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Normal 5-edge-colorings of a family of Loupekhine snarks

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ABSTRACT

In a proper edge-coloring of a cubic graph an edge uv is called poor or rich, if the set of colors of the edges incident to u and v contains exactly three or five colors, respectively. An edge-coloring of a graph is normal, if any edge of the graph is either poor or rich. In this note, we show that some snarks constructed by using a method introduced by Loupekhine admit a normal edge-coloring with five colors. The existence of a Berge-Fulkerson Covering for a part of the snarks considered in this paper was recently proved by Manuel and Shanthi (2015). Since the existence of a normal edge-coloring with five colors implies the existence of a Berge-Fulkerson Covering, our main theorem can be viewed as a generalization of their result.

KEYWORDS

Cubic graph; Petersen coloring conjecture; normal edge-coloring; class of snarks

1. Introduction

In graph theory the Petersen coloring conjecture asserts that the edges of every bridgeless cubic graph G can be colored by using as set of colors the edge-set of the Petersen graph P , such that adjacent edges of G are colored with adjacent edges of P . The conjecture is well-known and is considered hard to prove as it implies classical conjectures in the field such as the Berge-Fulkerson conjecture and the Cycle Double Cover conjecture (see [5, 8, 19]). Jaeger in [8] introduced an equivalent formulation of the Petersen coloring conjecture. More precisely, he showed that a bridgeless cubic graph satisfies this conjecture, if and only if, it admits a normal edge-coloring with at most five colors (see exact definitions later). Let $\chi'_N(G)$ denote the minimum number of colors in a normal edge-coloring of G . Usually, it is called the normal chromatic index of G . In this terms, the Petersen coloring conjecture amounts to proving that every bridgeless cubic graph has normal chromatic index at most five. To the best of our knowledge, the smallest known upper bound for $\chi'_N(G)$ in an arbitrary bridgeless cubic graph is 7 (see [2, 12]). The situation is similar in the larger class of all simple cubic graphs (not necessarily bridgeless): there are cubic graphs with normal chromatic index 7, on the other hand, in [12] two of the authors have shown that any simple cubic graph admits a normal 7-edge-coloring. One may wonder whether the upper bound 7 can be improved in other interesting subclasses of cubic graphs. Related with this question, in this paper we show that a class of snarks obtained with a method introduced by Loupekhine, admits a normal edge-coloring with five colors. Let us remark that this result implies the main result in [11].

Now, let us give all necessary definitions and notions used in the paper. We consider finite and undirected graphs. They contain neither loops nor parallel edges.

For a graph G , the set of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. Let $\partial_G(v)$ be the set of edges of G that are incident to the vertex v of G . Assume that G and H are two cubic graphs. If there is a mapping $\phi : E(G) \rightarrow E(H)$, such that for each $v \in V(G)$ there is $w \in V(H)$ such that $\phi(\partial_G(v)) = \partial_H(w)$, then we refer to ϕ as an H -coloring of G . If G admits an H -coloring, then we write $H \prec G$.

Let P be the Petersen graph (Figure 1). The Petersen coloring conjecture of Jaeger states:

Conjecture 1. (Jaeger, 1988 [9]) *For any bridgeless cubic graph G , we have $P \prec G$.*

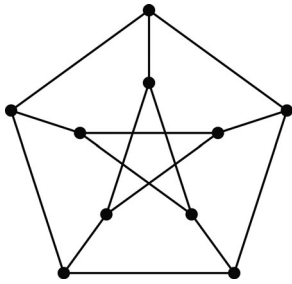
It is shown in [13] that the Petersen graph is the only connected, bridgeless cubic graph that can color all bridgeless cubic graphs. The conjecture is difficult to prove, since it implies the classical Berge-Fulkerson conjecture [5, 18] and (5,2)-cycle-cover conjecture [3, 15]. For our aim, we only need to recall the statement of the former one.

Conjecture 2. (Berge-Fulkerson, 1972) *Any bridgeless cubic graph G contains six (not necessarily distinct) perfect matchings such that any edge of G belongs to exactly two of them.*

We will call Berge-Fulkerson Covering of G any set of six perfect matchings which satisfy the condition in Conjecture 2.

2. Normal 5-edge-colorings

A k -edge-coloring of a graph G is an assignment of colors $\{1, \dots, k\}$ to edges of G , such that adjacent edges receive

Figure 1. The graph P .

different colors. For an edge-coloring c of G and a vertex v of G , let $S_c(v)$ be the set of colors that edges incident to v receive.

Definition 1. Assume that uv is an edge of a cubic graph G , and let c be an edge-coloring of G . We say that the edge uv is **poor** or **rich** with respect to c , if $|S_c(u) \cup S_c(v)| = 3$ or $|S_c(u) \cup S_c(v)| = 5$, respectively. An edge is **normal** (with respect to c) if it is poor or rich.

Edge-colorings in which all edges are poor are trivially 3-edge-colorings of G . On the other hand, edge-colorings having only rich edges have been considered before, and they are called strong edge-colorings [1]. In the present paper, we focus on the case when all edges must be normal.

Definition 2. An edge-coloring c of a cubic graph is **normal**, if any edge is normal with respect to c .

In [8], Jaeger has shown that:

Proposition 1. (Jaeger, [8]) Let G be any cubic graph. Then $P \prec G$, if and only if G admits a normal 5-edge-coloring.

This implies that **Conjecture 1** can be re-stated in the following way:

Conjecture 3. For any bridgeless cubic graph G , $\chi'_N(G) \leq 5$.

Our previous considerations imply that **Conjecture 3** holds for 3-edge-colorable cubic graphs. This means that the non-3-edge-colorable cubic graphs are the main obstacle for proving **Conjecture 3**. Let us note that in [6] **Conjecture 3** is verified for some non-3-edge-colorable bridgeless cubic graphs. Also, in [16] the percentage of edges of a bridgeless cubic graph, which can be made normal in a 5-edge-coloring, is estimated.

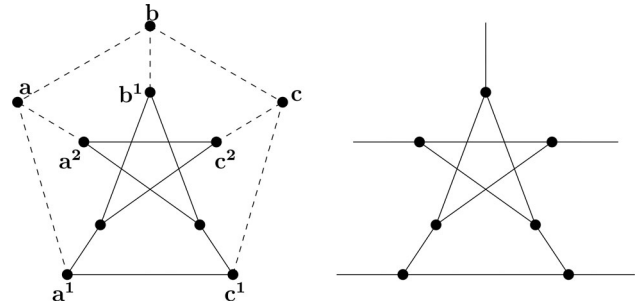
The following well-known result says that **Conjecture 3** implies **Conjecture 2**.

Proposition 2. Let G be a bridgeless cubic graph. If G admits a normal 5-edge-coloring, then it admits a Berge-Fulkerson covering.

Proof. (Sketch) The existence of a normal 5-edge-coloring is equivalent to the existence of a Petersen Coloring $\phi : E(G) \rightarrow E(P)$ of G . The preimages of the six distinct perfect matchings of P give a Berge-Fulkerson covering of G (see also [13]). \square

3. Loupekhine snarks

In many contexts, among all non-3-edge-colorable cubic graphs, ones with some additional restrictions are more

Figure 2. The component $B(P)$.

studied in order to avoid trivial cases. Even if there are many interesting definitions of what non-trivial could mean (see for instance [4, 14]), here we stick to the largely used definition of snark as a non-3-edge-colorable cyclically 4-edge-connected cubic graph of girth at least 5.

A classical method to construct snarks was proposed by Loupekhine in early 70s and it appears firstly in a paper of Isaacs in 1976 (see [7]).

Here, we present such a method in a bit more general context, in order to simplify our presentation. In particular, we do not really care whether the resulting graph is a snark or not. We will make use of the same terminology and notations in [10].

Let G be a bridgeless cubic graph and let abc be a path of length two in G . Consider the induced subgraph $B(G)$ of G obtained by removing the vertices a , b and c . Assume that the path abc is chosen in a way that $B(G)$ has exactly 5 vertices of degree 2, while every other vertex has degree equal to 3. We denote by a^1, a^2 the two vertices of degree 2 in $B(G)$ corresponding to the vertices distinct from b and adjacent to a in G . Analogously, we denote by b^1, c^1 and c^2 the other vertices of degree 2 in $B(G)$ (see **Figure 2** for an example).

Let $k > 1$ and G_1, G_2, \dots, G_k be arbitrary bridgeless cubic graphs. For all $i = 1, \dots, k$, consider the graph $B_i = B(G_i)$ and denote by $a_i^1, a_i^2, b_i^1, c_i^1, c_i^2$ its vertices of degree 2, according to the notation introduced above. We assume that for each i , the path $a^i b^i c^i$ is chosen so that these vertices exist in B_i . Consider the disjoint union of the k graphs $B_i = B(G_i)$. For all $i = 1, \dots, k$, add the pair of edges $a_i^1 c_{i+1}^1$ and $a_i^2 c_{i+1}^2$ between two of the vertices of degree 2 in B_i and two of the vertices of degree 2 in B_{i+1} (indices larger than k are taken modulo k).

The new graph G_k^B has exactly k vertices of degree 2, say $b_1^1, b_2^1, \dots, b_k^1$. Let G_k^C be a graph with exactly k vertices of degree one, say d_1, d_2, \dots, d_k , and all other vertices of degree 3. If we identify every vertex b_i^1 of G_k^B with a vertex d_i of G_k^C , we obtain a cubic graph, denoted by G_k^L .

Proposition 3. Let $k > 1$. Assume that G_k^C contains no edge whose removal leaves a graph containing a component with all vertices of degree two or three. Then G_k^L is a bridgeless cubic graph.

Loupekhine proved that, for a suitable choice of the graph G_k^C , if $k > 1$ is odd and the graphs G_1, G_2, \dots, G_k are snarks, then G_k^L is a snark as well. For this reason, all snarks obtained in this way are usually called Loupekhine snarks or

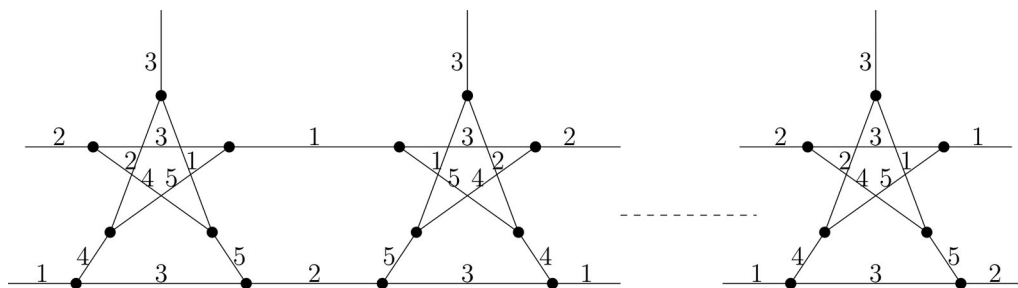


Figure 3. A normal 5 edge-coloring in the first subsequence of G_k^B .

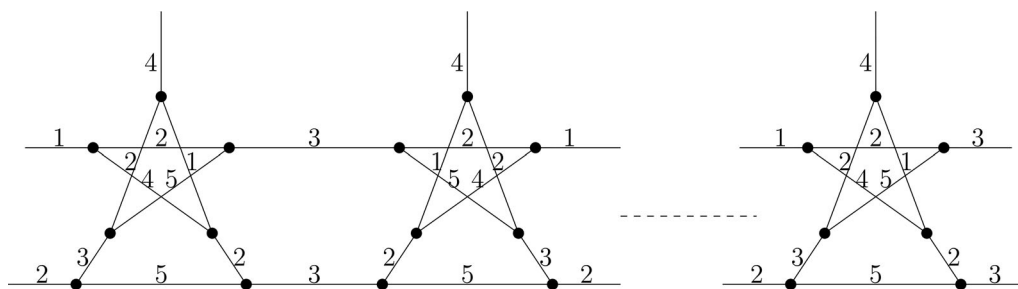


Figure 4. A normal 5 edge-coloring in the second subsequence of G_k^B .

L -snarks. Here, we extend this terminology to all graphs obtained by using the described method and in what follows we refer to them as L -graphs.

Definition 3. Let G be an L -graph.

- If every connected component of G_k^C is either an isolated edge or a star with three vertices of degree 1, then G is said to be an L_1 -graph.
- If every subgraph B_i is isomorphic to $B(P)$ (see Figure 2), where P is the Petersen graph, then G is said to be an LP -graph.
- An LP_1 -graph is a graph which is both an L_1 -graph and an LP -graph.

In [11], the authors prove that a very special family of LP_1 -graphs admits a Berge-Fulkerson covering. In the next section, we strongly generalize their result by proving the existence of a normal 5-edge-coloring for a larger class of LP_1 -graphs which properly contains the class of LP_1 -graphs studied in [11]. Hence, by Proposition 2, we have that all of them admit also a Berge-Fulkerson covering.

4. Main result

Now we construct a normal 5-edge-colorings for a class of LP_1 -graphs. Later, we will prove that our class properly contains all graphs considered in [11].

For any odd $k > 1$, let G_k^L be an LP_1 -graph, and, as described in Section 3, we consider its decomposition in the two subgraphs G_k^B and G_k^C . Recall that, since G_k^L is an LP_1 -graph, G_k^B is the disjoint union of copies of $B(P)$ and every connected component of G_k^C is either K_2 or a star $K_{1,3}$ with four vertices.

Firstly, we construct a normal 5-edge-coloring of G_k^B . We consider the k consecutive copies of $B(P)$, say B_1, \dots, B_k , and we subdivide them in three subsequences B_1, \dots, B_r ,

B_{r+1}, \dots, B_{r+s} and B_{r+s+1}, \dots, B_k with r and s odd. Note that all the three subsequences have an odd number of copies of $B(P)$ since k is also odd.

Color the subgraph of G_k^B induced by the subsequence B_1, \dots, B_r as in Figure 3.

Color the subgraph of G_k^B induced by the subsequence B_{r+1}, \dots, B_{r+s} as in Figure 4.

Color the subgraph of G_k^B induced by the subsequence B_{r+s+1}, \dots, B_k as in Figure 5.

By a direct check, the defined coloring is normal for each of the previous subgraphs and it remains normal also when we consider the entire subgraph G_k^B . Moreover, all the k vertices of degree 2 are incident to edges with colors 1 and 2.

Now, consider a subgraph G_k^C which satisfies the following additional properties:

P1) every copy of K_2 in G_k^C connects two vertices of G_k^B in the same subsequence;

P2) every copy of $K_{1,3}$ in G_k^C connects its vertex of degree 3 with three vertices of G_k^B in three different subsequences.

Color every edge of G_k^C incident a vertex of the first subsequence with the color 3, incident a vertex of the second subsequence with the color 4 and incident a vertex of the third subsequence with the color 5.

For all G_k^C which satisfy the two additional constraints, the presented coloring is a normal 5-edge-coloring of G_k^L . Then, we are now in position to state our main result.

Theorem 1. Let G_k^L be an LP_1 -graph and let $B_1 \dots B_k$ be the k blocks in G_k^B . If the k blocks can be partitioned in three subsequences of odd length in such a way that properties P1 and P2 hold, then G_k^L admits a normal 5-edge-coloring.

Now, let us show that the class of LP_1 -graphs presented in [11] satisfy the assumptions of Theorem 1.

For any odd $k > 1$, the LP_1 -graph considered in [11] is the one having the following decomposition in G_k^C and G_k^B .

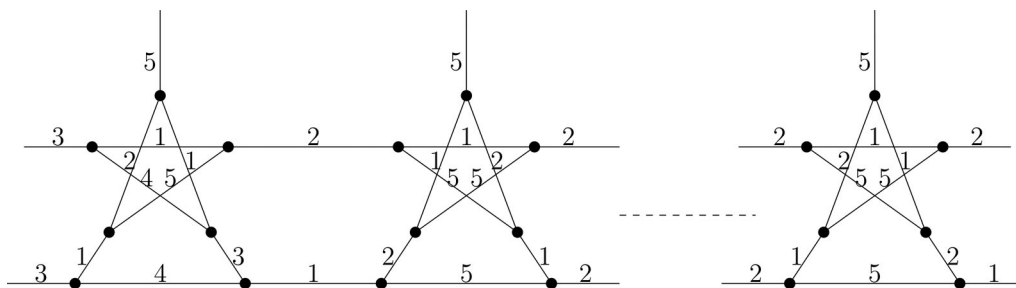


Figure 5. A normal 5 edge-coloring in the third subsequence of G_k^B .

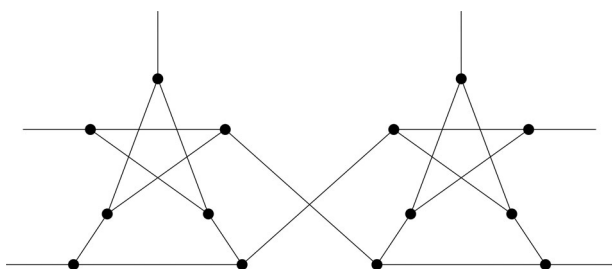


Figure 6. A twist between two copies of $B(P)$.

The subgraph G_k^C has a unique star and denote its three vertices of degree one by $z_1, z_{\lfloor k/2 \rfloor + 1}$ and $z_{\lfloor k/2 \rfloor + 2}$. Moreover, each copy of K_2 connects pairwise the vertices z_{i+2}, z_{k-i} , for $i = 0, \dots, \lfloor k/2 \rfloor - 2$.

We use the same patterns of coloring in Figures 3, 4, 5 where $B_{\lfloor k/2 \rfloor + 3}, \dots, B_1, \dots, B_{\lfloor k/2 \rfloor}$ is the first subsequence, the unique block $B_{\lfloor k/2 \rfloor + 1}$ gives the second subsequence and the unique block $B_{\lfloor k/2 \rfloor + 2}$ is the third one.

Hence, every element of this family of LP_1 -graphs satisfies both assumptions of Theorem 1 and then it is contained in our larger class of LP_1 -graphs.

Now, we briefly explain how we can slightly modify the coloring presented above in order to exhibit a normal 5-edge-coloring also for another class of Loupekhine graphs which contains the so called Loupekhine snarks of the second kind [17]. First of all, let us define this class. We will call *twist* the operation that replaces two edges between two consecutive blocks in G_k^B in the way described in Figure 6.

Definition 4. An LP_2 -graph is a graph which is obtained from an LP_1 -graph with a twist of the two edges between two consecutive selected blocks.

Observe that we do not really construct new examples of graphs if we perform additional twists between other blocks of the graphs so obtained. Indeed, it is easy to prove that an even number of twists produces a graph isomorphic to the original one. Hence, by applying an odd number of twists we obtain the same graph obtained by using a unique twist. Moreover, it is not relevant the pair of blocks where the twist is applied: indeed, given an LP_1 -graph G_k^L , all LP_2 -graphs obtained starting from G_k^L with a twist between two arbitrary consecutive blocks are isomorphic.

Hence, without loss of generality, we can always apply the twist to the two edges with color 3 between the last block in the second subsequence and the first block in the

third subsequence. With such a configuration, we maintain clearly the same pattern of coloring described before for the entire graph and the following holds.

Theorem 2. Let G_k^L be an LP_2 -graph and let $B_1 \dots B_k$ be the k blocks in G_k^B . If the k blocks can be partitioned in three subsequences of odd length in such a way that properties P1 and P2 hold, then G_k^L admits a normal 5-edge-coloring.

We would like to stress that it remains largely open the general problem of proving that the normal chromatic number of any Loupekhine snark is five, and it is still open even if we restrict our attention to the class of snarks among LP_1 -graphs.

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