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ABSTRACT
A Roman domination function on a graph G is a function $r : V(G) \to \{0,1,2\}$ satisfying the condition that every vertex u for which $r(u) = 0$ is adjacent to at least one vertex v for which $r(v) = 2$. The weight of a Roman domination function is the value $r(V(G)) = \sum_{u\in V(G)} r(u)$. The Roman domination number $\gamma_R(G)$ of G is the minimum weight of a Roman domination function on G. "Roman Criticality" often refers to the study of graphs where the Roman domination number decreases when adding an edge or removing a vertex of the graph. In this paper we add some condition to this notion of criticality and give a complete characterization of critical graphs with Roman Domination number $\gamma_R(G) = 4$.

KEYWORDS
Roman domination; critical

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n, this is, a graph without loops and multiple edges and with $|V(G)| = n$. Notice that herein we are not assuming that the graph is connected. Let $V(G)$ and $E(G)$ denote, as usual, the sets of vertices and edges respectively. The degree of a vertex v, $deg_G(v)$ or simply $deg(v)$, is the number of edges in $E(G)$ incident to it. We denote by $N_G[v]$, or just $N[v]$ the closed neighborhood of a vertex v in G. As usual, $G\backslash\{v\}$ denotes the graph which is obtained by removing vertex $\{v\}$ together with all edges containing it.

A function $r : V(G) \to \{0,1,2\}$ is a *Roman domination function* if for every $u \in V(G)$ such that $r(u) = 0$, then there is a vertex v adjacent to u so that $r(v) = 2$. The weight of a Roman domination function is the value $r(V(G)) = \sum_{u\in V} r(u)$. The *Roman domination number* (or RDN) of a graph G, denoted by $\gamma_R(G)$ is the minimum weight of all possible Roman domination functions on G.

If a Roman domination function r on G satisfies that $\gamma_R(G) = \sum_{u\in V(G)} r(u)$ we say that r is a minimal Roman domination function or a γ_R-function.

Given a graph G and a Roman domination function $r : V(G) \to \{0,1,2\}$ let $P := (V_0; V_1; V_2)$ be the partition of $V(G)$ induced by r, where $V_i = \{u \in V(G) \mid r(u) = i\}$. Then, P is called a *Roman partition*. Clearly there is a one to one correspondence between Roman domination functions $r : V(G) \to \{0,1,2\}$ and Roman partitions $(V_0; V_1; V_2)$. Then, we may denote $r = (V_0; V_1; V_2)$.

For basic properties on Roman domination functions, see for instance [4] and [8] and for an excellent motivation on the topic see [11] and [13].

In the last years, Roman domination has been intensely studied. Some papers study extremal problems on the RDN of a graph giving upper and lower bounds for it. See, for example [1] or [5]. Other authors are interested in how the RDN is changed or unchanged by removing a vertex or by adding or removing an edge. These properties are usually called criticality properties. For example, in [10], the authors study how the RDN changes by removing a vertex or an edge from a graph. Similarly, in [7], the authors study graphs for which removing a vertex or adding a new edge decreases the RDN. On the contrary, in [2], the authors study graphs for which adding a new edge does not change the RDN. A good compilation of all these properties can be found in [12]. For other works on criticality properties of the RDN see also [3, 6, 9] and the references therein.

A graph is called *vertex critical* if removing any vertex decreases the Roman domination number. A graph is called *edge critical* if adding any edge decreases the Roman domination number. If a graph is vertex critical then the elimination of any edge does not change the RDN. A vertex critical graph is called *edge-vertex critical* if eliminating any edge the graph is not vertex-critical anymore.

In this paper, we give a complete identification of the family of graphs which are vertex critical, edge critical and edge-vertex critical with Roman domination number four. In fact, we obtain that there is a graph G_n with n vertices (for $n > 5$) and $\gamma_R(G_n) = 4$ satisfying these three properties if and only if n is even. Furthermore, G_n is unique for each even n. We also prove that it can be checked if any graph G with $\gamma_R(G) = 4$ satisfies these criticality properties by looking at the induced subgraphs with 8 vertices

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2. Some critical properties on Roman domination

Definition 2.1. A graph G is nonelementary if $\gamma_R(G) < |V(G)|$.

It is well known the following characterization of nonelementary graphs.

Proposition 2.2. *A graph G is nonelementary if and only if there is a connected component with at least 3 vertices.*

Remark 2.1. If G is nonelementary, $\gamma_R(G) \leq 3$ if and only if there exists a vertex $v \in V(G)$ with $\deg(v) \geq n-2$.

A graph G is called *vertex critical* or *v-critical* if for every $v \in V(G), \gamma_R(G\backslash\{v\}) = \gamma_R(G) - 1$.

The following useful result appears as Proposition 11 in [10] and can be found also in [7].

Lemma 2.3. *A graph $G = (V(G), E(G))$ is v-critical if and only if for every vertex v there is a γ_R-function $r = (V_0; V_1; V_2)$ such that $v \in V_1$.*

A graph $G = (V(G), E(G))$ is called *edge critical* or *e-critical* if for any pair of nonadjacent vertices v, w, the graph $G' = (V(G), E(G) \cup [v, w])$ satisfies that $\gamma_R(G') = \gamma_R(G) - 1$.

The following proposition was proved in [7].

Proposition 2.4. *A graph $G = (V(G), E(G))$ is e-critical if and only if for every pair of vertices v, w such that $[v, w] \notin E(G)$ there exists a γ_R-function $r = (V_0; V_1; V_2)$ such that $v \in V_1$ and $w \in V_2$ or $w \in V_1$ and $v \in V_2$.*

Proposition 2.5. *If $G = (V(G), E(G))$ is a v-critical graph, then for any $e \in E(G), \gamma_R(G \setminus e) = \gamma_R(G)$.*

Proof. It is readily seen that $\gamma_R(G \setminus e) \geq \gamma_R(G)$. Consider any $e = [v, w] \in E(G)$. By Lemma 2.3, there is a γ_R-function of G, $r = (V_0; V_1; V_2)$, such that $v \in V_1$. Hence, r is a Roman function on $G \setminus e$ and $\gamma_R(G \setminus e) \leq \gamma_R(G)$. Therefore, $\gamma_R(G \setminus e) = \gamma_R(G)$. □

Definition 2.6. A v-critical graph $G = (V(G), E(G))$ is edge-vertex critical or *e-v-critical* if for every edge $e \in E(G)$ the graph $G \setminus e$ is not v-critical.

Proposition 2.7. *A v-critical graph $G = (V(G), E(G))$ is e-v-critical if and only if for every edge e there exists a vertex v_e such that for any γ_R-function $r = (V_0; V_1; V_2)$ with $v_e \in V_1$, then $e = [v, w]$ with $v \in V_0$ and $N[v] \cap V_2 = \{w\}$.*

Proof. By Proposition 2.5, $\gamma_R(G\backslash\{e\}) = n$. If G is e-v-critical, then for every $e \in E(G)$, $G\backslash\{e\}$ is not v-critical. Therefore, there exists v_e such that $\gamma_R(G\backslash\{e, v_e\}) = n$. Since G is v-critical, $\gamma_R(G\backslash\{v_e\}) = n-1$. Then, for any Roman function on $G\backslash\{v_e\}, r' = (V_0'; V_1'; V_2')$, with $|V_1'| + 2|V_2'| = n-1$ (in particular, for any Roman function of G, $r = (V_0; V_1; V_2)$, with $|V_1| + 2|V_2| = n$ and $v_e \in V_1$), $(V_0'; V_1'; V_2')$ (resp. $(V_0; V_1\backslash\{v_e\}; V_2)$) is not Roman on $G\backslash\{e, v_e\}$. Therefore, $e \in [V_0', V_2']$ (resp. $e \in [V_0, V_2]$). Moreover, if $e = [v, w]$ with $v \in V_0$, then the only vertex in V_2 adjacent to v is w.

Now, consider any $e \in E(G)$ and let v_e be such that for any γ_R-function $r = (V_0; V_1; V_2)$ with $v_e \in V_1$, then $e \in [V_0, V_2]$ and if $e = [v, w]$ with $v \in V_0$ then w is the only vertex in V_2 which is adjacent to v. Hence, r is not Roman for $G\backslash\{e\}$. Thus, there is no Roman function $r = (V_0; V_1; V_2)$ on $G\backslash\{e\}$ with $v_e \in V_1$ and $|V_1| + 2|V_2| = n$. Hence, by Lemma 2.3, $G\backslash\{e\}$ is not v-critical. □

3. Roman domination number 4

Notice that e-critical, v-critical and e-v-critical graphs with Roman domination number less or equal than 3 are trivially classified. In this section we study the first non-trivial case: graphs satisfying these criticality properties with Roman domination number 4. We obtain a complete identification of this (infinite) family proving that the graph is determined (up to isomorphism) by the number of vertices. (We do not distinguish between isomorphic graphs. Thus, given two graphs $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$, if there is a bijection $\varphi : V(G) \to V(G')$ such that $[v, w] \in E(G)$ if and only if $[\varphi(v), \varphi(w)] \in E(G')$ we consider them the same graph and simply write $G = G'$.)

It is immediate to check that the only elementary v-critical graphs G with $\gamma_R(G) = 4$ are $(V(G_1)) = \{a, b, c, d\}$, $E(G_1) = \emptyset)$, $(V(G_2) = \{a, b, c, d\}$, $E(G_2) = \{[a, b]\})$ and $(V(G_3) = \{a, b, c, d\}, E(G_3) = \{[a, b], [c, d]\})$.

Lemma 3.1. *Let G be a nonelementary graph with $\gamma_R(G) = 4$. Then, G is v-critical if and only if for every $x \in V(G)$ there exist two vertices a_x, b_x so that $a_x \neq x \neq b_x$ and such that $N[a_x] = G\backslash\{x, b_x\}$.*

Proof. The if part is clear. For every $x \in V(G)$, it suffices to define $r_x : V(G)\backslash\{x\} \to \{0, 1, 2\}$ so that $r_x(a_x) = 2, r_x(b_x) = 1$ and $r_x(y) = 0$ for all $y \neq a_x, b_x$. Then, r_x is Roman and $\gamma_R(G\backslash\{x\}) \leq 3$. (In fact, since $\gamma_R(G) = 4, \gamma_R(G\backslash\{x\}) = 3$).

Now let G be a nonelementary v-critical graph with $\gamma_R(G) = 4$. Since it is v-critical, for any vertex $x \in V(G)$ there is a Roman domination function $r_x : V(G)\backslash\{x\} \to \{0, 1, 2\}$ such that $\sum_{u \in V(G)\backslash\{x\}} r_x(u) = 3$. Since it is nonelementary, $|V(G)\backslash\{x\}| > 3$ and hence, there are two vertices $y, z \in V(G)\backslash\{x\}$ so that $r_x(y) = 2$ and $r_x(z) = 1$. Since $\gamma_R(G\backslash\{x\}) = 3$, $[y, z] \notin E(G)$.

Then, let $a_x = y$ and $b_x = z$. If $[a_x, x] \in E(G)$, then $r_x' : V(G) \to \{0, 1, 2\}$ such that $r_x'|_{V(G)\backslash\{x\}} := r_x$ and $r_x'(x) = 0$ is Roman and $\gamma_R(G) \leq 3$ which is a contradiction. Therefore, $N[a_x] = G\backslash\{x, b_x\}$. □

Corollary 3.2. *Let G be a nonelementary v-critical graph with $\gamma_R(G) = 4$ and consider x, a_x, b_x such that $N[a_x] = G\backslash\{x, b_x\}$. Then, either $N[x] = V(G)\backslash\{a_x, y\}$ or $N[b_x] = V(G)\backslash\{a_x, y\}$ for some $y \in V(G)$.*

Proof. It suffices to apply Lemma 3.1 to a_x. There are edges from a_x to every vertex different from x, b_x. Therefore, $a_{a_x} \in \{x, b_x\}$. □

Lemma 3.1 implies the following theorem which appears as Theorem 14 in [10].

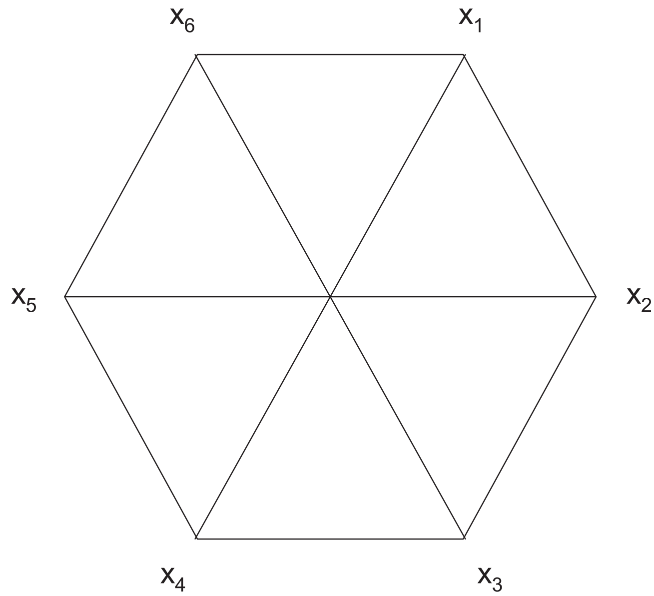

Figure 1. X_6 is v-critical.

Theorem 3.1. *Let G be a nonelementary graph of order n and Roman domination number $\gamma_R(G) = 4$. Then, G is v-critical if and only if for every $x \in V(G)$ there exists a nonadjacent vertex a_x with $\deg(a_x) = n-3$.*

The following proposition establishes that in a v-critical graph with Roman domination number four at least half of the vertices have degree $n - 3$.

Proposition 3.3. *Let G be a nonelementary v-critical graph of order n and Roman domination number $\gamma_R(G) = 4$. If $W_1 := \{v \in V(G) | \deg(v) = n-3\}$, then $|W_1| \geq \frac{n}{2}$.*

Proof. By Theorem 3.1 we know that there is at least one vertex, a_1 such that $\deg(a_1) = n-3$. Then, there are exactly two vertices x_1, y_1 so that $N[a_1] = V(G) \backslash \{x_1, y_1\}$. Consider any point $x_2 \notin \{x_1, y_1\}$. By Lemma 3.1, there is some vertex a_2 such that $N[a_2] = V(G) \backslash \{x_2, y_2\}$ for some y_2 (not necessarily different from x_1, y_1). Since $x_2 \notin \{x_1, y_1\}$, then $a_2 \neq a_1$ and $|\{x_1, y_1, x_2, y_2\}| \leq 4$. Then, we consider some vertex $x_3 \notin \{x_1, y_1, x_2, y_2\}$ and repeat the process until $\{x_1, y_1, ..., x_k, y_k\} = V(G)$. Then $k \geq \frac{n}{2}$ and $a_1, ..., a_k$ are k different vertices with degree $n - 3$. □

Remark 3.4. Using Theorem 3.1 it is easy to build an infinite family of different nonelementary v-critical graphs with $\gamma_R(G) = 4$.

Consider a cycle, C_n, of length $n \geq 5$ where the vertices $V(C_n)$ are ordered in the natural way by $x_1, ..., x_n$, and $x_i, x_{i+1} \in E(C_n)$ for every $i \equiv 1, ..., n (mod\ n)$. As we already know, if $n=5$ $\gamma_R(C_5) = 4$ and the graph is v-critical. Assume $n \geq 6$ and let us define X_n by attaching some extra edges to C_n so that, for every vertex x_i, $N[x_i] = V(C_n) \backslash \{x_{i-2}, x_{i+2}\}$ (where the vertices are taken (mod n)). See Figure 1. By Lemma 3.1, the resulting graph X_n is a nonelementary v-critical graph with $\gamma_R(X_n) = 4$.

Observe that these are not the only examples of v-critical graphs with Roman domination number four. Take any of the X_n above, with $n \geq 6$. Then, if we remove any of those extra edges (consider, for example, an hexagon with two diagonals, $[x_1, x_4]$ and $[x_2, x_5]$) it is still a nonelementary v-critical graph with $\gamma_R(G) = 4$.

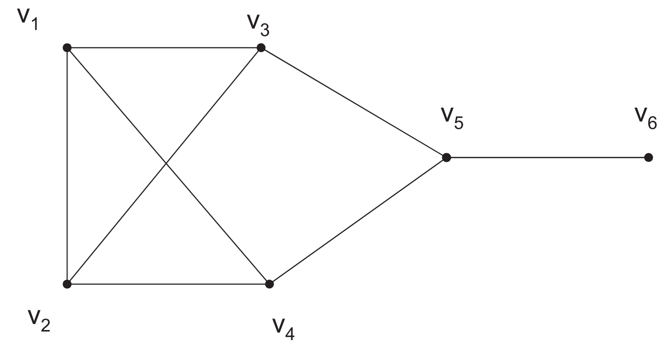

Figure 2. A v-critical graph with a cut vertex v_5.

Let us recall that a *cut vertex* in a connected graph G is a vertex v such that $G \backslash \{v\}$ is not connected. The following lemma is trivial.

Lemma 3.5. *If G is a nonelementary v-critical graph with $\gamma_R(G) = 4$ and v is a cut vertex, then one of the connected components of $G \backslash \{v\}$ has only one vertex.*

Example 3.6. There exists an infinite family of nonelementary critical graphs with cut vertices. See, for example, the graph represented on Figure 2. See also the family D_n defined below and represented in Figure 3.

Proposition 3.7. *A nonelementary graph $G = (V(G), E(G))$ of order n and $\gamma_R(G) = 4$ is e-critical if and only if given any pair of vertices v_1, v_2 with $\deg(v_1), \deg(v_2) < n-3$, then $[v_1, v_2] \in E(G)$.*

Proof. Suppose v_1, v_2 with $deg_G(v_1), deg_G(v_2) < n-3$ and $[v_1, v_2] \notin E(G)$. Let $G' = (V(G), E(G) \cup [v_1, v_2])$. Clearly, every vertex in G' has degree less or equal than $n - 3$. Then, by Remark 2.1, $\gamma_R(G') > 3$ and G is not e-critical.

Assume that for any pair of vertices v_1, v_2 with $deg_G(v_1), deg_G(v_2) < n-3$, then $[v_1, v_2] \in E(G)$. Let $G' = (V(G), E(G) \cup [v, w])$ for some $v, w \in V(G)$ with $[v, w] \notin E(G)$. Then one of them, say v, satisfies that $deg_G(v) \geq n-3$. Thus, $deg_{G'}(v) \geq n-2$ and by Remark 2.1, $\gamma_R(G) \leq 3$ (Figure 4). □

Lemma 3.8. *Let $G = (V(G), E(G))$ be a nonelementary e-critical, v-critical graph of order n and $\gamma_R(G) = 4$. If $W_1 := \{v \in V(G) | \deg(v) = n-3\}$, then $|W_1| \geq \frac{3n}{4}$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and assume $W_1 = \{v_1, ..., v_k\}$. By Proposition 3.7, and Remark 2.1 we know that for all v_r, v_s with $r, s > k$ then $[v_r, v_s] \in E(G)$. Therefore, since $\deg(v_j) < n-3$ for every $j > k$, there are at least three vertices $v_{j_1}, v_{j_2}, v_{j_3}$ with $j_1, j_2, j_3 \leq k$ such that $[v_{j_1}, v_j], [v_{j_2}, v_j], [v_{j_3}, v_j] \notin E(G)$.

Also, by Lemma 3.1, every v_i with $i \leq k$ is joined at most to k – 2 vertices in W_1. Hence, there is at most one vertex v_j with $j > k$ such that $[v_i, v_j] \notin E(G)$. Therefore, given

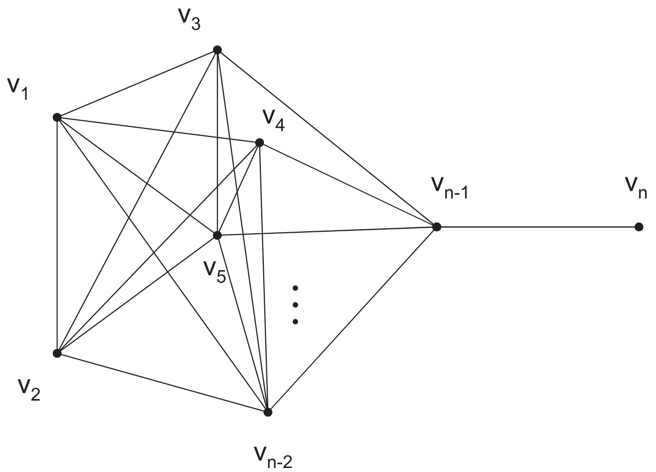

Figure 3. If $n > 5$ with n even, then there is a unique v-critical, e-v-critical and Roman complete graph, D_n, with $\gamma_R(D_n) = 4$.

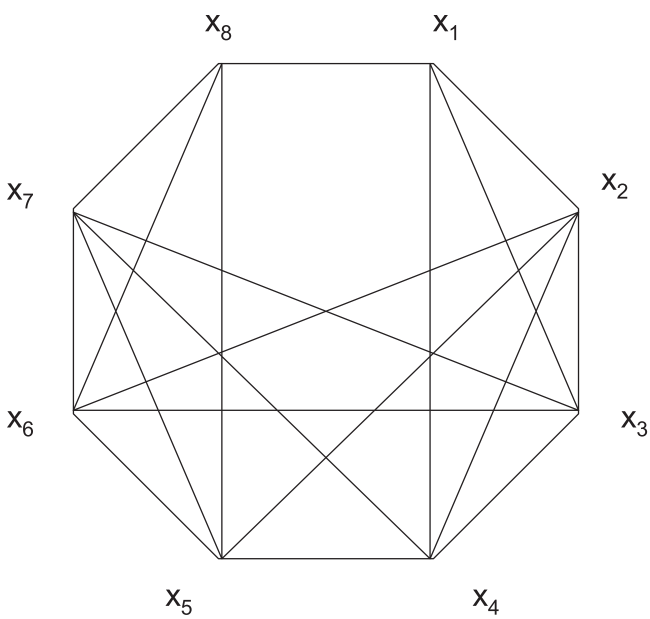

Figure 4. A nonelementary v-critical, e-critical graph with 8 vertices and two adjacent vertices of degree 4.

$v_j, v_{j_1}, v_{j_2}, v_{j_3}$ as above, $v_{j_i} \neq v_{j'_{i'}}$ for any $j \neq j'$ or $i \neq i'$. Thus, $k \geq 3(n-k)$ and $k \geq \frac{3n}{4}$. □

Remark 3.9. Notice that the graphs in the family described in Remark 3.4 are e-critical since every vertex has degree $n-3$. Then, this is an infinite family of different nonelementary v-critical, e-critical graphs with $\gamma_R(G) = 4$.

Proposition 3.10. *A nonelementary v-critical graph $G = (V(G), E(G))$ with $\gamma_R(G) = 4$ is e-v-critical if and only if for every edge $e \in E(G)$ there is a vertex $v_e \in V(G)$ such that every vertex with degree $n - 3$ is either contained in e or adjacent to v_e.*

Proof. Suppose G is e-v-critical. Then, for every edge e there is some v_e such that $\gamma_R(G \setminus \{e, v_e\}) = 4$. Let w be any vertex with $deg_G(w) = n-3$ and suppose $[v_e, w] \notin E(G)$. Then, $N_G[w] = G \setminus \{v_e, y\}$ for some vertex y. Let $r_w : V(G)\setminus\{v\} \to \{0,1,2\}$ such that $r_w(w) = 2, r_w(y) = 1$ and $r_w(x) = 0$ for every $x \in V(G)\setminus\{v_e, w, y\}$. Since $\gamma_R(G \setminus \{e, v_e\}) = 4$, it follows that r_w is not Roman on $G \setminus \{e\}$, this is, there is a vertex x_0 labelled with 0 which is not adjacent to w on $G \setminus \{e\}$. Thus, $e = [w, x_0]$.

Now suppose that for any edge $e \in E(G)$ there is a vertex $v_e \in V(G)$ such that every vertex w with degree $n - 3$ is either contained in e or adjacent to v_e. Let us see that $G \setminus \{e\}$ is not v-critical. By Proposition 2.5, we know that $\gamma_R(G\setminus\{e\}) = 4$. Suppose $\gamma_R(G \setminus \{e, v_e\}) = 3$ and let r be any Roman domination function on $G\setminus\{e, v_e\}$ such that $\sum_{u \in V(G)\setminus\{v_e\}} r(u) = 3$. Then, r also defines a Roman domination function on $G\setminus\{v_e\}$. If $r^{-1}(2) = \{w\}$, since $\gamma_R(G) = 4, deg_G(w) = n-3$ and $N_G(w) = G\setminus\{v_e, y\}$ for some $y \in V(G)$. Also, $r^{-1}(1) = \{y\}$ and $r^{-1}(0) = V(G)\setminus\{v_e, y\}$. By hypothesis, since w is nonadjacent to v_e, $e = [w, w']$ with w' different from v_e, y. Therefore, since $r(w') = 0$, r is not a Roman domination function on $G\setminus\{e, v_e\}$, leading to contradiction. □

Remark 3.11. Consider the family $\{X_n | n \geq 5\}$ described in Remark 3.4. For each n, consider $X_n = X_n^0, X_n^1, \ldots, X_n^{k_n}$ a sequence of graphs where X_n^i is obtained from X_n^{i-1} by removing one edge so that X_n^i is again v-critical until $X_n^{k_n}$ is e-v-critical (since $\mathrm{X_n}$ is nonelementary, by Proposition 2.5, there is always such a $\mathrm{k_n}$). Since for each n, $|V(X_n)| = n$, the graphs in $\{X_n^{k_n} | n \geq 5\}$ define an infinite family of different v-critical and e-v-critical graphs.

Proposition 3.12. *Let $G = (V(G), E(G))$ be a nonelementary e-critical, v-critical and e-v-critical graph with $\gamma_R(G) = 4$. Let $W_1 := \{v \in V(G) | \deg(v) = n-3\} = \{v_1, \ldots, v_k\}$ and $W_2 := \{v \in V(G) | \deg(v) < n-3\} = \{v_{k+1}, \ldots, v_n\}$. Then, either $G = C_5$ or $|W_2| = n-k = 1, \deg(v_n) = 1$ and there is a cut vertex.*

Proof. Claim: $|W_2| \leq 1$. Otherwise, suppose there exist at least two vertices $v_r, v_s \in W_2$. By Proposition 3.7, for all $v_r, v_s \in W_2, e = [v_r, v_s] \in E(G)$. By Proposition 3.10, there is a vertex $v_e \in V(G)$ such that for every $v \in W_1$, either $v = v_r$, $v = v_s$ or v is adjacent to v_e. In this case, since $v_r, v_s \in W_2$, every $v \in W_1\setminus\{v_e\}$ is adjacent to v_e. However, by Theorem 3.1, there exists some vertex $a_{v_e} \neq v_e$ in W_1 such that a_{v_e} is not adjacent to v_e leading to contradiction. Therefore, $|W_2| \leq 1$.

Claim: If $|W_2| = 1$, then $\deg(v_n) = 1$ and there is a cut vertex. Suppose $n - k = 1$ and $\deg(v_n) \geq 2$. There is no loss of generality if we assume that $[v_{n-1}, v_n], [v_{n-2}, v_n] \in E(G)$. Applying Proposition 3.10 for $e_1 = [v_{n-1}, v_n]$ and $e_2 = [v_{n-2}, v_n]$ we know that there exist two vertices, $v_{e_1}, v_{e_2} \in \{v_1, \ldots, v_{n-3}\}$ such that for every $v \in W_1$, if $v \neq v_{e_1}, v_{n-1}$ then $[v, v_{e_1}] \in E(G)$ and if $v \neq v_{e_2}, v_{n-2}$ then $[v, v_{e_2}] \in E(G)$. See Figure 5. Let us assume that $v_{e_1} = v_1$ and $v_{e_2} = v_2$. Hence, $N[v_1] = V(E)\setminus\{v_{n-1}, v_n\}$ and $N[v_2] = V(E)\setminus\{v_{n-2}, v_n\}$. Notice that this implies that $v_1 \neq v_{n-2}, v_n$ and $v_2 \neq v_{n-1}, v_n$. In particular, $e_3 = [v_1, v_2] \in E(G)$. Let us apply again Proposition 3.10 for e_3. Then, there is some vertex v_{e_3}

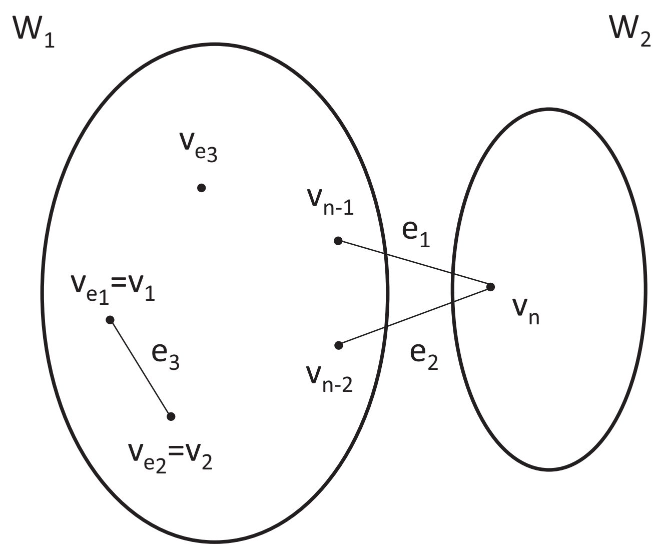

Figure 5. If $|W_2| = 1$ and $\deg(v_n) > 1$, then there is a contradiction: $v_{e_3} \neq v_n$ and $\deg(v_{e_3}) = n-2$.

such that for every $v \neq v_1, v_2, v_{e_3}, v_n$, then $[v, v_{e_3}] \in E(G)$. Let us distinguish the following three cases:

- If $v_{e_3} = v_n$, then v_n is adjacent to every vertex $v \neq v_1, v_2$ and $\deg(v_n) = n-3$ which is a contradiction.
- If $[v_{e_3}, v_n] \in E(G)$. By the election of v_1, v_2, either $[v_{e_3}, v_1] \in E(G)$ or $[v_{e_3}, v_2] \in E(G)$. Therefore, $\deg(v_{e_3}) \geq 1+1+n-4 = n-2$ which implies that $\gamma_R(G) = 3$ leading to contradiction.
- If $v_{e_3} \neq v_n$ and $[v_{e_3}, v_n] \notin E(G)$. Then $v_{e_3} \neq v_{n-1}, v_{n-2}$ and, therefore, $[v_{e_3}, v_1] \in E(G)$ and $[v_{e_3}, v_2] \in E(G)$. Thus, v_{e_3} is adjacent to every vertex $v \neq v_n, v_{e_3}$ and $\deg(v_{e_3}) = n-2$ which implies that $\gamma_R(G) = 3$ leading to contradiction.

Thus, there is a unique edge incident to v_n, let's say $[v_{n-1}, v_n]$, and v_{n-1} is a cut vertex.

Claim: If $|W_2| = 0$, then $G = C_5$.

Suppose $k = n$, this is, $\deg(v) = n-3$ for every $v \in V(G)$. Consider all the edges incident to v_1: $e_1, \dots, e_{n-3}$ and let $e_i = [v_1, w_i], i = 1, n-3)$. Then, by Proposition 3.10, there are $n-3$ different vertices $v_2, \dots, v_{n-2}$ such that $N[v_{i+1}] = G \backslash \{v_1, w_i\}$ for $i = 1, n-3$. In particular, $[v_1, v_{i+1}] \notin E(G)$ for $i = 1, n-3$. But since $\deg(v_1) = n-3$, it follows that $n-3 \leq 2$ and $n \leq 5$. Since we assumed that G is nonelementary then $n = 5$ and $\deg(v_i) = 2$ for every $1 \leq i \leq n$.

Let us assume with no loss of generality that $[v_1, v_2], [v_2, v_3] \in E(G)$. Then, if $[v_1, v_3] \in E(G)$ these vertices have already degree 2. Hence, $\deg(v_4), \deg(v_5) < 2$ leading to contradiction. So, we may assume, again without loss of generality, that $[v_3, v_4] \in E(G)$. Again, if $[v_1, v_4] \in E(G)$, then $\deg(v_5) < 2$. Therefore, $G = C_5$. □

Corollary 3.13. *C_5 is the unique nonelementary graph, G, with $\gamma_R(G) = 4$ which is e-critical, v-critical and e-v-critical with no cut vertices.*

For any even number $n \geq 6$, let us denote by $D_n = (V(D_n), E(D_n))$ the graph such that $V(D_n) = \{v_1, \dots, v_n\}$ and $E(D_n) = \{[v_1, v_2]\} \cup \{[v_{n-1}, v_n]\} \cup \{[v_j, v_{n-1}] : 3 \leq j \leq n-2\} \cup \{[v_r, v_j] : r = 1, 2 \text{ and } 3 \leq j \leq n-2\} \cup \{[v_i, v_j] : 3 \leq i, j \leq n-2\} \backslash \{[v_{2j-1}, v_{2j}] : 2 \leq j \leq \frac{n-2}{2}\}$. See Figure 3.

Notice that for every $1 \leq i \leq n-1, \deg(v_i) = n-3$ and $\deg(v_n) = 1$. In fact, if $r = 1, 2$, $N(v_r) = V(E) \backslash \{v_{n-1}, v_n\}$, if $3 \leq j \leq n-2$ with j odd, then $N(v_j) = V(E) \backslash \{v_{j+1}, v_n\}$, if $3 \leq j \leq n-2$ with j even, then $N(v_j) = V(E) \backslash \{v_{j-1}, v_n\}$, $N(v_{n-1}) = V(E) \backslash \{v_1, v_2\}$ and $N(v_n) = \{v_{n-1}, v_n\}$.

Proposition 3.14. *For every even number $n \geq 6$, D_n is a nonelementary e-critical, v-critical and e-v-critical graph with $\gamma_R(G) = 4$.*

Proof. Since $\deg(v_i) = n-3$ for every $i \neq n$ it follows immediately that G is v-critical, by Theorem 3.1, and G is e-critical.

To check that G is e-v-critical, by Proposition 3.10, it suffices to find for every edge $e = [x, y]$ a vertex v_e which is adjacent to every vertex in $\{v_1, \dots, v_{n-1}\} \backslash \{x, y\}$.

If $e = [v_{n-1}, v_n]$, then $v_e = v_1$.

If $e = [v_1, v_2]$, then $v_e = v_{n-1}$.

For the rest of the edges, note that if $v_{2j-1} \in e$, then $v_{2j} \notin e$ and v_{2j} is adjacent to every vertex in $\{v_1, \dots, v_{n-1}\} \backslash \{v_{2j-1}\}$ and if $v_{2j} \in e$, then $v_{2j-1} \notin e$ and v_{2j-1} is adjacent to every vertex in $\{v_1, \dots, v_{n-1}\} \backslash \{v_{2j}\}$ for every $2 \leq j \leq \frac{n-2}{2}$. □

Thus, there is an infinite family of e-critical, v-critical and e-v-critical graphs.

Theorem 3.15. *If $G = (V(G), E(G))$ is a nonelementary e-critical, v-critical and e-v-critical graph with $\gamma_R(G) = 4$, then:*
a) If $|V(G)| = 5$, then $G = C_5$
b) If $|V(G)| = n > 5$, then n is even and $G = D_n$.

Proof. If $|W_2| = 0$, it follows from the proof of Proposition 3.12 that $G = C_5$.

Now, let us see that if $|W_2| = 1$, then $n > 5$ is even and, up to relabeling the vertices, $G = D_n$.

As we saw, there is a unique edge incident to v_n, $[v_n, v_{n-1}]$. Since $\deg(v_{n-1}) = n-3$ then there are two vertices, let's say v_1, v_2 such that $N[v_{n-1}] = G \backslash \{v_1, v_2\}$ and for every $3 \leq j \leq n-2, [v_j, v_{n-1}] \in E(G)$. Since $\deg(v_1) = \deg(v_2) = n-3$, and $[v_r, v_k] \notin E(G)$ for $r = 1, 2$ and $k = n, n-1$, it follows that $[v_1, v_2] \cup \{[v_r, v_j] : r = 1, 2 \text{ and } 3 \leq j \leq n-2\} \subset E(G)$. Finally, for every $3 \leq i \leq n-2, \deg(v_i) = n-3$. Therefore, for each $3 \leq i \leq n-2$ there is exactly one $j \neq i, 3 \leq j \leq n-2$, such that $[v_i, v_j] \notin E(G)$. This forces n to be even. We may assume, relabeling if necessary, that those missing edges are $\{v_{2j-1}, v_{2j}\}$ with $2 \leq j \leq \frac{n-2}{2}$. Thus, G is isomorphic to D_n. □

In the following characterization we prove that for an arbitrary graph G with $\gamma_R(G) = 4$, the Roman criticality can be studied just by looking at small induced subgraphs. In fact, it suffices to check some properties on the induced subgraphs with 8 vertices to determine if the graph is e-critical, v-critical and e-v-critical.

Theorem 3.16. *Let $G = (V(G), E(G))$ be a graph with $\gamma_R(G) = 4$ and $|V(G)| = n \geq 8$. Then, n is even and $G = D_n$ if and only if the following conditions hold:*

(a) *There exist* $\{v_1, \ldots, v_4\} \in V(G)$ *such that* $[v_1, v_i] \notin E(G)$ *for every* $2 \le i \le 4$.
(b) *For every* $\{v_1, \ldots, v_8\} \in V(G)$, *if* $[v_1, v_i] \notin E(G)$, *for all* $2 \le i \le 4$, *then* $|\{v_k : 1 \le k \le 7 \text{ such that } [v_k, v_8] \in E(G)\}| \ge 5$.
(c) *For every* $\{v_1, \ldots, v_6\} \in V(G)$, *if* $[v_1, v_i] \notin E(G)$ *for every* $2 \le i \le 4$ *then* $|\{v_k : 5 \le k \le 6 \text{ such that } [v_1, v_k] \in E(G)\}| \le 1$.

Proof. Property $a)$ holds for some subset $\{v_1, \ldots, v_4\} \in V(G)$ if and only if there is at least one vertex v such that $\deg(v) < n - 3$.

Property $b)$ holds for every subset $\{v_1, \ldots, v_8\} \in V(G)$ if and only if there is at most one vertex v such that $\deg(v) < n - 3$.

Property $c)$ holds for every subset $\{v_1, \ldots, v_6\} \in V(G)$ if and only if for every vertex v with $\deg(v) < n - 3$ then $deg(v) = 1$.

Finally, as we saw in the proof of Theorem 3.15, if $n > 5$ and the unique vertex v such that $\deg(v) < n - 3$ satisfies that $deg(v) = 1$, then $G = D_n$. □

Thus, from Theorems 3.15 and 3.16 we obtain the following:

Corollary 3.17. *Let* $G = (V(G), E(G))$ *be a graph with* $\gamma_R(G) = 4$ *and* $|V(G)| = n \ge 8$. *Then, G is a e-critical, v-critical and e-v-critical graph if and only if the following conditions hold:*

(a) *There exist* $(v_1, \ldots, v_4) \in V(G)$ *such that* $[v_1, v_i] \notin E(G)$ *for every* $2 \le i \le 4$.
(b) *For every* $(v_1, \ldots, v_8) \in V(G)$, *if* $[v_1, v_i] \notin E(G)$, *for every* $2 \le i \le 4$, *then* $|\{v_k : 1 \le k \le 7 \text{ such that } [v_k, v_8] \in E(G)\}| \ge 5$.
(c) *For every* $(v_1, \ldots, v_6) \in V(G)$, *if* $[v_1, v_i] \notin E(G)$ *for every* $2 \le i \le 4$ *then* $|\{v_k : 5 \le k \le 6 \text{ such that } [v_1, v_k] \in E(G)\}| \le 1$.

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