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On central max-point-tolerance graphs

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ABSTRACT

Max-point-tolerance graphs (MPTG) were studied by Catanzaro et al. in 2017 and the same class of graphs were introduced in the name of p-BOX(1) graphs by Soto and Caro in 2015. This class has a wide application in genome studies as well as in telecommunication networks. In our article, we consider central max-point-tolerance graphs (central MPTG) by taking the points of MPTG as center points of their corresponding intervals. In the course of study on this class of graphs, we show that the class of central MPTG is same as the class of unit max-tolerance graphs. We also prove that the class of unit central MPTG is same as that of proper central MPTG and both of them are equivalent to the class of proper interval graphs.

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1. Introduction

The class of interval graphs was initially posed by Hajös in 1957 [8] as a study of intersection graphs of intervals on real line. In 1959, the molecular biological scientist Benzer [2] used the model of interval graphs to obtain a physical map from information on pairwise overlaps of the fragments of DNA. Interval graphs were well studied by many people in Computer Science and Discrete Mathematics for their wide application. Many combinatorial problems have been solved for interval graphs in linear time.

This class finds application in many theoretical and practical situations. For this the graph class was generalized to several variations. In one direction it went in developing concepts of probe interval graphs [4], circular-arc graphs [1], and interval digraphs [12]. On the other hand, in 1982, Golumbic and Monma introduced the concept of min-tolerance graphs (commonly known as *tolerance graphs*) [6]. We denote the length of an interval I on the real line by $|I|$. A simple undirected graph $G=(V,E)$ is a *min-tolerance* graph if each vertex $u \in V$ corresponds to a real interval I_u and a positive real number t_u, called *tolerance*, such that uv is an edge of G if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$. Golumbic and Trenk [7] introduced *max-tolerance* graphs where each vertex $u \in V$ corresponds to a real interval I_u and a positive real number t_u (known as tolerance) such that uv is an edge of G if and only if $|I_u \cap I_v| \geqslant \max\{t_u, t_v\}$. For max-tolerance graphs, we may assume $t_u \leq |I_u|$ for each $u \in V$ otherwise u becomes isolated. A max-tolerance graph is a *unit-max-tolerance graph* if $|I_u| = |I_v|$ for all $u, v \in V$. Some combinatorial problems like finding maximal cliques were obtained in polynomial time whereas the recognition problem was proved to be NP-hard [9] for max-tolerance graphs in 2006. Also a geometrical connection of max-tolerance graphs to semi-squares was obtained by Kaufmann et al. [9]. For further details of tolerance graphs one can refer the work by Golumbic and Trenk [7].

In 2015, Soto and Caro [13] introduced a new graph class, namely p-BOX graphs where each vertex corresponds to a box and a point within it in the d-dimensional Euclidean space. Any two vertices are adjacent if and only if the intersection of their corresponding boxes contains both the corresponding points. When the dimension is one the graph class is denoted by p-BOX(1). In 2017, this dimension one graphs were studied independently by Catanzaro et al. [3], but with a different name, *max-point tolerance graphs* (MPTG) where each vertex $u \in V$ corresponds to a pair of an interval and a point (I_u, p_u), where I_u is an interval on the real line and $p_u \in I_u$, such that uv is an edge of G if and only if $\{p_u, p_v\} \subseteq I_u \cap I_v$. The graphs MPTG have many practical applications in human genome studies and modeling of telecommunication networks [3]. A graph $G = (V, E)$ is called *central-max-point-tolerance graph* (central MPTG) if p_u is the center point of I_u for each $u \in V$. This graph class actually matches with the graph defined as *c-p*-BOX(1) *graph* by Soto and Caro [13]. It is known that *c-p*-BOX(1) graphs are max-tolerance graphs. We use the terms MPTG and central MPTG for p-BOX(1) and *c-p*-BOX(1) graphs throughout this article.

In our article we prove that central MPTG are same as unit max-tolerance graphs. Incidentally this settles a question raised in the book by Golumbic and Trenk [7] that whether interval graphs are unit max-tolerance graphs or not. Moreover we show that a *unit central MPTG* is same as a *proper central MPTG* and also is same as a proper interval graph. In Conclusion section, we show the relations between

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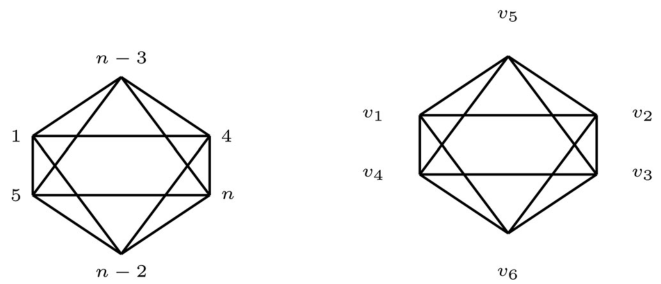

Figure 1. The graphs G_1 in Observation 2.5 and G_2 in Example 2.6.

the subclasses of max-tolerance graphs related to central MPTG and list major open problems in this area.

2. Preliminaries

A matrix whose entries are only zeros and ones is a *binary matrix*. A binary matrix is said to satisfy *consecutive* 1's *property for rows* if its columns can be permuted in such a way that 1's in each row occur consecutively [5]. For a simple undirected graph $G = (V, E)$, a matrix is known as the *augmented adjacency matrix* of G if we replace all principal diagonal elements of the adjacency matrix of G by 1 [5]. Among many characterizations of proper interval graphs, we list the following which will serve our purpose. Let $G = (V, E)$ be a simple undirected graph and $u \in V$. Then $N(u) = \{x \in V | ux \in E\}$ is the set of (open) neighbors of u and $N[u] = N(u) \cup \{u\}$ is the set of (closed) neighbors of u. The reduced graph $\tilde{G}$ is obtained from G by merging vertices having same closed neighborhood. $G(n, r)$ is a graph with n vertices $x_1, x_2, ..., x_n$ such that x_i is adjacent to x_j if and only if $0 < |i-j| \leq r$, where r is a positive integer.

Theorem 2.1. [10, 11] *Let $G = (V,E)$ be an interval graph, then the following are equivalent:*

1. *G is a proper interval graph.*
2. *G is a unit interval graph.*
3. *There exist a linear ordering $<$ on V such that for every choice of vertices u, v, w*
   $u < v < w$ and $uw \in E$ implies $uv, vw \in E$.
4. *$\tilde{G}$ is an induced subgraph of $G(n, r)$ for some positive integers n, r with $n > r$.*

The following characterization of MPTG is known:

Theorem 2.2. [3] *Let $G = (V, E)$ be a simple undirected graph. Then G is an MPTG if and only if there is an ordering $\prec$ of vertices of G such that the following condition holds:*

$$\text{For any } x \prec u \prec v \prec y, xv, uy \in E \Rightarrow uv \in E. \qquad (2.1)$$

Definition 2.3. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ binary matrices. We define $A \wedge B = (c_{ij})$ where $c_{ij} = a_{ij} \wedge b_{ij}$ with the rules: $0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0$ and $1 \wedge 1 = 1$.

The above characterization leads to the following observations.

Observation 2.4. *Let G be a simple undirected graph. Then following are equivalent:*

1. *G is an MPTG.*
2. *There is an ordering of vertices of G such that for any $u < v$, $u, v \in V$,*

$$uv \notin E \Rightarrow uw \notin E \text{ for all } w > v \text{ or, } wv \notin E \text{ for all } w < u. \qquad (2.2)$$

3. *There exists an ordering of vertices such that every 0 above the principal diagonal of the augmented adjacency matrix $A(G)$ has either all entries right to it are 0 or, all entries above it are 0.*
4. *There exists a binary matrix M with consecutive 1's property for rows such that the augmented adjacency matrix $A(G) = M \wedge M^T$.*

Proof. The condition 2 is equivalent to Equation (2.1) in the other way. Condition 3 is a matrix version of condition 2. Condition 4 follows from definition of MPTG. □

Let $G = (V, E)$ be a simple undirected graph and $\emptyset \neq X \subseteq V$. Then $G[X]$ denotes the subgraph of G induced by X. In Proposition 6.7 of reference [3] it is proved that if G is an MPTG with non-adjacent vertices u and v, then $G[N(u) \cap N(v)]$ is an interval graph. Also in Proposition 7.1 of reference [9] it is shown that $\overline{C_n}$, $n > 9$ is not a max-tolerance graph. We show that these graphs are not MPTG as well.

Observation 2.5. *The complement of a cycle of length greater than 9 is not an MPTG.*

Proof. Suppose on contrary $\overline{C_n}$, $n > 9$ is an MPTG. Now as $\overline{C_n}$, $n > 9$ contain the graph G_1 with vertices $\{1, 4, 5, n-3, n-2, n\}$ in Figure 1 (left) as induced subgraph where common neighbors of n – 2 and n – 3 form a chordless 4-cycle and so is not an interval graph. Thus, G_1 is not an MPTG. Since any subgraph of an MPTG must be an MPTG, hence the result follows. □

In the following example we show that MPTG and max-tolerance graphs are not same.

Example 2.6. The graph G_2 in Figure 1 (right) with vertex set $V = \{v_i | 1 \leq i \leq 6\}$ is not an MPTG as aforementioned. But it is a max-tolerance graph with the following interval and tolerance representation.

$$I_1 = [10, 46], t_1 = 21, I_2 = [20, 50], t_2 = 18, I_3 = [18, 49.5],$$
$$t_3 = 28.5, I_4 = [15, 60], t_4 = 31, I_5 = [21, 52], t_5 = 10,$$
$$I_6 = [12, 50], t_6 = 30.$$

Theorem 2.7. [13] *The graph class central MPTG properly contains the class of interval graphs.*

3. Central max-point-tolerance graphs

We begin with a trivial but important observation which will be used throughout the rest of the article.

Observation 3.1. Let $\{I_u|u \in V\}$ be a collection of intervals, where $I_u = [l_u, r_u], h_u = |I_u| = r_u - l_u$ and $c_u = \frac{l_u+r_u}{2}$. Then $\{c_u, c_v\} \subseteq I_u \cap I_v \iff |c_v - c_u| \leqslant \frac{1}{2}\min\{h_u, h_v\} \iff l_v \leqslant c_u \leqslant c_v \leqslant r_u$ (for $c_u \leq c_v$).

Proof. We have $c_u \in I_v = [l_v, r_v] = [c_v - \frac{h_v}{2}, c_v + \frac{h_v}{2}] \iff c_v - \frac{h_v}{2} \leq c_u \leq c_v + \frac{h_v}{2} \iff -\frac{h_v}{2} \leq c_u - c_v \leq \frac{h_v}{2} \iff |c_u - c_v| \leq \frac{h_v}{2}$. Thus $\{c_u, c_v\} \subseteq I_u \cap I_v \iff |c_u - c_v| \leq \frac{1}{2}\min\{h_u, h_v\}$. Also it is clear that $c_u \in I_v = [l_v, r_v] \iff l_v \leq c_u \leq r_v$. □

In the aforementioned section, we observed that the graph classes of max-point-tolerance graphs and max tolerance graphs are not same. Now it is interesting to see that the classes of central MPTG and unit max-tolerance graphs are same.

Theorem 3.2. *Let G be a simple undirected graph. Then G is a central MPTG if and only if G is a unit max-tolerance graph.*

Proof. Let $G = (V, E)$ be a central MPTG with a central MPTG representation (I_u, c_u) where $I_u = [l_u, r_u]$, c_u be the center point of I_u for each vertex $u \in V$. Let $h_u = r_u - l_u$ for all $u \in V$. Choose $h_0 > \max\{h_u|u \in V\}$. Define $t_u = \frac{h_0 - h_u}{2}, y_u = c_u + \frac{h_0}{2}$ and $T_u = [c_u, y_u]$. Note that $t_u > 0$ and $|T_u| = y_u - c_u = \frac{h_0}{2}$ which is a constant for all $u \in V$.

Suppose $uv \in E$ and $c_v \leq c_u$. Then $c_u - c_v \leq \frac{1}{2}\min\{h_u, h_v\} \leq \frac{h_0}{2}$. So $c_u \leq c_v + \frac{h_0}{2} = y_v$. This implies $c_v \leq c_u \leq y_v$. So $T_u \cap T_v = [c_u, y_v] \neq \emptyset$ and $|T_u \cap T_v| = y_v - c_u = c_v + \frac{h_0}{2} - c_u = \frac{h_0}{2} - (c_u - c_v) \geq \frac{h_0 - h_u}{2}, \frac{h_0 - h_v}{2}$. So $y_v - c_u \geq t_u, t_v$, i.e.,

$$|T_u \cap T_v| \geq \max\{t_u, t_v\}. \quad (3.1)$$

On the other hand, Equation (3.1) implies $T_u \cap T_v \neq \emptyset$ and $c_v \leq c_u \leq y_v \leq y_u$. So $|T_u \cap T_v| = y_v - c_u$. Now $y_v - c_u \geq \max\{t_u, t_v\}$ implies $\frac{h_0}{2} - (c_u - c_v) \geq \max\{\frac{h_0 - h_u}{2}, \frac{h_0 - h_v}{2}\}$. Thus $c_u - c_v \leq \frac{1}{2}\min\{h_u, h_v\}$, i.e., $uv \in E$. Therefore, G is a unit max-tolerance graph with interval representation $\{T_u = [c_u, y_u]|u \in V\}$ and tolerances $\{t_u|u \in V\}$ as defined earlier.

Conversely, let $G = (V, E)$ be a unit max-tolerance graph with interval representation $\{T_u = [l_u, r_u]|u \in V\}$ and tolerances $\{t_u|u \in V\}$. Let $h = |T_u|$ for all $u \in V$. Define $I_u = [l_u - (h - t_u), l_u + (h - t_u)]$. Then c_u, the center of $I_u = l_u$ and $h_u = |I_u| = 2(h - t_u) < 2h$. Suppose $uv \in E$. Then $|T_u \cap T_v| \geq \max\{t_u, t_v\}$. Now for $l_u \leq l_v, |T_u \cap T_v| = r_u - l_v$. Then $r_u - l_v \geq \max\{t_u, t_v\}$, i.e., $h + l_u - l_v \geq \max\{t_u, t_v\}$. This implies $l_v - l_u \leq \min\{h - t_u, h - t_v\}$, i.e., $c_v - c_u \leq \frac{1}{2}\min\{h_u, h_v\}$. Finally, the condition that $0 < c_v - c_u \leq \frac{1}{2}\min\{h_u, h_v\} \Rightarrow l_v - l_u < h$ and $l_u \leq l_v < l_u + h = r_u$. So $T_u \cap T_v \neq \emptyset$ and $|T_u \cap T_v| = r_u - l_v$. Then $l_v - l_u = c_v - c_u \leq \frac{1}{2}\min\{h_u, h_v\}$ implies $|T_u \cap T_v| \geq \max\{t_u, t_v\}$, i.e., $uv \in E$. Thus $\{(I_u, c_u)|u \in V\}$ is a central MPTG representation of G. □

Remark 3.3. In reference [7, p. 215], Golumbic wrote that "Every interval graph is a proper max-tolerance graph. It is not yet known whether this can be strengthened to unit max-tolerance." The aforementioned theorem shows central MPTG and unit max-tolerance graphs denote the same graph class. Hence from Theorem 2.7 one can easily conclude that every interval graph is a unit max-tolerance graph. Thus, we settle the above query posed in the book of Golumbic.

In the sequel, we show that the class of max-tolerance graphs properly contains the class of central MPTG. We begin with the following definition which unfolds more insight in the structure of a central MPTG.

Definition 3.4. (C-order) Let $G = (V, E)$ be a central MPTG with (distinct) center points $\{c_u|u \in V\}$ of the intervals $\{I_u|u \in V\}$ in its central MPTG representation $\{(I_u, c_u)|u \in V\}$. The C-order of the set V is the total order induced by the center points. For convenience abusing notation, henceforth we write $u < v$ if and only if $c_u < c_v$.

In the following we present a necessary condition for central MPTG.

Theorem 3.5. *Let $G = (V, E)$ be a central MPTG. Then there is an ordering $\prec^*$ of vertices of G such that the following condition holds:*

$$\text{For any } x \prec^* u \prec^* v \prec^* y, xv, uy \in E \Rightarrow uv \in E \text{ and } (xu \in E \text{ or } vy \in E \text{ or } xu, vy \in E). \quad (3.2)$$

Proof. Let $G = (V, E)$ be a central MPTG with a central MPTG representation (I_u, c_u) for each $u \in V$. We arrange vertices according to the increasing order of center points (i.e., in C-order) of representing intervals. Suppose in this ordering we have $x < u < v < y$ and $xv, uy \in E$. Then $c_v, c_x \in I_v \cap I_x$ and $c_u, c_y \in I_u \cap I_y$. Also we have $c_x < c_u < c_v < c_y$. Now $c_x, c_v \in I_v \Rightarrow c_u \in I_v$ and $c_u, c_y \in I_u \Rightarrow c_v \in I_u$. Therefore $uv \in E$. Again $c_x, c_v \in I_x \Rightarrow c_u \in I_x$ and $c_u, c_y \in I_y \Rightarrow c_v \in I_y$. Thus if $xu, vy \notin E$, then $c_x \notin I_u$ and $c_y \notin I_v$. But then $c_u - c_x > c_y - c_u$ as $c_x \notin I_u$ but $c_y \in I_u$, and $c_y - c_v > c_v - c_x$ as $c_y \notin I_v$ but $c_x \in I_v$. Combining these inequalities we have $c_v < \frac{c_x + c_y}{2} < c_u$ which is a contradiction. Therefore $xu \in E$ or, $vy \in E$ or, $xu, vy \in E$. □

In reference [13], it is shown that every cycle C_n of length $n \geq 3$ is a central MPTG.

Definition 3.6. A cycle C_n is said to be *circularly consecutive* C-ordered if starting from a fixed vertex (say u) one can order all its vertices in a circularly consecutive way in clockwise (or anticlockwise) direction until u is reached in a C-order.

In particular, for $n = 4$, one can obtain a central MPTG representation of C_4 from the work by Soto and Caro [13] where vertices are circularly consecutive C-ordered. We state a stronger version in the following corollary.

Corollary 3.7. *Any induced C_4 in central MPTG must be circularly consecutive C-ordered.*

Proof. All other possible C-orderings of vertices will violate Equation (3.2). Hence the proof follows. □

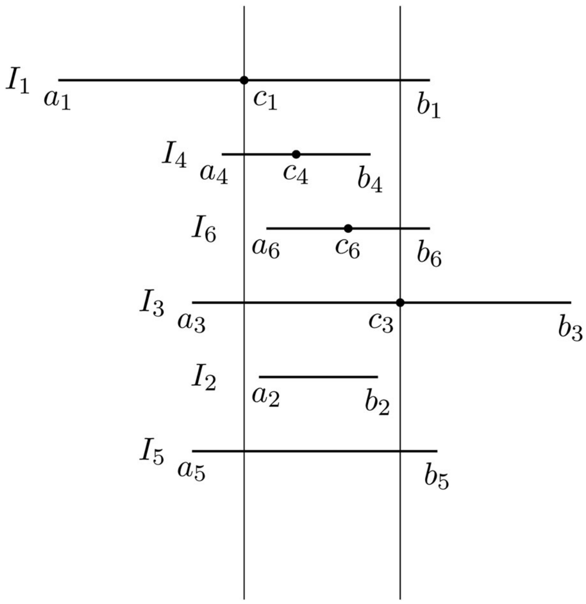

Figure 2. Relative positions of intervals described in the proof of Theorem 3.9.

Similarly one can obtain the following:

Corollary 3.8. *Any induced P_4 in central MPTG must have vertex consecutive ending edges i.e., vertices corresponding to ending edges of P_4 occur consecutively in a C-order (up to permutations between them) at least in one end.*

Theorem 3.9. *$\overline{C_6}$ is a max-tolerance graph but it is not a central MPTG.*

Proof. Let $\{v_i | 1 \leq i \leq 6\}$ be the vertices occurred in circularly consecutive way in clockwise (or anticlockwise) order in C_6. We assign the following intervals and tolerances for all the vertices so that they satisfy max-tolerance representation in $\overline{C_6}$. $I_{v_1} = [0, 20], t_{v_1} = 10, I_{v_2} = [12, 24], t_{v_2} = 6, I_{v_3} = [0, 22], t_{v_3} = 11, I_{v_4} = [9.5, 19.5], t_{v_4} = 5, I_{v_5} = [7.5, 30.5], t_{v_5} = 11.5, I_{v_6} = [10.5, 21.5], t_{v_6} = 5.5$.

Now suppose $\overline{C_6} = (V, E)$ is a central MPTG with central MPTG representation (I_v, c_v) where $I_v = [a_v, b_v]$, c_v be the center point of I_v for each $v \in V$. Let $\{v_i | 1 \leq i \leq 6\}$ be the vertices occurred in circularly consecutive way in clockwise (or anticlockwise) order in C_6. It is easy to check that the subgraph induced by deleting the vertices $\{v_2, v_5\}$ from $\overline{C_6}$ is a C_4. Now from Corollary 3.7 we can conclude that the vertices in $C_4 = \{v_1, v_4, v_6, v_3\}$ are circularly consecutive C-ordered. Without loss of generality we can take $c_1 < c_4 < c_6 < c_3$. As $v_1v_5, v_3v_5 \in E, a_5 \leq c_1$ and $c_3 \leq b_5$. Thus we get $[c_1, c_3] \subseteq [a_5, b_5]$. Hence $c_4 \in [c_1, c_3]$ imply $c_4 \in I_5$. Below we will show $c_5 \in I_4$ (see Figure 2) which lead us to contradiction as v_4, v_5 are nonadjacent in $\overline{C_6}$.

As $v_3v_6 \in E, c_3 \leq b_6$. Again $v_1v_3 \in E$ imply $[c_1, c_3] \subseteq I_3$ as $c_1 < c_3$. Moreover $c_4 \in [c_1, c_3] \subseteq I_3$. Hence $c_3 \notin I_4$ as $v_3v_4 \notin E$, which imply $b_4 < c_3$ as $c_4 < c_3$. Combining we get

$$b_4 < c_3 \leq b_6 \tag{3.3}$$

Since $v_1v_5, v_3v_5 \in E$, we get $[c_1, c_3] \subseteq I_5$ as $c_1 < c_3$. Now as $c_6 \in [c_1, c_3], c_6 \in I_5$. Hence $c_5 \notin I_6$ as $v_5v_6 \notin E$. Hence $c_5 < a_6$ or $c_5 > b_6$. Again $c_5 \leq b_4$ (as $v_4v_5 \in E$) and hence c_5 cannot be greater than b_6 as $b_4 < b_6$ from Equation (3.3). Thus we get

$$c_5 < a_6 \tag{3.4}$$

Note that $v_1v_3 \in E$ imply $[c_1, c_3] \subseteq I_1$ as $c_1 < c_3$. Again $c_6 \in [c_1, c_3] \subseteq I_1$. But as $v_1v_6 \notin E$, $c_1 < a_6$ as $c_1 < c_6$. Again $v_1v_4 \in E$ imply $a_4 \leq c_1$. Hence combining we get $a_4 \leq c_1 < a_6$. Again $v_4v_6 \in E$ imply $a_6 \leq c_4$ and $c_6 \leq b_4$. Hence $a_6 \leq c_4 < c_6 \leq b_4$. Thus combining these inequalities and using Equation (3.3) one can conclude $a_4 \leq c_1 < a_6 \leq c_4 < c_6 \leq b_4 < c_3 \leq b_6$. Thus we get

$$a_4 \leq c_1 < a_6 < b_4 < c_3 \leq b_6 \tag{3.5}$$

As $v_2v_4, v_2v_6 \in E$ imply $c_2 \in I_4 \cap I_6 = [a_6, b_4] \subseteq [c_1, c_3] \subseteq I_1, I_3$ (from Equation (3.5)) which imply $c_2 \in I_1, I_3$. Now $v_1v_2 \notin E$ imply $a_2 > c_1$ as $c_1 < c_2$ as aforementioned. Again $v_2v_3 \notin E$ imply $b_2 < c_3$ as $c_2 < c_3$ as aforementioned. Thus we get $c_1 < a_2 \leq c_2 \leq b_2 < c_3$, i.e.,

$$[a_2, b_2] \subseteq [c_1, c_3]. \tag{3.6}$$

We now show $c_5 \in I_4$. As $v_2v_5 \in E, c_5 \in [a_2, b_2] \subseteq [c_1, c_3] \subseteq [a_4, b_6]$ from Equations (3.5) and (3.6). Now using Equation (3.4) one can conclude that c_5 must belong to I_4.

From earlier observations, we can conclude now that no central MPTG representation of $\overline{C_6}$ can be found with respect to the earlier C-ordering. For other possible C-orderings following similar type argument one can reach to contradiction. □

Now we present a sufficient condition for an MPTG to be a central MPTG.

Theorem 3.10. *Let $G = (V, E)$ be an MPTG with n vertices. Let the ordering $\{v_1, v_2, ..., v_n\}$ of vertices of G that satisfies* Equation *(2.1) and each v_i corresponds to a natural number x_i such that $x_1 < x_2 < \cdots < x_n$ and the following conditions hold for all $i = 1, 2, ..., n$:*

$$x_{i_2+1} - x_i > x_i - x_{i_1} \text{ when } i_2 < n \tag{3.7}$$

$$x_i - x_{i_1-1} > x_{i_2} - x_i \text{ when } i_1 > 1 \tag{3.8}$$

where i_1 and i_2 be the least and the highest indices such that $i_1 = i$ or, $v_iv_{i_1} \in E$ and $i_2 = i$ or, $v_iv_{i_2} \in E$. Then G is a central MPTG.

Proof. Suppose the conditions hold. Define $r_i = \max\{x_i - x_{i_1}, x_{i_2} - x_i\}$ and $I_i = [x_i - r_i, x_i + r_i]$ for $i = 1, 2, ..., n$. We show that $G = (V, E)$ is a central MPTG with an interval representation $\{I_{v_i} | i = 1, 2, ..., n\}$ where $V = \{v_1, v_2, ..., v_n\}$ and this ordering of vertices satisfies Equation (2.1). Suppose $v_iv_j \in E$. Then by definition of i_1 and i_2, we have $x_{i_1} \leq x_j \leq x_{i_2}$ and $x_{j_1} \leq x_i \leq x_{j_2}$. Then $x_i - x_{i_1} \geq x_i - x_j$ and $x_{i_2} - x_i \geq x_j - x_i$ which imply $|x_i - x_j| \leq r_i$ and so $x_j \in I_{v_i}$. Similarly $x_i \in I_{v_j}$. Hence $\{x_i, x_j\} \subseteq I_{v_i} \cap I_{v_j}$. Now let $v_iv_j \notin E$. Without loss of generality we assume $i < j$. Suppose $j_1 < i$ and $j < i_2$. Then we have $j_1 < i < j < i_2$ and $v_{j_1}v_j, v_iv_{i_2} \in E$. Then by Equation (2.1), $v_iv_j \in E$, which is a contradiction. Thus either $i < j_1$ or $j > i_2$. Then $i \leq j_1 - 1$ or $j \geq i_2 + 1$. For the first inequality by Equation (3.7), we have $x_j - x_i \geq$

$x_j - x_{j_1-1} > x_{j_2} - x_j$. Also $x_j - x_i > x_j - x_{j_1}$, as $x_i < x_{j_1}$. Thus $x_j - x_i > r_j$ which implies $x_i \notin I_{v_j}$. Similarly $j \geq i_2 + 1$ implies $x_j \notin I_{v_i}$. Therefore G is a central MPTG. □

Definition 3.11. A central MPTG $G = (V, E)$ is called *proper* if it has an interval representation with the required condition such that no interval contains another properly. We call it *proper-central-max-point tolerance graph* (in brief, proper central MPTG). Similarly, a central MPTG $G = (V, E)$ is called unit if it has an interval representation with the required condition such that every interval has unit (or, same) length. We call it *unit-central-max-point tolerance graph* (in brief, unit central MPTG).

Theorem 3.12. *Let G be a simple undirected graph. Then the following are equivalent.*

1. *G is a proper central MPTG.*
2. *G is a unit central MPTG.*
3. *G is a proper interval graph.*

Proof. $(\mathbf{1} \iff \mathbf{2})$: Let G be a proper central MPTG with respect to the representation (I_i, c_i) where $I_i = [a_i, b_i]$, c_i be the center point of I_i for each vertex $i \in V$. First, we arrange the intervals according to increasing order of left end points. As no interval properly contains another, the right end points have the same order as left end points and so as the center points as well. We process the representation from left to right, adjusting all intervals to length l where l is the length of first interval (i.e., $|I_1| = l$). At each step until all intervals have been adjusted I_x be the leftmost unadjusted interval.

Let I_j be an adjusted interval occurs before I_x, then one of the following things happen.

1. $c_j \notin I_x$.
2. $c_j \in I_x, c_x \in I_j$.
3. $c_j \in I_x$ but $c_x \notin I_j$.

Let I_{j_1} and I_{j_2} be any two adjusted intervals referred in conditions (2) and (3), respectively. Then I_{j_2} must occur before I_{j_1} otherwise the right end point of I_{j_2} would occur before c_x and so I_{j_2} would be properly contained in I_{j_1} as I_{j_1} contains c_x. But this is a contradiction.

Now if I_x does not contain center point of any adjusted intervals then take $\alpha = a_x$. If I_x contains center points of some adjusted intervals and I_i be the leftmost among them, then $c_l \in I_x$ for all $i \leq l \leq x$ as all of them have same length. Now if $c_x \in I_i$, then take $\alpha = c_i$. It follows from the last paragraph that $c_x \in I_l$ for all $i \leq l \leq x$ in this case. Now if $c_x \notin I_i, c_x \notin I_l$ for any $l < i$. Let I_j be the leftmost interval for which $c_x \in I_j$. Then $i < j < x$. Take $\alpha = c_j$ in this case. Now if no such I_j exists between I_i and I_x i.e., if $c_x \notin I_j$ for all $i \leq j < x$ then $c_x \notin I_l$ for any $l < x$. Take $\alpha = b_l$ in this case where b_l is the rightmost endpoint for which $c_l \in I_x, c_x \notin I_l$. Clearly $i \leq l < x$. We adjust the portion $[a_x, \infty)$ by shrinking or expanding $[a_x, b_x]$ to $[\alpha, \alpha + l]$ and scaling and shifting $[b_x, \infty)$ to $[\alpha + l, \infty)$. Iterating this operation produces the unit central MPTG representation.

Now it is sufficient to show adjusting I_x in above way does not affect the adjacency of vertex x with previous intervals. When $\alpha = a_x$, then $\alpha = a_x > c_l$ for all $l < x$. Hence $c_l \notin I_x$ after adjustment. When $\alpha = c_i$, then $c_i = \alpha \in I_x$ and $c_x = \alpha + \frac{l}{2} = c_i + \frac{l}{2} = b_i \in I_i$. Moreover, for all $i < l < x$, $c_x \in I_l$ and $\alpha = c_i < c_l < b_i = \alpha + \frac{l}{2}$ (i.e., $c_l \in I_x$) and $c_x \notin I_l$ for all $l < i$ after adjustment. When $\alpha = c_j$ then the arguments are similar as mentioned earlier. Again when $\alpha = b_l$, then $c_x = \alpha + \frac{l}{2} = b_l + \frac{l}{2} > b_l$ which imply $c_x \notin I_l$. Hence $c_x \notin I_k$ for all $k < l$.

Conversely, if G is a unit central MPTG then all intervals associated to the vertices of G must be of the same length. Thus, none of them contains other properly and so G is a proper central MPTG with the same interval representation.

$(\mathbf{3} \Rightarrow \mathbf{1})$: Let $G = (V, E)$ be a proper interval graph. So the reduced graph $\hat{G} = (\hat{V}, \hat{E})$ is an induced subgraph of $G(n, r) = (V_n, E')$ for some $n, r \in \mathbb{N}$ with $n > r$, where $V_n = \{v_1, v_2, ..., v_n\}$ and $v_i \leftrightarrow v_j$ if and only if $|i - j| \leq r$ by condition 4 of Theorem 2.1. Let $\hat{V} = \{v_{i_1}, v_{i_2}, ..., v_{i_m}\} \subseteq V_n$. Now for each $u \in V$, define $p_u = i_j$ if u is a copy of v_{i_j} and $I_u = [p_u - r, p_u + r]$. First, all intervals I_u are of same length $2r$ and so none of them properly contains other.

Next let $u, v \in V$. Suppose $p_u = i_j$ and $p_v = i_k$. Then u is a copy of v_{i_j} and v is a copy of v_{i_k}. If $uv \in E$, then $v_{i_j}v_{i_k} \in \hat{E} \subseteq E'$. Therefore $|i_j - i_k| \leq r \Rightarrow |p_u - p_v| \leq r \Rightarrow p_v \in I_u$ and $p_u \in I_v \Rightarrow p_u, p_v \in I_u \cap I_v$. Finally, let $uv \notin E$. Then $v_{i_j}v_{i_k} \notin \hat{E}$. Since $\hat{G}$ is an induced subgraph of $G(n, r)$, we have $v_{i_j}v_{i_k} \notin E'$. Then $|i_j - i_k| > r \Rightarrow |p_u - p_v| > r \Rightarrow p_v \notin I_u$ and $p_u \notin I_v$. Thus G is a proper central MPTG.

$(\mathbf{1} \Rightarrow \mathbf{3})$: Let $G = (V, E)$ be a proper central MPTG with a proper central MPTG representation (I_u, c_u) where $I_u = [l_u, r_u]$, c_u be the center point of I_u for each $u \in V$. We arrange vertices according to the increasing order of center points, $V = \{v_1, v_2, ..., v_n\}$. To prove that G is a proper interval graph we show that vertices of G satisfy condition 3 of Theorem 2.1 with respect to the above ordering.

Denote $I_{u_i} = [l_{u_i}, r_{u_i}]$ by $[l_i, r_i]$ and $c_i = \frac{l_i + r_i}{2}$ for $i = 1, 2, ..., n$. Let $i < j < k$ and $u_iu_k \in E$. Then $c_i < c_j < c_k$. Now since G is a central MPTG, $c_k - c_i \leq \min\{c_i - l_i, c_k - l_k\}$. Now $c_j - c_i < c_k - c_i \leq c_i - l_i$. Now if $l_j > c_i$, then $l_k \leq c_i < l_j < c_j < c_k$ as $c_k - c_i \leq c_k - l_k$. So $[l_j, c_j] \subsetneq [l_k, c_k]$. But this implies $[l_j, r_j] \subsetneq [l_k, r_k]$ which contradicts the fact that G is a proper central MPTG. Thus $l_j \leq c_i$. So we have $c_j - c_i \leq c_j - l_j$. Hence $c_j - c_i \leq \min\{c_i - l_i, c_j - l_j\}$. Therefore $u_iu_j \in E$, as required. Similarly, it can be shown that $u_ju_k \in E$. Thus G is a proper interval graph. □

It is proved by Catanzaro et al. [3] that if G is an MPTG with non-adjacent vertices u and v, then $G[N(u) \cap N(v)]$ is an interval graph. We found the following analogous result for central MPTG.

Proposition 3.13. *If G is a central MPTG with non-adjacent vertices u and v, then $G[N(u) \cap N(v)]$ is a proper interval graph.*

Proof. Let $G = (V, E)$ be a central MPTG with an interval representation $\{I_i = [a_i, b_i] | i \in V\}$ where vertices are

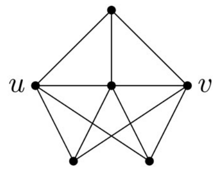

Figure 3. The graph G in Example 3.14.

arranged according to C-order. Let c_i be the center point of I_i. Suppose $u<v$. Then vertices of $G[N(u)\cap N(v)]$ which occur between u, v form a clique from Equation (2.1). Again if there exist a vertex of $G[N(u)\cap N(v)]$ occurs before u then no vertex can occur after v which belongs to $G[N(u)\cap N(v)]$ and conversely follows from Equation (2.1) and the fact $uv\notin E$. Moreover vertices of $G[N(u)\cap N(v)]$ that occur before u form a clique. Let $x<y<u<v$ such that $x,y\in G[N(u)\cap N(v)]$ then $c_x<c_y<c_u<c_v\leq b_x$ as $vx\in E$. This implies $c_y\in[c_x,b_x]\subset I_x$. Now if $a_y>c_x$, then $a_u\leq c_x<a_y<c_y<c_u$ (as $ux\in E$) which implies $c_y-a_y<c_u-a_u$. Hence $|I_y|<|I_u|$. Again as $a_y,c_y\in[a_u,c_u]$ from above we can conclude that $b_y\leq b_u$. But $yv\in E$ implies $a_v<c_y<c_u<c_v\leq b_y$ which imply $c_u\in[a_v,c_v]\subset I_v$. Also as u is not adjacent to v, $c_v>b_u$. Hence from above we get $b_u<c_v\leq b_y$ which is a contradiction. Therefore $a_y\leq c_x$. So $c_x\in[a_y,c_y]\subset I_y$. Hence $xy\in E$. Similarly one can show vertices of $G[N(u)\cap N(v)]$ which occur after v form a clique.

Now let $\{u_i|u_i<u\}$ be the vertices of $G[N(u)\cap N(v)]$ arranged in C-order and $\{x_j|u<x_j<v\}$ be the vertices of $G[N(u)\cap N(v)]$ arranged according to increasing order of left end points. From the above observations it is clear that $u_i[x_j]$'s form clique for $u_i<u[u<x_j<v]$. Let u_k $[x_m]$ be the last vertices occurred before u [between u and v] respectively. Now we show that $G[N(u)\cap N(v)]$ becomes a proper interval graph with respect to the ordering $\{u_1,\dots,u_k,x_1,\dots,x_m\}$. In this ordering by $p\prec q$ we mean p occurs before q. In fact we will show that the vertices satisfy condition 3 of Theorem 2.1 with respect to the ordering $\prec$. Let $u_l\prec x_i\prec x_j$ where $1\leq l\leq k, 1\leq i,j\leq m$ such that $u_lx_j\in E$. Then $c_{u_l}<c_u<c_{x_i},c_{x_j}<c_v<b_{u_l}$ as $vu_l\in E$. This implies $c_{x_i}\in[c_{u_l},b_{u_l}]\subset I_{u_l}$. Now as $u_lx_j\in E, a_{x_j}\leq c_{u_l}<c_u<c_{x_i}$ implies $a_{x_i}<a_{x_j}\leq c_{u_l}<c_{x_i}$ (as $x_i\prec x_j\iff a_{x_i}<a_{x_j}$) which imply $c_{u_l}\in[a_{x_i},c_{x_i}]\subset I_{x_i}$. Hence $u_lx_i\in E$. Let $u_i\prec u_j\prec x_l$ where $1\leq i,j\leq k, 1\leq l\leq m$ such that $u_ix_l\in E$. Then $u_i\prec u_j\prec x_l\prec v$ clearly. Now from Equation (2.1) one can conclude $u_jx_l\in E$. Similarly, one can show if there exists vertices of $G[N(u)\cap N(v)]$ that occurs after v, then with respect to the ordering $\{x_1,\dots,x_m,v_1,\dots,v_k\}$ (use $p\prec' q$ if and only if p occurs before q in this ordering) $G[N(u)\cap N(v)]$ forms a proper interval graph where $\{x_i|u<x_i<v\}$ are vertices of $G[N(u)\cap N(v)]$ arranged according to increasing order of right end points, and $\{v_j|v<v_j\}$ are vertices of $G[N(u)\cap N(v)]$ arranged in C-order. □

The aforementioned proposition leads to a construction of the following forbidden graph for the class of central MPTG.

Example 3.14. By Proposition 3.13 we see that the graph G (see Figure 3) formed by taking $K_{1,3}$ together with two non-adjacent vertices (say, u, v) which are adjacent to each vertex of $K_{1,3}$ is not a central MPTG.

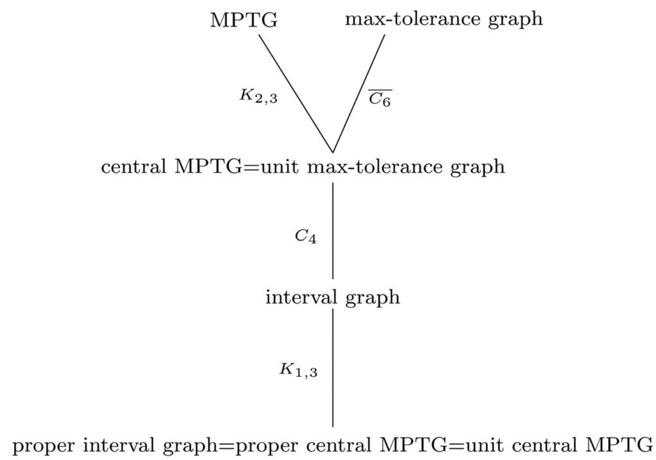

Figure 4. Hierarchy of subclasses of the class of max-tolerance graph.

4. Conclusion

It was proved by Soto and Caro [13] that interval graphs $\subset$ central MPTG $\subseteq$ max-tolerance graph. Combining with these we establish the relations between some subclasses of max-tolerance graphs related to central MPTG in Figure 4. Note that $C_4\in$ unit max-tolerance $\setminus$ interval graphs. C_4 has a unit-max-tolerance representation having intervals $[1,5],[2,6],[3,7],[4,8]$ and corresponding tolerances $1,3,3,1$ for its consecutive vertices (clockwise or anticlockwise). But C_4 is not an interval graph. In Theorem 3.9, we have shown that $\overline{C_6}$ is a max-tolerance graph which is not a central MPTG. Also we note that $K_{2,3}$ is not a central MPTG (by Lemma 7 of reference [13]) but it is a MPTG (by Lemma 8 of reference [13]). Finally we end up by listing the major unsolved problems in this area

1. Recognition algorithm and forbidden subgraph characterization of MPTG.
2. Combinatorial characterization, adjacency matrix characterization, recognition algorithm, and forbidden subgraph characterization of central MPTG.

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