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A fixed-point approach for azimuthal equatorial ocean flows

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ABSTRACT
We present a study of equatorial ocean flows exhibiting geophysical effects, a complex vertical structure and moving in the azimuthal direction, with no variation in this direction. By means of a fixed-point approach, we study the relation between the pressure at the free surface of such flows and the resulting distortion of that free surface. The method in the present paper allows us to enlarge the class of nonlinearities from our previous works.

1. Introduction

This paper is concerned with a study of the ocean dynamics in the equatorial region of the Pacific ocean, situated within a band of about $2°$ latitude from the Equator. Equatorial waves are believed to be the main triggers of the El Niño phenomenon – an event associated with the appearance around the Christmas season of an ocean anomaly manifesting itself as a warm equatorial water flow approaching the western coast of South America. This occurrence was named by early fishermen 'El Niño' (Spanish for 'The Christ Child') and is one of the main factors in global climate change, cf. the discussion in [1].

For an appropriate analytical treatment that catches observed particularities of equatorial ocean dynamics, one needs to take into account a series of features like nonlinear effects arising from the conservation of momentum equations, as well as from the intricate boundary conditions, geophysical effects stemming from the Earth's rotation or the involved vertical structure of the ocean currents; e.g. while in a near-surface layer, and within about 150 km on each side of the Equator, there is the westward current (driven by the prevailing trade winds), confined to depths of no more than about 100–200 m, lies the Equatorial Undercurrent (EUC)–an eastward flowing jet whose core resides on the thermocline, cf. [2,3].

A rigorous mathematical framework for the study of geophysical ocean flows was established by Constantin [4–7] and Constantin and Johnson [1,2,8] by means of providing exact solutions describing geophysical water flows. It is to note that exact solutions are extremely rare in fluid mechanics, in general. While unable to capture all flow peculiarities, exact solutions might confirm the correctness of the governing equations. On the other hand, they supply the foundations of more direct and relevant analyses by means of asymptotic or perturbative methods. Several recent developments

pertaining to geophysical fluid dynamics were obtained by means of investigating exact solutions [9–27].

The latter aspect of exact solutions is also the prospect that we pursue here. More precisely, we work with the governing equations for an inviscid, incompressible fluid, written in cylindrical coordinates (with the Equator 'straightened' to become a generator of the cylinder) together with the free surface and the rigid boundary conditions. For this system, an exact solution is presented; it describes a steady flow which is moving only in the azimuthal direction, without variations in this direction. However, our study is able to accommodate an EUC, since the azimuthal velocity component has an arbitrary variation with respect to the depth (i.e. radius).

Using the pressure boundary condition at the free surface, we derive a relation between this pressure and the distortion of the free surface. This relation is shown to be amenable to a fixed-point approach for partially ordered sets as developed in [28–31]. Using the latter approach, we then prove existence and uniqueness for the function describing the shape of the free surface, under the (reasonable) assumption that the pressure acting on the surface is not too big. This method applies for larger classes of nonlinearities (allowing for more general flows) than those in [23].

2. The model for equatorial ocean waves

We give here a presentation of the geometry and of the variables associated with the rotating system. The coordinate system is chosen so that the Equator is 'straightened' and replaced by a line parallel to the z-axis, while the body of the sphere is represented by a circular disc described in the corresponding polar coordinates. Therefore, in a right-handed system, our coordinates are (r, θ, z), where r is the distance to the center of the disc (representing the Earth), $\theta \in (-\pi/2, \pi/2)$ is increasing from North to South and measures the deflection from the Equator, and the positive z-axis points from West to East. The equation $\theta = 0$ describes the line of the Equator. The corresponding unit vectors in the (r, θ, z) system are $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ and the velocity components are (u, v, w). Throughout the paper $R \approx 6378$ km will denote the Earth's radius.

Remark 2.1: The range of the polar angle θ is $[-\varepsilon, \varepsilon]$, with $\varepsilon = 0.016$, choice that accommodates a strip of the width of about 100 km centered about the Equator.

With regard to the previous considerations, the governing equations in a coordinate system with its origin at the center of the sphere are Euler's equations, which in cylindrical coordinates are written, cf. [2], as

$$u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} = -\frac{1}{\rho}p_r + F_r$$

$$v_t + uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}p_\theta + F_\theta \tag{1}$$

$$w_t + uw_r + \frac{v}{r}w_\theta + ww_z = -\frac{1}{\rho}p_z + F_z,$$

(where $p(r, \theta, z)$ denotes the pressure in the fluid and (F_r, F_θ, F_z) is the body-force vector) and the equation of mass conservation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}v_\theta + w_z = 0. \tag{2}$$

To include the effects of the Earth's rotation in our setting, we associate $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ to a point fixed on the sphere which is rotating about its polar axis. This means that we need to add in the left of Euler's

equations the Coriolis force

$$2\Omega \times \mathbf{u}$$

and the centripetal acceleration

$$\Omega \times (\Omega \times \mathbf{r}),$$

with

$$\Omega = -\Omega((\sin\theta)\mathbf{e}_r + (\cos\theta)\mathbf{e}_\theta),$$
$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\varphi,$$
$$\mathbf{r} = r\mathbf{e}_r,$$

where $\Omega \approx 7.29 \times 10^{-5}$ rad s^{-1} is the rotation speed of the Earth. Adding the joint contributions of the Coriolis and of the centripetal acceleration

$$2\Omega(-w\cos\theta, w\sin\theta, u\cos\theta - v\cos\theta) + r\Omega^2(-\cos^2\theta, \sin\theta\cos\theta, 0), \qquad (3)$$

to Equation (1), and taking into account that, cf. [2], the body force is due only to gravity (denoted with g), we have that the water motion is driven by the system

$$u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} - 2w\Omega\cos\theta - r\Omega^2\cos^2\theta = -\frac{1}{\rho}p_r - g$$

$$v_t + uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} + 2w\Omega\sin\theta + r\Omega^2\sin\theta\cos\theta = -\frac{1}{\rho}\frac{1}{r}p_\theta \qquad (4)$$

$$w_t + uw_r + \frac{v}{r}w_\theta + ww_z + 2\Omega(u\cos\theta - v\sin\theta) = -\frac{1}{\rho}p_z,$$

together with the equation of mass conservation (2). The specification of the water wave problem becomes complete after we impose the kinematic boundary conditions

$$u = wh_z + \frac{1}{r}vh_\theta \qquad (5)$$

on the free surface $r = R + h(\theta, z)$, and

$$u = wd_z + \frac{1}{r}vd_\theta \qquad (6)$$

on the bed $r = d(\theta, z)$, respectively, while, at the free surface, we also require the dynamic boundary condition

$$p = P(\theta, z). \qquad (7)$$

3. Existence and uniqueness of solutions

The solutions that we seek represent water flows that exhibit a pred direction of propagation. Namely, we will prove the existence of flows that are purely in the azimuthal direction with no variation in this

direction. This means that the velocity field (u, v, w) satisfies

$$u = v = 0 \quad \text{and} \quad w = w(r, \theta), \tag{8}$$

while the free surface is described by

$$r = R + h(\theta),$$

for some unknown function $\theta \to h(\theta)$, and the bed is represented by

$$r = d(\theta),$$

for some given function $\theta \to d(\theta)$. Amounting to the previous requirements, the system (4) becomes

$$- 2w\Omega \cos \theta - r\Omega^2 \cos^2 \theta = -\frac{1}{\rho} p_r - g$$

$$2w\Omega \sin \theta + r\Omega^2 \sin \theta \cos \theta = -\frac{1}{\rho} \frac{1}{r} p_\theta \tag{9}$$

$$0 = p_z.$$

Moreover, the equation of mass conservation (2), as well as the kinematic boundary conditions (5) and (6), are automatically satisfied.

Noticing from (9) that $p = p(r, \theta)$, we can eliminate the pressure from the first two equations and obtain that w satisfies the partial differential equation

$$r w_r \sin \theta + w_\theta \cos \theta = 0. \tag{10}$$

which can be solved by the method of characteristics. More precisely, we obtain

$$w(r, \theta) = f(r \cos \theta), \tag{11}$$

for some arbitrary function f. From the latter, we obtain the formula for the pressure

$$p(r, \theta) = A - \rho g r + 2\rho \Omega F(r \cos \theta) + \tfrac{1}{2} \rho r^2 \Omega^2 \cos^2 \theta, \tag{12}$$

where A is a constant, F is an anti derivative of f with $F(0) = 0$. Of course, in this circumstance

$$w(r, \theta) = F'(r \cos \theta).$$

The lack of the z dependence of the pressure function yields that the dynamic boundary condition (7) reads now

$$p = P(\theta) \quad \text{on } r = R + h(\theta).$$

The latter yields

$$P(\theta) = A - \rho g [R + h(\theta)] + \tfrac{1}{2} \rho [R + h(\theta)]^2 \Omega^2 \cos^2 \theta + 2\rho \Omega F([R + h(\theta)] \cos \theta), \tag{13}$$

equation that represents a link between the imposed pressure at the ocean's surface to the result-ing deformation of that surface. The pressure needed to maintain the free surface undisturbed and

following the Earth's curvature is obtained by setting $h \equiv 0$ in (13). We denote it by $P_0(\theta)$. Thus,

$$P_0(\theta) = A - \rho g R + \tfrac{1}{2}\rho R^2 \Omega^2 \cos^2 \theta + 2\rho \Omega F(R \cos \theta).$$

Moreover, assuming that the pressure at the Equator (that is at the location $\theta = 0$) is atmospheric, then

$$P_{\mathrm{atm}} = A - \rho g R + \tfrac{1}{2}\rho R^2 \Omega^2 + 2\rho \Omega F(R).$$

To non-dimensionalize (13), we divide it by P_{atm} and obtain

$$\alpha - \beta[1 + \mathfrak{h}(\theta)] + \gamma[1 + \mathfrak{h}(\theta)]^2 \cos^2 \theta + \delta \mathfrak{f}\big([1 + \mathfrak{h}(\theta)] \cos \theta\big) - \mathfrak{P}(\theta) = 0, \qquad (14)$$

with the non-dimensional constants $\alpha, \beta, \gamma, \delta > 0$ defined by means of

$$\alpha = \frac{A}{P_{\mathrm{atm}}}, \quad \beta = \frac{\rho g R}{P_{\mathrm{atm}}}, \quad \gamma = \frac{\rho R^2 \Omega^2}{2P_{\mathrm{atm}}}, \quad \delta = \frac{2\rho \Omega F(R)}{P_{\mathrm{atm}}}, \qquad (15)$$

and with the non-dimensional functions

$$\mathfrak{h}(\theta) := \frac{h(\theta)}{R},$$

$$\mathfrak{P}(\theta) := \frac{P(\theta)}{P_{\mathrm{atm}}},$$

$$\mathfrak{f}(s) := \frac{F(Rs)}{F(R)}.$$

Notice that Equation (14) can be written as the fixed-point equation

$$\mathcal{O}\mathfrak{H} = \mathfrak{H}, \qquad (16)$$

provided we denote $\mathfrak{H} := 1 + \mathfrak{h}$ and set

$$\mathcal{O}\mathfrak{H} = \frac{\alpha}{\beta} + \frac{\gamma}{\beta}\mathfrak{H}^2 \cos^2 \theta + \frac{\delta}{\beta}\mathfrak{f}(\mathfrak{H} \cos \theta) - \frac{1}{\beta}\mathfrak{P}(\theta). \qquad (17)$$

The latter equation allows for the utilization of a fixed-point theorem due to O'Regan and Petruşel [28] (see also Petruşel and Rus [29]), which improves upon a result of Ran and Reurings [31]. We quote below the notions related to this result, as well as the result itself and refer the reader to [28,29] for further details.

Let $(X, \preceq)$ be a partially ordered set, i.e. X is a nonempty set and $\preceq$ is a reflexive, transitive and anti-symmetric relation on X. In this context, we denote

$$X_{\preceq} := \{(x, y) \in X \times X \mid x \preceq y \text{ or } y \preceq x\}.$$

Let X be a nonempty set. Then, by definition $(X, d, \preceq)$ is an ordered metric space if and only if:

(i) (X, d) is an metric space;
(ii) $(X, \preceq)$ is a partially ordered set;
(iii) $(x_n)_{n \in \mathbb{N}} \to x$, $(y_n)_{n \in \mathbb{N}} \to y$ and $x_n \preceq y_n$, for each $n \in \mathbb{N} \Rightarrow x \preceq y$.

The following result (see [30]) will be applied in our main theorem.

Theorem 3.1: *Let $(X, d, \preceq)$ be an ordered metric space and $\mathcal{O} : X \to X$ be an operator. We suppose that*:

(i) *For each $x, y \in X$ with $(x, y) \notin X_\le$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_\le$ and $(y, c(x, y)) \in X_\le$;*

(ii) *$\mathcal{O} : (X, \preceq) \to (X, \preceq)$ is increasing;*

(iii) *there exists $x_0 \in X$ such that $x_0 \preceq \mathcal{O}(x_0)$;*

(iv)$_a$ *$\mathcal{O}$ is continuous*

      or

(iv)$_b$ *if (x_n) is an increasing sequence convergent in X to x, then $x_n \preceq x$, for all $n \in \mathbb{N}$;*

(v) *there exists $a \in]0, 1[$ such that $d(\mathcal{O}(x), \mathcal{O}(y)) \le a d(x, y)$, for each $x, y \in X$ with $x \preceq y$;*

(vi) *the metric d is complete.*

    Then $\mathcal{O}$ is a Picard operator, i.e. $\mathcal{O}$ has a unique fixed-point $x^* \in X$ and, for each $x \in X$, the sequence $(\mathcal{O}^n(x))_{n \in \mathbb{N}}$ of successive approximations of $\mathcal{O}$ starting from x converges to x^*.

    Now, we can prove our existence, uniqueness and approximation result for the considered problem.

Theorem 3.2: *We consider the problem (14) and the equivalent fixed-point problem (16), where the operator $\mathcal{O}$ is given by (17). We assume that $w \le (g/6\Omega) \, m \cdot s^{-1}$. Given any sufficiently small deviation $\mathfrak{P}$ from $\mathfrak{P}_0$ such that $\mathfrak{P}_0(\theta) \le \alpha$ for all $\theta \in [0, \varepsilon]$, there exists a unique $\mathfrak{H}^* \in C[0, \varepsilon]$ with*

$$\sup_{\theta \in [0,\varepsilon]} |\mathfrak{H}^*(\theta)| \le \frac{g}{2R\Omega^2},$$

satisfying $\mathcal{O}\mathfrak{H}^* = \mathfrak{H}^*$. Moreover, $\mathfrak{H}^*$ can be obtained as the limit of the sequence $(\mathcal{O}^n(\mathfrak{H}))_{n \in \mathbb{N}}$, for every $\mathfrak{H} \in C([0, \varepsilon], \mathbb{R}_+)$, with $\sup_{\theta \in [0,\varepsilon]} |\mathfrak{H}(\theta)| \le g/2R\Omega^2$.

Proof: Let

$$X := \left\{ \mathfrak{H} \in C([0, \varepsilon], \mathbb{R}_+) : \sup_{\theta \in [0,\varepsilon]} |\mathfrak{H}(\theta)| \le \frac{g}{2R\Omega^2} \right\}$$

be the set of positive valued continuous functions on $[0, \varepsilon]$ that do not exceed $g/2R\Omega^2$.

We define a partial ordering '$\preceq$' on X by requiring for $\mathfrak{H}, \mathfrak{G} \in X$, that

$$\mathfrak{H} \preceq \mathfrak{G} \text{ if and only if } \mathfrak{H}(\theta) \le \mathfrak{G}(\theta), \quad \text{for all } \theta \in [0, \varepsilon].$$

Moreover, the set X can be given the structure of a metric space by defining the functional $d : X \times X \to \mathbb{R}$ through

$$d(\mathfrak{H}, \mathfrak{G}) := \sup_{\theta \in [0,\varepsilon]} |\mathfrak{H}(\theta) - \mathfrak{G}(\theta)| \quad \text{for} \quad \mathfrak{H}, \mathfrak{G} \in X$$

Clearly, d defined above is a complete metric on X and $(X, d, \preceq)$ is an ordered metric space. Moreover, for any increasing sequence $(\mathfrak{H}_n)_{n \in \mathbb{N}}$ in X converging to a certain $\mathfrak{H}^* \in X$ we have $\mathfrak{H}_n(t) \le \mathfrak{H}^*(t)$, for any $t \in [0, \varepsilon]$. Also, for every $\mathfrak{H}, \mathfrak{G} \in X$ there exists $c(\mathfrak{H}, \mathfrak{G}) \in X$ which is comparable, with respect '$\preceq$', with $\mathfrak{H}$ and $\mathfrak{G}$.

Defining $\mathcal{O}$ by the formula (17), we will prove first that $\mathcal{O}$ is continuous and monotone. To this end, note that

$$\mathcal{O}\mathfrak{H}_1 - \mathcal{O}\mathfrak{H}_2 = \frac{\gamma}{\beta}\cos^2\theta(\mathfrak{H}_1^2 - \mathfrak{H}_2^2) + \frac{\delta}{\beta}(\mathfrak{f}(\mathfrak{H}_1\cos\theta) - \mathfrak{f}(\mathfrak{H}_1\cos\theta)). \qquad (18)$$

Utilizing the mean value theorem, we find that

$$\mathfrak{f}(\mathfrak{H}_1\cos\theta) - \mathfrak{f}(\mathfrak{H}_2\cos\theta) = \frac{R}{F(R)} \cdot f(Rc_\theta)(\mathfrak{H}_1 - \mathfrak{H}_2)\cos\theta,$$

for some c_θ lying between $\mathfrak{H}_1\cos\theta$ and $\mathfrak{H}_2\cos\theta$. Taking now into account the formulas for γ, δ and β from (15) we infer that

$$\mathcal{O}\mathfrak{H}_1 - \mathcal{O}\mathfrak{H}_2 = \frac{R\Omega^2}{2g}\cos^2\theta(\mathfrak{H}_1^2 - \mathfrak{H}_2^2) + \frac{2\Omega\cos\theta}{g}f(Rc_\theta)(\mathfrak{H}_1 - \mathfrak{H}_2), \qquad (19)$$

from which we see at once that $\mathcal{O}$ is continuous and monotone, since $\mathfrak{H}_1, \mathfrak{H}_2 \in X$ and since f is positive by (11).

To verify condition (v) from Theorem 3.1, we use the assumption that the w-component of the velocity field is less than $g/6\Omega$ and see from (19) that

$$|\mathcal{O}\mathfrak{H}_1 - \mathcal{O}\mathfrak{H}_2| \leq \tfrac{1}{2}|\mathfrak{H}_1 - \mathfrak{H}_2| + \tfrac{1}{3}|\mathfrak{H}_1 - \mathfrak{H}_2| = \tfrac{5}{6}|\mathfrak{H}_1 - \mathfrak{H}_2| \qquad (20)$$

Letting $\mathfrak{H}_0 \equiv 0$ we see that condition (iii) from Theorem 3.1 is satisfied. Indeed, the relation $\mathfrak{H}_0 \preceq \mathcal{O}(\mathfrak{H}_0)$ is equivalent with $\mathfrak{P}_0(\theta) \leq \alpha$, for all $\theta \in [0, \varepsilon]$, which means, according to [2], that A is of the order of 6.4×10^5 bar. This gives exactly our assumption. This concludes the proof. ∎

Remark 3.3: In the proof of the above result the operator $\mathcal{O}$ was continuous. However, by Theorem 3.1, we can see that a fixed-point result, in the above context, can be obtained for non-continuous mappings, using the assumption $(iv)_b$.

4. Conclusion

We showed the relevance of recent advances in fixed-point theory to the study of nonlinear ocean flows; we refer to [32] for a survey of fixed-point theorems. We believe that the presented approach can be further developed to study other problems of current interest in equatorial water flows – e.g. the inclusion of capillary effects [18] and accounting for the Earth's curvature [23]. Some aspects of geophysical flows in the Southern Ocean appear also to be amenable to the type of approach undertaken in [8,16,17,19].

Disclosure statement

No potential conflict of interest was reported by the author(s).

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