,\mathrm{e}^{2s}\|f\|^2_{L^2(-\ell,\ell)},$$

because $\mathrm{supp}(\partial_t\eta),\ \mathrm{supp}(\partial_t^2\eta) \subset [-T+\epsilon,-T+2\epsilon] \cup [T-2\epsilon,T-\epsilon]$, and for all $x \in [-\ell,\ell]$ and $t \in [-T+\epsilon,-T+2\epsilon] \cap [T-2\epsilon,T-\epsilon]$, we have $\varphi(x,t) = \mathrm{e}^{\gamma\psi(x,t)} < 1$. We therefore have

$$\begin{aligned}\int_{Q_\pm} & \mathrm{e}^{2s\varphi}(s|\partial_x w|^2 + s|\partial_t w|^2 + s^3|w|^2) \\ & \le C\int_{Q_\pm} \mathrm{e}^{2s\varphi}|f\partial_t R|^2 + C\,\mathrm{e}^{2s}\|f\|^2_{L^2(-\ell,\ell)} + Cs\,\mathrm{e}^{Cs}\mathcal{D}^2,\end{aligned} \tag{4}$$

for all $s \ge s_*$.

We next set $z := \mathrm{e}^{s\varphi}w$ and z satisfies

$$\begin{cases} Az = \mathrm{e}^{s\varphi}(Aw + G(x,t)) & \text{in } Q_\pm, \\ z(\cdot,0) = 0, \quad \partial_t z(\cdot,0) = \mathrm{e}^{s\varphi(\cdot,0)}f(\cdot)R(\cdot,0) & \text{on } (-\ell,\ell), \\ z(-\ell,\cdot) = z(\ell,\cdot) = 0 & \text{on } (-T,T), \end{cases}$$

where

$$\begin{aligned}G :=\ & (2s\partial_t\varphi + as\partial_x\varphi)\partial_t w + (-2s\partial_x\varphi + as\partial_t\varphi)\partial_x w \\ & + (s\partial_t^2\varphi + s^2|\partial_t\varphi|^2 - s\partial_x^2\varphi - s^2|\partial_x\varphi|^2 + as\partial_x\partial_t\varphi + as^2\partial_x\varphi\partial_t\varphi)w.\end{aligned}$$

From Lemma 2.3, we have

$$\begin{aligned}\|\mathrm{e}^{s\varphi(x,0)}f\|^2_{L^2(-\ell,\ell)} &\le C\int_{Q_\pm} \mathrm{e}^{2s\varphi}|\partial_t z|^2 \\ & + C\int_{Q_\pm} \mathrm{e}^{2s\varphi}|f\partial_t R|^2 + C\,\mathrm{e}^{2s}\|f\|^2_{L^2(-\ell,\ell)} + C\|\mathrm{e}^{s\varphi}G\partial_t z\|_{L^1(Q_\pm)}\end{aligned} \tag{5}$$

and we also find

$$\mathrm{e}^{s\varphi} G \partial_t z \leq C\,\mathrm{e}^{2s\varphi}(s|\partial_t w| + s|\partial_x w| + s^2|w|)(|\partial_t w| + s|w|)$$

and thus obtain

$$\|\mathrm{e}^{s\varphi} G \partial_t z\|_{L^1(Q_\pm)} \leq C \int_{Q_\pm} \mathrm{e}^{2s\varphi}(s|\partial_x w|^2 + s|\partial_t w|^2 + s^3|w|^2).$$

Furthermore, we have

$$\begin{aligned}\int_{Q_\pm} \mathrm{e}^{2s\varphi}|f\partial_t R|^2 &\leq C \int_{-\ell}^{\ell} \mathrm{e}^{2s\varphi(x,0)}|f(x)|^2 \left(\int_{-T}^{T} \mathrm{e}^{-2s(\varphi(x,0)-\varphi(x,t))}\,\mathrm{d}t\right)\mathrm{d}x\\ &\leq C \int_{-\ell}^{\ell} \mathrm{e}^{2s\varphi(x,0)}|f(x)|^2 \left(\int_{-T}^{T} \mathrm{e}^{-2s\,\mathrm{e}^{\gamma|x-x_0|^2}(1-\mathrm{e}^{-\gamma\beta t^2})}\,\mathrm{d}t\right)\mathrm{d}x\\ &\leq o(1)\|\mathrm{e}^{s\varphi(x,0)}f\|^2_{L^2(-\ell,\ell)}, \quad \text{as } s \to \infty,\end{aligned}$$

by the Lebesgue's dominated convergence theorem. We apply this estimate and (4) to (5) to get

$$\|\mathrm{e}^{s\varphi(x,0)}f\|^2_{L^2(-\ell,\ell)} \leq C\,\mathrm{e}^{2s}\|f\|^2_{L^2(-\ell,\ell)} + Cs\,\mathrm{e}^{Cs}\mathcal{D}^2.$$

There exists $\kappa > 1$ such that for all $x \in [-\ell, \ell]$, $\varphi(x, 0) = \mathrm{e}^{\gamma|x-x_0|} \geq \kappa > 1$ holds. Therefore, taking sufficiently large $s > s^*$, we finally have

$$\|f\|_{L^2(-\ell,\ell)} \leq C\mathcal{D},$$

for a constant $C > 0$ independent of f. ∎

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ORCID

Hiroshi Takase http://orcid.org/0000-0002-8540-5300

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Appendix

***Proof of Proposition* 2.1:** For simplicity, we write

$$p(x) := \partial_x\psi(x,t), \quad q(t) := \partial_t\psi(x,t).$$

Simple calculation yields

$$\begin{aligned}
\partial_x\varphi &= \gamma p\varphi, \quad \partial_t\varphi = \gamma q\varphi,\\
\partial_x^2\varphi &= (\gamma^2p^2 + 2\gamma)\varphi, \quad \partial_t^2\varphi = (\gamma^2q^2 - 2\beta\gamma)\varphi,\\
\partial_x\partial_t\varphi &= \gamma^2pq\varphi.
\end{aligned}$$

We introduce

$$w := e^{s\varphi}u, \quad Pw := e^{s\varphi}A\,e^{-s\varphi}w.$$

Then,

$$\begin{aligned}
Pw = {}& \partial_t^2w - \partial_x^2w + 2s(\partial_x\varphi\partial_xw - \partial_t\varphi\partial_tw) - s^2(|\partial_x\varphi|^2 - a\partial_x\varphi\partial_t\varphi - |\partial_t\varphi|^2)w\\
&+ s(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)w + a\partial_x\partial_tw - as(\partial_t\varphi\partial_xw + \partial_x\varphi\partial_tw).
\end{aligned}$$

We decompose Pw into P_+w and P_-w as follows.

$$P_+w := \partial_t^2w - \partial_x^2w + a\partial_x\partial_tw - s^2(|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)w$$

and

$$P_-w := s(2\partial_x\varphi - a\partial_t\varphi)\partial_xw - s(2\partial_t\varphi + a\partial_x\varphi)\partial_tw + s(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)w.$$

Note that $Pw = P_+w + P_-w$. We wish to make the lower bound of $\|Pw\|^2_{L^2(Q_\pm)}$, and we therefore try to estimate $(P_+w, P_-w)_{L^2(Q_\pm)}$.

$$\begin{aligned}
&(P_+w, P_-w)_{L^2(Q_\pm)}\\
&= \int_{Q_\pm}\partial_t^2w\cdot s(2\partial_x\varphi - a\partial_t\varphi)\partial_xw - \int_{Q_\pm}\partial_t^2w\cdot s(2\partial_t\varphi + a\partial_x\varphi)\partial_tw\\
&\quad + \int_{Q_\pm}\partial_t^2w\cdot s(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)w - \int_{Q_\pm}\partial_x^2w\cdot s(2\partial_x\varphi - a\partial_t\varphi)\partial_xw\\
&\quad + \int_{Q_\pm}\partial_x^2w\cdot s(2\partial_t\varphi + a\partial_x\varphi)\partial_tw - \int_{Q_\pm}\partial_x^2w\cdot s(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)w\\
&\quad + \int_{Q_\pm}a\partial_x\partial_tw\cdot s(2\partial_x\varphi - a\partial_t\varphi)\partial_xw - \int_{Q_\pm}a\partial_x\partial_tw\cdot s(2\partial_t\varphi + a\partial_x\varphi)\partial_tw\\
&\quad + \int_{Q_\pm}a\partial_x\partial_tw\cdot s(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)w\\
&\quad - \int_{Q_\pm}s^2(|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)w\cdot s(2\partial_x\varphi - a\partial_t\varphi)\partial_xw\\
&\quad + \int_{Q_\pm}s^2(|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)w\cdot s(2\partial_t\varphi + a\partial_x\varphi)\partial_tw\\
&\quad - \int_{Q_\pm}s^2(|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)w\cdot s(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)w\\
&=: \sum_{k=1}^{12}J_k.
\end{aligned}$$

Through integration by parts, we get

$$
\begin{aligned}
J_1 &= -s\int_{Q_\pm}(2\partial_x\varphi - a\partial_t\varphi)\frac{1}{2}\partial_x(|\partial_t w|^2) - s\int_{Q_\pm}\partial_t(2\partial_x\varphi - a\partial_t\varphi)\partial_t w\partial_x w\\
&= \frac{s}{2}\int_{Q_\pm}\partial_x(2\partial_x\varphi - a\partial_t\varphi)|\partial_t w|^2 - s\int_{Q_\pm}\partial_t(2\partial_x\varphi - a\partial_t\varphi)\partial_t w\partial_x w,\\
J_2 &= -s\int_{Q_\pm}(2\partial_t\varphi + a\partial_x\varphi)\frac{1}{2}\partial_t(|\partial_t w|^2) = \frac{s}{2}\int_{Q_\pm}\partial_t(2\partial_t\varphi + a\partial_x\varphi)|\partial_t w|^2,\\
J_3 &= -s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_t w|^2 - s\int_{Q_\pm}\partial_t(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\frac{1}{2}\partial_t(|w|^2)\\
&= -s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_t w|^2 + \frac{s}{2}\int_{Q_\pm}\partial_t^2(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2,\\
J_4 &= \frac{s}{2}\int_{Q_\pm}\partial_x(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2 - \frac{s}{2}\int_{-T}^{T}\left[(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2\right]_{x=-\ell}^{\ell}\,\mathrm{d}t,\\
J_5 &= -s\int_{Q_\pm}\partial_x(2\partial_t\varphi + a\partial_x\varphi)\partial_x w\partial_t w - s\int_{Q_\pm}(2\partial_t\varphi + a\partial_x\varphi)\frac{1}{2}\partial_t(|\partial_x w|^2)\\
&= -s\int_{Q_\pm}\partial_x(2\partial_t\varphi + a\partial_x\varphi)\partial_x w\partial_t w + \frac{s}{2}\int_{Q_\pm}\partial_t(2\partial_t\varphi + a\partial_x\varphi)|\partial_x w|^2,\\
J_6 &= s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_x w|^2 + s\int_{Q_\pm}\partial_x(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\frac{1}{2}\partial_x(|w|^2)\\
&= s\int_{Q_\pm}(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|\partial_x w|^2 - \frac{s}{2}\int_{Q_\pm}\partial_x^2(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2,\\
J_7 &= -\frac{s}{2}\int_{Q_\pm}a\partial_t(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2,\\
J_8 &= \frac{s}{2}\int_{Q_\pm}a\partial_x(2\partial_t\varphi + a\partial_x\varphi)|\partial_t w|^2,\\
J_9 &= -s\int_{Q_\pm}a(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\partial_x w\partial_t w - s\int_{Q_\pm}a\partial_x(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\frac{1}{2}\partial_t(|w|^2)\\
&= -s\int_{Q_\pm}a(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)\partial_x w\partial_t w + \frac{s}{2}\int_{Q_\pm}a\partial_t\partial_x(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2,\\
J_{10} &= \frac{s^3}{2}\int_{Q_\pm}\partial_x((|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)(2\partial_x\varphi - a\partial_t\varphi))|w|^2,\\
J_{11} &= -\frac{s^3}{2}\int_{Q_\pm}\partial_t((|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)(2\partial_t\varphi + a\partial_x\varphi))|w|^2,\\
J_{12} &= -s^3\int_{Q_\pm}(|\partial_x\varphi|^2 - |\partial_t\varphi|^2 - a\partial_x\varphi\partial_t\varphi)(\partial_x^2\varphi - a\partial_x\partial_t\varphi - \partial_t^2\varphi)|w|^2.
\end{aligned}
$$

First, we consider a part of the sum $\sum_{k=1}^{9} J_k$. Set $\sigma(x,t) := s\gamma\varphi(x,t)$ and

$$
\mathcal{B} := \frac{s}{2}\int_{-T}^{T}\left[(2\partial_x\varphi - a\partial_t\varphi)|\partial_x w|^2\right]_{x=-\ell}^{\ell}\,\mathrm{d}t.
$$

$$
\begin{aligned}
\sum_{k=1}^{9} J_k = &\int_{Q_\pm}\left[\sigma\gamma\left(2q^2 + 2apq + \frac{a^2}{2}p^2\right) + \sigma(-4\beta + a^2)\right]|\partial_t w|^2\\
&+\int_{Q_\pm}\left[\sigma\gamma\left(2p^2 - 2apq + \frac{a^2}{2}q^2\right) + \sigma(4 - \beta a^2)\right]|\partial_x w|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{Q_\pm}[\sigma\gamma((a^2-4)pq+2a(q^2-p^2))+\sigma(-4\beta a-4a)]\partial_t w\partial_x w\\
&+\int_{Q_\pm}\mathcal{O}(s\gamma^4\varphi)|w|^2-\mathcal{B}\\
&=2\int_{Q_\pm}\sigma\gamma\left(\left(q+\frac{ap}{2}\right)\partial_t w-\left(p-\frac{aq}{2}\right)\partial_x w\right)^2\\
&+\int_{Q_\pm}\sigma[(-4\beta+a^2)|\partial_t w|^2+(4-\beta a^2)|\partial_x w|^2-4a(\beta+1)\partial_t w\partial_x w]\\
&+\int_{Q_\pm}\mathcal{O}(s\gamma^4\varphi)|w|^2-\mathcal{B}\\
&\geq\int_{Q_\pm}\sigma[(-4\beta+a^2)|\partial_t w|^2+(4-\beta a^2)|\partial_x w|^2-4a(\beta+1)\partial_t w\partial_x w]\\
&+\int_{Q_\pm}\mathcal{O}(s\gamma^4\varphi)|w|^2-\mathcal{B}.
\end{aligned}
\tag{A1}
$$

To estimate the terms $\sigma\times$ (first-order terms), we use the next energy inequality.

$$
\begin{aligned}
(P_+w,-\sigma w)_{L^2(Q_\pm)}&=\int_{Q_\pm}P_+w\cdot(-\sigma w)\\
&=-\int_{Q_\pm}\sigma\partial_t^2 ww+\int_{Q_\pm}\sigma\partial_x^2 ww\\
&\quad+\int_{Q_\pm}\sigma s^2(|\partial_x\varphi|^2-a\partial_x\varphi\partial_t\varphi-|\partial_t\varphi|^2)|w|^2-\int_{Q_\pm}a\sigma\partial_x\partial_t ww\\
&=:\sum_{k=1}^4 I_k.
\end{aligned}
$$

Integration by parts yields

$$
\begin{aligned}
I_1&=\int_{Q_\pm}s\gamma\partial_t\varphi\frac{1}{2}\partial_t(|w|^2)+\int_{Q_\pm}s\gamma\varphi|\partial_t w|^2\\
&=\int_{Q_\pm}s\gamma\varphi|\partial_t w|^2-\frac{1}{2}\int_{Q_\pm}s\gamma\partial_t^2\varphi|w|^2,\\
I_2&=-\int_{Q_\pm}s\gamma\partial_x\varphi\frac{1}{2}\partial_x(|w|^2)-\int_{Q_\pm}s\gamma\varphi|\partial_x w|^2\\
&=-\int_{Q_\pm}s\gamma\varphi|\partial_x w|^2+\frac{1}{2}\int_{Q_\pm}s\gamma\partial_x^2\varphi|w|^2,\\
I_3&=\int_{Q_\pm}s^3\gamma\varphi(|\partial_x\varphi|^2-a\partial_t\varphi\partial_x\varphi-|\partial_t\varphi|^2)|w|^2,\\
I_4&=\int_{Q_\pm}as\gamma\partial_x\varphi\frac{1}{2}\partial_t(|w|^2)+\int_{Q_\pm}as\gamma\varphi\partial_t w\partial_x w\\
&=-\frac{1}{2}\int_{Q_\pm}as\gamma\partial_t\partial_x\varphi|w|^2+\int_{Q_\pm}as\gamma\varphi\partial_t w\partial_x w.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\int_{Q_\pm}P_+w\cdot(-\sigma w)&=\sum_{k=1}^4 I_k\\
&=\int_{Q_\pm}\sigma|\partial_t w|^2-\int_{Q_\pm}\sigma|\partial_x w|^2+\int_{Q_\pm}a\sigma\partial_t w\partial_x w+\int_{Q_\pm}\left[-\frac{1}{2}s\gamma\partial_t^2\varphi+\frac{1}{2}s\gamma\partial_x^2\varphi\right.\\
&\quad\left.+s^3\gamma\varphi(|\partial_x\varphi|^2-a\partial_x\varphi\partial_t\varphi-|\partial_t\varphi|^2)-\frac{a}{2}s\gamma\partial_t\partial_x\varphi\right]|w|^2
\end{aligned}
$$

$$= \int_{Q_\pm} \sigma|\partial_t w|^2 - \int_{Q_\pm} \sigma|\partial_x w|^2 + \int_{Q_\pm} a\sigma\,\partial_t w \partial_x w + \int_{Q_\pm} [\sigma^3(p^2 - q^2 - apq) + \mathcal{O}(s\gamma^3\varphi)]|w|^2. \tag{A2}$$

Let $\mu \in \mathbb{R}$ be a constant to be determined later. Multiplying (A2) by 2μ, and adding it to (A1) yields

$$\begin{aligned}
&\sum_{k=1}^{9} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B} \\
&\geq \int_{Q_\pm} \sigma(-4\beta + a^2 + 2\mu)|\partial_t w|^2 + \int_{Q_\pm} \sigma(4 - \beta a^2 - 2\mu)|\partial_x w|^2 \\
&\quad + \int_{Q_\pm} \sigma(-4\beta a - 4a + 2a\mu)\partial_t w \partial_x w \\
&\quad + \int_{Q_\pm} [\sigma^3 \cdot 2\mu(p^2 - apq - q^2) + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)]|w|^2 \\
&\geq \int_{Q_\pm} \sigma(-4\beta + a^2 + 2\mu - \epsilon| - 2\beta a - 2a + a\mu|)|\partial_t w|^2 \\
&\quad + \int_{Q_\pm} \sigma\left(4 - \beta a^2 - 2\mu - \frac{1}{\epsilon}| - 2\beta a - 2a + a\mu|\right)|\partial_x w|^2 \\
&\quad + \int_{Q_\pm} [\sigma^3 \cdot 2\mu(p^2 - apq - q^2) + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)]|w|^2,
\end{aligned}$$

for all $\epsilon > 0$ to be determined later. We wish to choose $\epsilon > 0$ such that both coefficients are positive, i.e.

$$\begin{aligned}
&-4\beta + a^2 + 2\mu - \epsilon| - 2\beta a - 2a + a\mu| > 0, \\
&4 - \beta a^2 - 2\mu - \frac{1}{\epsilon}| - 2\beta a - 2a + a\mu| > 0
\end{aligned} \tag{A3}$$

holds. This claim follows only if μ satisfies $\mu^2 - 2(\beta + 1)\mu + \beta(a^2 + 4) < 0 \Leftrightarrow (\mu - \beta - 1)^2 < \beta^2 - (a^2 + 2)\beta + 1$. Therefore, if we choose $\mu := \beta + 1$, by our assumption of $\beta > 0$, this inequality holds and (A3) also holds. Hence, there exists a positive constant $C > 0$ such that

$$\begin{aligned}
&\sum_{k=1}^{9} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B} \\
&\quad \geq C\int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + \int_{Q_\pm} [2\mu\sigma^3(p^2 - apq - q^2) + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)]|w|^2.
\end{aligned} \tag{A4}$$

Finally, we estimate zeroth-order terms. We have

$$\begin{aligned}
J_{10} &= \int_{Q_\pm} \sigma^3\left[\gamma\left(3p^4 + \frac{3(a^2-2)}{2}p^2q^2 - \frac{9a}{2}p^3q + \frac{3a}{2}pq^3\right) + (6p^2 + (a^2-2)q^2 - 6apq)\right]|w|^2, \\
J_{11} &= \int_{Q_\pm} \sigma^3\left[\gamma\left(3q^4 + \frac{3(a^2-2)}{2}p^2q^2 - \frac{3a}{2}p^3q + \frac{9a}{2}pq^3\right) - ((a^2-2)\beta p^2 + 6\beta q^2 + 6\beta apq)\right]|w|^2, \\
J_{12} &= \int_{Q_\pm} \sigma^3\Big[-\gamma(p^2 - apq - q^2)^2 - 2(\beta+1)(p^2 - apq - q^2)\Big]|w|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{k=10}^{12} J_k = \int_{Q_\pm} \sigma^3\Big[&\gamma \cdot 2(p^2 - apq - q^2)^2 \\
&+ (4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq\Big]|w|^2.
\end{aligned}$$

Adding this equality to (A4) yields

$$\begin{aligned}
&\sum_{k=1}^{12} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B} \\
&\quad = (P_+ w, P_- w)_{L^2(Q_\pm)} + 2(P_+ w, -\mu\sigma w)_{L^2(Q_\pm)} + \mathcal{B} \\
&\quad \geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + \int_{Q_\pm} [\sigma^3 \gamma \cdot 2(p^2 - apq - q^2)^2 \\
&\quad\quad + \sigma^3\{(4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + 2\mu(p^2 - apq - q^2)\} \\
&\quad\quad + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)]|w|^2. \qquad \text{(A5)}
\end{aligned}$$

We consider the coefficient of σ^3. Noting $\mu = \beta + 1$, we have

$$\begin{aligned}
&(4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + 2\mu(p^2 - apq - q^2) \\
&\quad = (4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + (2\mu - 2\nu)(p^2 - apq - q^2) + 2\nu(p^2 - apq - q^2) \\
&\quad \geq 2(\mu - \nu)(p^2 - apq - q^2) + (2\nu + 4 - a^2\beta - \delta a|\nu + 2 + 2\beta|)p^2 \\
&\quad\quad + \left(-2\nu + a^2 - 4\beta - \frac{a}{\delta}|\nu + 2 + 2\beta|\right) q^2,
\end{aligned}$$

for all $\delta > 0$ to be determined later. We wish to choose $\delta > 0$ such that

$$\begin{aligned}
&2\nu + 4 - a^2\beta - \delta a|\nu + 2 + 2\beta| > 0, \\
&-2\nu + a^2 - 4\beta - \tfrac{a}{\delta}|\nu + 2 + 2\beta| > 0.
\end{aligned} \qquad \text{(A6)}$$

Such δ exists only if ν satisfies $\nu^2 + 2(\beta+1)\nu + \beta(a^2+4) < 0 \Leftrightarrow (\nu + \beta + 1)^2 < \beta^2 - (a^2+2)\beta + 1$. Therefore, if we take $\nu = -\mu = -\beta - 1$, both the above inequality and (A6) hold. Hence, there exists constant a $C > 0$ such that

$$\begin{aligned}
&(4 - a^2\beta)p^2 + (a^2 - 4\beta)q^2 - 4(\beta+1)apq + 2\mu(p^2 - apq - q^2) \\
&\quad \geq 4(1+\beta)(p^2 - apq - q^2) + C(p^2 + q^2).
\end{aligned}$$

We apply this estimate to (A5) to obtain

$$\begin{aligned}
&\sum_{k=1}^{12} J_k + 2\mu \sum_{k=1}^{4} I_k + \mathcal{B} \\
&\quad \geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + \int_{Q_\pm} \Big[\sigma^3 \gamma \cdot 2(p^2 - apq - q^2)^2 \\
&\quad\quad + \sigma^3\{4(\beta+1)(p^2 - apq - q^2) + C(p^2+q^2)\} + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)\Big]|w|^2 \\
&\quad \geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) \\
&\quad\quad + \int_{Q_\pm} \left[\sigma^3 \cdot 2\gamma \left((p^2 - apq - q^2) + \frac{\beta+1}{\gamma}\right)^2 + \sigma^3 \left\{C(p^2+q^2) - \frac{2(\beta+1)^2}{\gamma}\right\}\right. \\
&\quad\quad \left. + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)\right] |w|^2 \\
&\quad \geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) \\
&\quad\quad + C\left(\min_{-\ell \leq x \leq \ell} p(x)^2 - \frac{2(\beta+1)^2}{\gamma}\right) \int_{Q_\pm} [\sigma^3 + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)]|w|^2. \\
&\quad \geq C \int_{Q_\pm} \sigma(|\partial_t w|^2 + |\partial_x w|^2) + C \int_{Q_\pm} [\sigma^3 + \mathcal{O}(s\gamma^4\varphi) + \mathcal{O}(s\gamma^3\varphi)]|w|^2,
\end{aligned}$$

for sufficiently large $\gamma > 0$.

The Cauchy–Schwarz inequality then yields

$$
\begin{aligned}
C\int_{Q_\pm}(\sigma|\partial_t w|^2+\sigma|\partial_x w|^2+\sigma^3|w|^2)+\int_{Q_\pm}[\mathcal{O}(s\gamma^4\varphi)+\mathcal{O}(s\gamma^3\varphi)]|w|^2 \\
\leq \frac{1}{2}\|Pw\|^2_{L^2(Q_\pm)}+\int_{Q_\pm}\mathcal{O}(\sigma^2)|w|^2+\mathcal{B}.
\end{aligned}
$$

We can choose $s>0$ large enough to ensure

$$
\int_{Q_\pm}(\sigma|\partial_t w|^2+\sigma|\partial_x w|^2+\sigma^3|w|^2)\leq C\|Pw\|^2_{L^2(Q_\pm)}+C\mathcal{B}.
$$

We define $w=\mathrm{e}^{s\varphi}u$ and thus obtain $\partial_t w=\mathrm{e}^{s\varphi}(\partial_t u+s\partial_t\varphi u)$ and $\partial_x w=\mathrm{e}^{s\varphi}(\partial_x u+s\partial_x\varphi u)$. Again by choosing $s>0$ large, we can rewrite the inequality in terms of $u=w\,\mathrm{e}^{-s\varphi}$:

$$
\int_{Q_\pm}\mathrm{e}^{2s\varphi}(\sigma|\partial_x u|^2+\sigma|\partial_t u|^2+\sigma^3|u|^2)\leq C\int_{Q_\pm}\mathrm{e}^{2s\varphi}|Au|^2+C\mathcal{B}.
$$

Here, we get

$$
\begin{aligned}
\mathcal{B}&=\int_{-T}^{\mathrm{T}}\left[\sigma\left(p-\frac{a}{2}q\right)|\partial_x w|^2\right]_{-\ell}^{\ell}\,\mathrm{d}t \\
&\leq C\int_{-T}^{\mathrm{T}}\left(\mathrm{e}^{2s\varphi(\ell,t)}\sigma(\ell,t)|\partial_x u(\ell,t)|^2+\mathrm{e}^{2s\varphi(-\ell,t)}\sigma(-\ell,t)|\partial_x u(-\ell,t)|^2\right)\mathrm{d}t.
\end{aligned}
$$

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