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On balanced bipartitions of graphs

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ABSTRACT
Bollobás and Scott conjectured that every graph G has a balanced bipartite spanning subgraph H such that for each $v \in V(G), d_H(v) \geq (d_G(v) - 1)/2$, for each $v \in V(G)$. In this paper, we consider the contrary side and show that every graphic sequence has a realization G which admits a balanced bipartite spanning subgraph H such that $d_H(v) \leq \lceil (d_G(v) + 1)/2 \rceil$ for each $v \in V(G)$, and we show that the bound is sharp.

KEYWORDS
Bisection; balanced bipartition; degree sequence

AMS SUBJECT CLASSIFICATIONS (2000)
05C75

1. Introduction

For a graph G and for any $v \in V(G)$, we use $d_G(v)$ to denote the degree of the vertice v in G. Let G be a graph and $V_1, \cdots, V_k$ be a partition of $V(G)$. When $k = 2$, such a partition is said to be bipartition of G. A subgraph H of G is said to be a *bisection* of G if H is bipartite subgraph of G and the partition sets of H differs at most one. A bisection is also said to be balanced bipartition. Bollobás and Scott [1] conjectured that every graph G has a bisection H such that

$$d_H(v) \geq (d_G(v) - 1)/2 \text{ for all } v \in V(G)$$

In [3], Ji et al. give an infinite family of counterexamples to this Bollobás and Scott conjecture, which indicates that $\lfloor (d_G(v) - 1)/2 \rfloor$ rather $(d_G(v) - 1)/2$ is probably the correct lowered bound.

Similar to Bollobás and Scott conjecture, we propose the following conjecture.

Conjecture 1. *Every graph G has a bisection H such that*

$$d_H(v) \leq \lceil (d_G(v) + 1)/2 \rceil \text{ for all } v \in V(G)$$

This conjecture is concerned with the Bollobás and Scott conjecture. Let G^c denote the complement of a graph G. After a simple calculation, we can prove that if G^c admits a balanced bipartite spanning subgraph satisfied Conjecture 1, the same bipartition of $V(G)$ can induce a balanced spanning subgraph H of G satisfied

$$d_H(v) \geq \lfloor (d_G(v) - 1)/2 \rfloor \text{ for all } v \in V(G).$$

The bound is just Ji et al. mentioned. But the reverse may not be true.

Hartke and Seacrest [2] studied a degree sequence version of this Bollobás and Scott conjecture, and they proved that for any graphic sequence π with even length, π has a realization G which admits a bisection H such that $d_H(v) \geq (d_G(v) - 1)/2$ for all $v \in V(G)$. Ji et al. [3] extended the result to all degree sequences, they proved that for any graphic sequence π, π has a realization G which admits a bisection H such that $d_H(v) \geq (d_G(v) - 1)/2$ for all $v \in V(G)$.

For any positive integer k, let $[k] := \{1, 2, ..., k\}$. For a graph G and a labelling of its vertices $V(G) = \{v_1, ... v_n\}$, we defined the *parity bisection* of G to be the bisection with partition sets V_1 and V_2, where $V_i = \{v_j \in V(G) : j \equiv i \text{ mod } 2\}$ for each $i \in [2]$, and $E(H) = \{uv \in E(G) : u \in V_1, v \in V_2\}$.

In this paper, we obtain the following result.

Theorem 1. *Let $\pi = (d_1, ..., d_n)$ be any graphic sequence with $d_1 \geq ... \geq d_n$. Then there exists a realization G of π with $V(G) = \{v_1, ..., v_n\}$ and $d_G(v_i) = d_i$ for $i \in [n]$, such that if H denotes the parity bisection of G then $d_H(v) \leq \lceil (d_G(v) + 1)/2 \rceil$, for $v \in V(G)$.*

The bound in Theorem 1 is sharp as shown by the following examples. Let $\pi = (2k, 2k, \underbrace{2, ..., 2}_{2k-1})$ be a graphic sequence where k is a positive integer, let G be a realization of π, and $V(G) = \{v_1, v_2, ..., v_{2k+1}\}$, $d_G(v_1) = d_G(v_2) = 2k$, $d_G(v_3) = ... = d_G(v_{2k+1}) = 2$. We can see that v_1 and v_2 adjacent to all the other vertices, excluding itself, and G is unique up to isomorphism. Let H be an arbitrary bisection of G with parts A and B, without loss of generality, we may assume that $|A| = k$, and $|B| = k + 1$. If v_1 and v_2 are in the same part, there exists a vertex v such that $d_H(v) = 2$, and $(d_G(v) + 1)/2 < d_H(v) = \lceil (d_G(v) + 1)/2 \rceil$. If v_1 and v_2 are in the different parts, assume that $v_1 \in A$ and $v_2 \in B$, then $d_H(v_1) = k + 1$, and $(d_G(v_1) + 1)/2 < d_H(v_1) = \lceil (d_G(v_1) + 1)/2 \rceil$.

2. Proof of Theorem 1

For the proof of Theorem 1, we need two operations on a sequence of nonnegative integers. Let $\pi = (d_1, ..., d_n)$ with

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$d_1 \geq \ldots \geq d_n$. By removing d_i from π and subtracting 1 from the d_i remaining elements of π with lowest indices. We obtain a new sequence $\pi' = (d'_1, \ldots, d'_{i-1}, d'_{i+1}, \ldots, d'_n)$, and we say that π' is obtained from π by *laying off* d_i, and the laying off operation is introduced by Kleitman and Wang [4].

It is easy to see the sequence π' laying off d_i from π need not non-increasing. In order to avoid this case, Hartke and Seacrest [2] introduce a variation of laying off operation. Choose a fix $i \in [n]$, assume that s is the smallest value among the d_i largest elements in π, not including d_i itself. Let $L = \{i \in [n] - i : d_i > s\}$. Obviously, $|L| < d_i$. Let M be the set of $d_i - |L|$ largest indices j with $j \neq i$ and $d_j = s$. We removed d_i from π, subtract 1 from d_j for all $j \in L \cup M$. This operation is called *laying off with order*. In the following, we give a good result about laying off with order.

Lemma 1. (Harket-Seacrest [2]). *For any $i \in [n]$, the sequence $\pi = (d_1, \ldots, d_n)$ with $d_1 \geq \ldots \geq d_n$ is graphic if and only if the sequence π' obtained from π by laying off d_i with order is graphic.*

Having finished all the necessary preparations, we are ready to prove Theorem 1. First, we give a brief outline of our proof. Choose two consecutive elements d_l and d_{l+1} of π, by Lemma 1, we can obtain a new graphic sequence π'' with length n – 2 by laying off d_l and d_{l+1} with order. By induction, π'' has an $(n-2)$ vertex realization F whose parity bisection J has the desired property. We then show that one can form G from F by adding two new vertices and choosing their neighbors so that the parity bisection of G satisfies Theorem 1.

Proof of Theorem 1. We apply induction on the length n of the graphic sequence $\pi = (d_1, \ldots, d_n)$ with $d_1 \geq d_2 \geq \ldots \geq d_n$. The assertion is holds when $n = 1$, 2. So we may assume that $n \geq 3$ and the assertion holds for all graphic sequence with length less than n. There exists two consecutive elements of π that are identical; Let $l \in [n-1]$ be fixed such that

$$d_l = d_{l+1} = k$$

Let $\pi' = (d'_1, \ldots, d'_l, d'_{l+2}, \ldots d'_n)$ be the sequence obtained from π by laying off d_{l+1} with order. Let $\pi'' = (d''_1, \ldots, d''_{l-1}, d''_{l+2}, \ldots d''_n)$ be the sequence obtained from π' by laying off d'_l with order. By Lemma 1, π' and π'' are all graphic sequence.

Let $\omega = (f_1, f_2, \ldots, f_{n-2})$ be the sequence obtained from π with d_l and d_{l+1} removed, and re-indexed so that the indices are consecutive, i.e., $f_i = d_i$ for $i \in [l-1]$ and $f_i = d_{i+2}$ for $i \in [n-2] \setminus [l-1]$. Let $\omega' = (f'_1, f'_2, \ldots, f'_{n-2})$ be the sequence obtained from π' with d'_l removed, and re-indexed so that the indices are consecutive. Also, let $\omega'' = (f''_1, f''_2, \ldots, f''_{n-2})$ be the sequence obtained from π'' by re-indexing so that the indices are consecutive. Note that ω'' is graphic sequence.

To turn a realization of ω'' to a realization of π, we need to track the changes between f_i and f''_i for all $i \in [n-2]$. Note that $0 \leq f_i - f''_i \leq 2$. Let

$$X_1 = \{i \in [n-2] : f''_i = f_i - 1\},$$
$$X_2 = \{i \in [n-2] : f''_i = f_i - 2\}$$

and

$$K = d'_l = \begin{cases} k-1 & \text{if } d'_l = d_l - 1 \\ k & \text{if } d'_l = d_l \end{cases}$$

So $K = \Sigma_{i \in [n-2]} |f_i - f'_i| = \Sigma_{i \in [n-2]} |f'_i - f''_i|$; then

$$|X_1| + 2|X_2| = 2K$$

For convenience, we introduce some notation. For nonempty sets A and B of integers, we write $A < B$ if the maximum integer in A is less than the minimum integer in B. A set S of integers is *consecutive* if it consists of consecutive integers. A sequence of pairwise disjoint sets, $A_1, \ldots, A_t$, of integers is said to be *consecutive* if $A_1 \cup \ldots \cup A_t$ is consecutive and, for any $i, j \in [t]$ with $i < j$ and A_i and A_j nonempty, we have $A_i < A_j$.

Ji, Ma, Yan and Yu [3] proved the following Claim 1 and Claim 2, they are also essential in our proof of Theorem 1. For better understanding of the proof of Theorem 1, we now present the proof of Claim 1 and the method of this proof comes from [3].

Claim 1. (Ji, Ma, Yan and Yu [3]). *There exist consecutive sets R_1, R_2, R'_1, R'_2, Q such that $X_1 = R'_1 \cup R'_2$ and $X_2 = R_1 \cup R_2$ such that*

(a) *the sequence R_1, R'_1, Q, R'_2 is consecutive,*
(b) *either $R_2 = \emptyset$ or $R_2 = Q$, and*
(c) *$f''_i = f''_j + 1$ for all $i \in R'_1, j \in R'_2$.*

Proof of Claim 1. Let s be the minimum of the largest k numbers in π. In order to track the connection among ω, ω' and ω'', we divide $[n-2]$ into six pairwise disjoint sets:

$$A = \{i \in [n-2] : f_i \geq s+2\},$$
$$B = \{i \in [n-2] : f_i = s+1\},$$
$$C = \{i \in [n-2] : f_i = s, f'_i = f_i\},$$
$$D = \{i \in [n-2] : f_i = s, f'_i = f_i - 1\},$$
$$E = \{i \in [n-2] : f_i = s-1\},$$
$$F = \{i \in [n-2] : f_i \leq s-2\}.$$

By the definition of π' and ω', we can see that A, B, C, D, E, F are consecutive and

$$\forall i \in A, \quad f'_i = f_i - 1 \geq s+1,$$
$$\forall i \in B, \quad f'_i = f_i - 1 = s,$$
$$\forall i \in C, \quad f'_i = f_i = s,$$
$$\forall i \in D, \quad f'_i = f_i - 1 = s-1,$$
$$\forall i \in E, \quad f'_i = f_i = s-1,$$
$$\forall i \in F, \quad f'_i = f_i \leq s-1.$$

Thus, we can see that $A \cup B \cup D = \{i \in [n-2] : f'_i = f_i - 1\}$, and then $|A| + |B| + |D| = K$.

Let $Y = \{i \in [n-2] : f''_i = f'_i - 1\}$. Then it follows that

$$A \subseteq Y \text{ and } |Y| = K = |A| + |B| + |D|$$

To complete the proof of Claim 1, we distinguish four cases based on the relations among the size of B, C, D, E.

First, suppose $|C| \geq |B| + |D|$. Let C'' consists of the $|B| + |D|$ largest integers in C, and $C' = C \setminus C''$. It is easy to see that $Y = A \cup C''$. Let $R_1 = A$, $R_2 = \emptyset$, $R'_1 = B$, $R'_2 = C'' \cup D$ and $Q = C'$. We can see that $X_1 = R'_1 \cup R'_2$ and $X_2 = R_1 \cup R_2$, and (a) and (b) hold. Note that $f''_i = s$ for $i \in R'_1$, and $f''_i = s - 1$ for $i \in R'_2$; Then (c) holds.

Next, suppose that $|D| \leq |C| < |B| + |D|$. Let B'' consists of the $|B| + |D| - |C|$ largest integers in B, and $B' = B \setminus B''$. Then $Y = A \cup B'' \cup C$. Let $R_1 = A$, $R_2 = Q = B''$, $R'_1 = B'$, $R'_2 = C \cup D$. It is easy to see that $X_1 = R'_1 \cup R'_2$ and $X_2 = R_1 \cup R_2$, and (a) and (b) hold. Note that $f''_i = s$ for $i \in R'_1$, and $f''_i = s - 1$ for $i \in R'_2$; Then (c) holds.

Now, assume that $|C| < |D| \leq |C| + |E|$. Let E'' consists of the $|D| - |C|$ largest integers in E, and $E' = E \setminus E''$. Then $Y = A \cup B \cup C \cup E''$. Let $R_1 = A \cup B$, $R_2 = \emptyset$, $R'_1 = C \cup D$, $R'_2 = E''$, and $Q = E'$. It is easy to see that $X_1 = R'_1 \cup R'_2$ and $X_2 = R_1 \cup R_2$, and (a) and (b) hold. Note that $f''_i = s - 1$ for $i \in R'_1$, and $f''_i = s - 2$ for $i \in R'_2$; Then (c) holds.

Finally, we consider the case $|D| > |C| + |E|$. Let D'' consists of the $|D| - |C| - |E|$ largest integers in D, and $D' = D \setminus D''$. Then $Y = A \cup B \cup C \cup D'' \cup E$. Let $R_1 = A \cup B$, $R_2 = Q = D''$, $R'_1 = C \cup D'$, $R'_2 = E$. It is easy to see that $X_1 = R'_1 \cup R'_2$ and $X_2 = R_1 \cup R_2$, and (a) and (b) hold. Note that $f''_i = s - 1$ for $i \in R'_1$, and $f''_i = s - 2$ for $i \in R'_2$; Then (c) holds. □

Let $I_1 = \{i \in [n-2] : i \equiv 1 \ mod\ 2\}$ and $I_2 = \{i \in [n-2] : i \equiv 0 \ mod\ 2\}$.

Claim 2. (Ji, Ma, Yan and Yu [3]). $|X_1 \cap I_1| - |X_1 \cap I_2| \in \{-2, 0, 2\}$. *Moreover,* $|X_1 \cap I_1| - |X_1 \cap I_2| = 0$ *implies* $|X_2 \cap I_1| - |X_2 \cap I_2| \in \{-1, 0, 1\}$.

Now, we construct a realization of $\pi = (d_1, \ldots, d_n)$. Note that $\omega'' = (f''_1, \ldots, f''_{n-2})$ is a graphic sequence. By induction hypothesis, there exists a realization F of ω'' with $V(F) = \{w_1, .., w_{n-2}\}$, and $d_F(w_i) = f''_i$ for $i \in [n-2]$, such that the parity bisection J of F satisfies

$$d_J(w_i) \leq \lceil (d_F(w_i) + 1)/2 \rceil \ \textit{for all}\ i \in [n-2]. \tag{1}$$

Let $W_j = \{w_i : i \equiv j \ mod\ 2\}$ for $j \in [n-2]$.

In what follows, we will construct a graph G as a realization of π such that its parity bisection H of G satisfies $d_H(v) \leq \lceil (d_G(v) + 1)/2 \rceil$ for all $v \in V(G)$, by adding two new vertices a, b (so $V(G) = V(F) \cup \{a, b\}$) and some edges from these two vertices to F. Notice that if $K = k - 1$, we will add edge ab, for convenience, let

$$\epsilon = \begin{cases} 1 & \textit{if}\ K = k - 1 \\ 0 & \textit{if}\ K = k \end{cases}$$

We write $V(G) = \{v_1, \ldots, v_n\}$ such that $v_i = w_i$ for $i \in [l-1]$, $\{v_l, v_{l+1}\} = \{a, b\}$, $v_i = w_{i-2}$ for $i \in [n] \setminus [l+1]$.

In the view of Claim 2, we divide the proof of Theorem 1 into three cases. In each of the three cases, we use a to represent the vertex in $\{v_l, v_{l+1}\}$ with odd index, so the parity partition of $V(G)$ is

$$V_1 = W_1 \cup \{a\}, V_2 = W_2 \cup \{b\}.$$

Case 1. $|X_1 \cap I_1| - |X_1 \cap I_2| = 0$

We know that $F \subseteq G$ and $V(G) = V(F) \cup \{a, b\}$, and we need to add edges at a and b to form G, a realization of π. Add ab if $\epsilon = 1$, add av_i for all $i \in X_2 \cup (X_1 \cap I_1)$, and add bv_j for all $j \in X_2 \cup (X_1 \cap I_2)$, G is a realization of π. Let H denote the parity bisection of G, so V_1, V_2 are the partition sets of H. Now, we show that $d_H(v) \leq \lceil (d_G(v) + 1)/2 \rceil$ for all $v \in V(G)$.

In a particular case, when $X_1 = \emptyset$, by Claim 1, we can see $R_2 = \emptyset$ and $X_2 = R_1$. Since R_1 is start from the integer 1, we have

$$0 \leq |X_2 \cap I_1| - |X_2 \cap I_2| \leq 1, \tag{2}$$

For each w_i with $i \notin X_1 \cup X_2$, its neighborhoods in F and G are the same, by (1), $d_H(w_i) = d_J(w_i) \leq \lceil (d_F(w_i) + 1)/2 \rceil = \lceil (d_G(w_i) + 1)/2 \rceil$.

For w_i with $i \in X_2$, then $d_G(w_i) = d_F(w_i) + 2$, $d_H(w_i) = d_J(w_i) + 1$, by (1), $d_H(w_i) = d_J(w_i) + 1 \leq \lceil (d_F(w_i) + 1)/2 \rceil + 1 = \lceil (d_G(w_i) + 1)/2 \rceil$.

For w_i with $i \in X_1$, then $d_G(w_i) = d_F(w_i) + 1$, $d_H(w_i) = d_J(w_i)$, by (1), $d_H(w_i) = d_J(w_i) \leq \lceil (d_F(w_i) + 1)/2 \rceil = \lceil d_G(w_i)/2 \rceil$.

For the vertices a, $d_G(a) = |X_2| + |X_1 \cap I_1| + \epsilon$ and $d_H(a) = |X_2 \cap I_2| + \epsilon$. So

$$2d_H(a) - d_G(a) = (|X_2 \cap I_2| - |X_2 \cap I_1|) - |X_1 \cap I_1| + \epsilon$$

By Claim 2, we have $|X_2 \cap I_2| - |X_2 \cap I_1| \leq 1$. If $\epsilon = 0$, we have $2d_H(a) - d_G(a) \leq 1$, and $d_H(a) \leq (d_G(a) + 1)/2$; If $\epsilon = 1$ and $X_1 \neq \emptyset$, by $X_1 \neq \emptyset$, we have $|X_1 \cap I_1| = |X_1 \cap I_2| \geq 1$, then $2d_H(a) - d_G(a) \leq 1$, and $d_H(a) \leq (d_G(a) + 1)/2$; We can see that if $\epsilon = 0$ or $X_1 \neq \emptyset$, $d_H(a) \leq (d_G(a) + 1)/2$; If $\epsilon = 1$ and $X_1 = \emptyset$, by (2), we have $2d_H(a) - d_G(a) \leq 1$, and $d_H(a) \leq (d_G(a) + 1)/2$.

For the vertices b, then $d_G(b) = |X_2| + |X_1 \cap I_2| + \epsilon$ and $d_H(b) = |X_2 \cap I_1| + \epsilon$. So

$$2d_H(b) - d_G(b) = (|X_2 \cap I_1| - |X_2 \cap I_2|) - |X_1 \cap I_2| + \epsilon$$

By Claim 2, we have $|X_2 \cap I_1| - |X_2 \cap I_2| \leq 1$. Using the similar argument above, we can see that if $\epsilon = 0$ or $X_1 \neq \emptyset$, $2d_H(b) - d_G(b) \leq 1$, so $d_H(b) \leq (d_G(b) + 1)/2$; If $\epsilon = 1$ and $X_1 = \emptyset$, by (2), we have $|X_2 \cap I_1| - |X_2 \cap I_2| \leq 1$. When $|X_2 \cap I_1| - |X_2 \cap I_2| \leq 0$, it is easy to see $2d_H(a) - d_G(a) \leq 1$, so $d_H(a) \leq (d_G(a) + 1)/2$, and when $|X_2 \cap I_1| - |X_2 \cap I_2| = 1$, $d_G(b) = |X_2| + \epsilon = 2|X_2 \cap I_2| + 2$ and $d_H(b) = |X_2 \cap I_1| + \epsilon = |X_2 \cap I_2| + 2$, then $d_H(b) = \lceil (d_G(b) + 1)/2 \rceil$.

Case 2. $|X_1 \cap I_2| - |X_1 \cap I_1| = 2$

Recall that $X_1 = R'_1 \cup R'_2$, where each R'_i is consecutive. Thus $|R'_i \cap I_2| = |R'_i \cap I_1| + 1$ for $i \in [2]$. Since the sequence R_1, R'_1, Q, R'_2 is consecutive and start from integer 1, we see that $R_1 \cap I_1 \neq \emptyset$, $|R_1 \cap I_2| = |R_1 \cap I_1| - 1$, and $|Q \cap I_2| = |Q \cap I_1| - 1$. Since $R_2 = \emptyset$ or $R_2 = Q$ (by (b) of claim 1), we have

$$-2 \leq |X_2 \cap I_2| - |X_2 \cap I_1| \leq -1 \tag{3}$$

In a particular case, when $X_1 \cap I_1 = \emptyset$, $|X_1 \cap I_2| = 2$, and there is only one even integer in R'_1 and R'_2, respectively.

We claim that there exists some $z \in X_1 \cap I_2$ with $d_J(w_z) \leq (d_F(w_z) + 1)/2$. To see this, choose $x \in R'_1 \cap I_2$ and $y \in R'_2 \cap I_2$. By (1), we have $d_J(w_x) \leq \lceil (d_F(w_x) + 1)/2 \rceil$

and $d_J(w_y) \le \lceil (d_F(w_y)+1)/2 \rceil$. By (c) of Claim 1, $d_F(w_x) = d_F(w_y)+1$. Observe that $d_F(w_x)$ and $d_F(w_y)$ are of different parities. So there exists $z \in \{x,y\}$ and $d_F(w_z)$ is odd such that $d_J(w_z) \le (d_F(w_z)+1)/2$.

We now add edges at a and b to form G from F, add ab if $\epsilon = 1$, add av_i for all $i \in X_2 \cup (X_1 \cap I_1) \cup \{z\}$, and bv_j for all $j \in X_2 \cup (X_1 \cap I_2) \setminus \{z\}$, G is a realization of π. Next we show that the parity bisection H of G satisfied that $d_H(v) \le \lceil (d_G(v)+1)/2 \rceil$ for all $v \in V(G)$.

For each w_i with $i \notin X_1 \cup X_2$, its neighborhoods in F and G are the same, by (1), $d_H(w_i) = d_J(w_i) \le \lceil (d_F(w_i)+1)/2 \rceil = \lceil (d_G(w_i)+1)/2 \rceil$.

For w_i with $i \in X_2$, then then $d_G(w_i) = d_F(w_i)+2$, $d_H(w_i) = d_J(w_i)+1$, by (1), $d_H(w_i) = d_J(w_i)+1 \le \lceil (d_F(w_i)+1)/2 \rceil + 1 = \lceil (d_G(w_i)+1)/2 \rceil$.

For w_i with $i \in X_1 \setminus \{z\}$, then $d_G(w_i) = d_F(w_i)+1$, $d_H(w_i) = d_J(w_i)$, by (1), $d_H(w_i) = d_J(w_i) \le \lceil (d_F(w_i)+1)/2 \rceil = \lceil d_G(w_i)/2 \rceil$.

For the vertex w_z and note that $d_F(w_z)$ is odd, then $d_G(w_z) = d_F(w_z)+1$, $d_H(w_z) = d_J(w_z)+1$, by (1), $d_H(w_z) = d_J(w_i)+1 \le (d_F(w_z)+1)/2+1 = \lceil (d_G(w_z)+1)/2 \rceil$.

For the vertex a, $d_G(a) = |X_2| + |X_1 \cap I_1| + 1 + \epsilon$ and $d_H(a) = |X_2 \cap I_2| + 1 + \epsilon$. By (3), we have

$$2d_H(a) - d_G(a) = (|X_2 \cap I_2| - |X_2 \cap I_1|) - |X_1 \cap I_1| + 1 + \epsilon \le 1.$$

Then $d_H(a) \le (d_G(a)+1)/2$.

And for the vertex b, $d_G(b) = |X_2| + |X_1 \cap I_2| - 1 + \epsilon$ and $d_H(b) = |X_2 \cap I_1| + \epsilon$. we have

$$2d_H(b) - d_G(b) = (|X_2 \cap I_1| - |X_2 \cap I_2|) - |X_1 \cap I_2| + 1 + \epsilon$$

If $\epsilon = 0$, by (3) and notice that $|X_1 \cap I_2| \ge 2$, $2d_H(b) - d_G(b) \le 1$. Then $d_H(b) \le (d_G(b)+1)/2$.

If $\epsilon = 1$ and $X_1 \cap I_1 \ne \emptyset$, notice that $X_1 \cap I_1 \ne \emptyset$, then $|X_1 \cap I_2| \ge 3$, and by (3), $2d_H(b) - d_G(b) \le 1$. Then $d_H(b) \le (d_G(b)+1)/2$.

If $\epsilon = 1$ and $X_1 \cap I_1 = \emptyset$, $d_G(b) = |X_2| + 1 + \epsilon = 2(|R_1 \cap I_2| + |R_2 \cap I_2|) + 4$, $d_H(b) = |X_2 \cap I_1| + \epsilon = (|R_1 \cap I_2| + |R_2 \cap I_2|) + 3$. Then $d_H(b) = \lceil (d_G(b)+1)/2 \rceil$.

Case 3. $|X_1 \cap I_1| - |X_1 \cap I_2| = 2$

In this case, we have $|R'_i \cap I_1| = |R'_i \cap I_2| + 1$, for $i \in [2]$. Since R_1, R'_1, Q, R'_2 is consecutive, it follows that $|R_1 \cap I_1| = |R_1 \cap I_2|$, and $|Q \cap I_1| = |Q \cap I_2| - 1$. Since $R_2 = \emptyset$ or $R_2 = Q$, we have

$$0 \le |X_2 \cap I_2| - |X_2 \cap I_1| \le 1 \tag{4}$$

In a particular case, when $X_1 \cap I_2 = \emptyset$, $|X_1 \cap I_1| = 2$, and there is only one odd integer in R'_1 and R'_2, respectively.

Since X_1 is even and $|R'_i \cap I_1| = |R'_i \cap I_2| + 1$, for $i \in [2]$, there exist $x \in R'_1 \cap I_1$ and $y \in R'_2 \cap I_1$. By (1), we have $d_J(w_x) \le \lceil (d_F(w_x)+1)/2 \rceil$ and $d_J(w_y) \le \lceil (d_F(w_y)+1)/2 \rceil$. By (c) of Claim 1, $d_F(w_x) = d_F(w_y)+1$. Observe that $d_F(w_x)$ and $d_F(w_y)$ are of different parities. So there exists $z \in \{x,y\}$ and $d_F(w_z)$ is odd such that $d_J(w_z) \le (d_F(w_z)+1)/2$.

We now add edges at a and b to form G from F, add ab if $\epsilon = 1$, add av_i for all $i \in X_2 \cup (X_1 \cap I_1) \setminus \{z\}$, and bv_j for all $j \in X_2 \cup (X_1 \cap I_2) \cup \{z\}$, G is a realization of π. Next we show that the parity bisection H of G satisfied that $d_H(v) \le \lceil (d_G(v)+1)/2 \rceil$ for all $v \in V(G)$.

For each w_i with $i \notin X_1 \cup X_2$, its neighborhoods in F and G are the same, by (1), $d_H(w_i) = d_J(w_i) \le \lceil (d_F(w_i)+1)/2 \rceil = \lceil (d_G(w_i)+1)/2 \rceil$.

For w_i with $i \in X_2$, then $d_G(w_i) = d_F(w_i)+2$, $d_H(w_i) = d_J(w_i)+1$, by (1), $d_H(w_i) = d_J(w_i)+1 \le \lceil (d_F(w_i)+1)/2 \rceil + 1 = \lceil (d_G(w_i)+1)/2 \rceil$.

For w_i with $i \in X_1 \setminus \{z\}$, then $d_G(w_i) = d_F(w_i)+1$, $d_H(w_i) = d_J(w_i)$, by (1), $d_H(w_i) = d_J(w_i) \le \lceil (d_F(w_i)+1)/2 \rceil = \lceil d_G(w_i)/2 \rceil$.

For the vertex w_z and note that $d_F(w_z)$ is odd, then $d_G(w_z) = d_F(w_z)+1$, $d_H(w_z) = d_J(w_z)+1$, by (1), $d_H(w_z) = d_J(w_i)+1 \le (d_F(w_z)+1)/2+1 = \lceil (d_G(w_z)+1)/2 \rceil$.

For the vertex a, $d_G(a) = |X_2| + |X_1 \cap I_1| - 1 + \epsilon$ and $d_H(a) = |X_2 \cap I_2| + \epsilon$. By (4) and the fact $|X_1 \cap I_1| \ge 2$ we have

$$2d_H(a) - d_G(a) = (|X_2 \cap I_2| - |X_2 \cap I_1|) - |X_1 \cap I_1| + 1 + \epsilon \le 1$$

Then $d_H(a) \le (d_G(a)+1)/2$. And for the vertex b, $d_G(b) = |X_2| + |X_1 \cap I_2| + 1 + \epsilon$ and $d_H(b) = |X_2 \cap I_1| + 1 + \epsilon$. We have

$$2d_H(b) - d_G(b) = (|X_2 \cap I_1| - |X_2 \cap I_2|) - |X_1 \cap I_2| + 1 + \epsilon;$$

If $\epsilon = 0$, by (4), $2d_H(b) - d_G(b) \le 1$, then $d_H(b) \le (d_G(b)+1)/2$.

If $\epsilon = 1$ and $X_1 \cap I_2 \ne \emptyset$, notice that $X_1 \cap I_2 \ne \emptyset$, then $|X_1 \cap I_2| \ge 1$, we have $2d_H(b) - d_G(b) \le 1$, then $d_H(b) \le (d_G(b)+1)/2$.

If $\epsilon = 1$ and $X_1 \cap I_2 = \emptyset$, $d_G(b) = |X_2| + 1 + \epsilon = 2(|R_1 \cap I_2| + |R_2 \cap I_2|) + 1$, $d_H(b) = |X_2 \cap I_1| + 1 + \epsilon = (|R_1 \cap I_2| + |R_2 \cap I_2|) + 1$. Then $d_H(b) = (d_G(b)+1)/2$. □

Disclosure statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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