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Cycle stochastic graphs: Structural and forbidden graph characterizations

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ABSTRACT
A vertex (respectively, edge) cycle stochastic function of a graph G is a labeling of vertices (respectively, edges) by a non-negative real valued function $f_V : V(G) \to \mathbb{R}^+ \cup \{0\}$ (respectively, $f_E : E(G) \to \mathbb{R}^+ \cup \{0\}$) such that for every cycle of G, the sum of labels of its vertices (respectively, edges) is 1. The graphs where we can define such a function are called vertex cycle stochastic graphs (respectively, edge cycle stochastic graphs). In this paper, we provide a structure theorem for biconnected cycle stochastic graphs, which is extended to characterize edge cycle stochastic graphs. We also find a minimal forbidden graph characterization for biconnected vertex cycle stochastic graphs and its description for vertex cycle stochastic graphs.

KEYWORDS
Cycle stochastic graphs; structural characterization; minimal forbidden graph characterization

1. Introduction

Graph labeling is well studied in graph theory with wide applications in communications networks, astronomy, database management, secret sharing schemes etc. (see [3] for a survey). Berge [2] observed such an application in *strongly perfect graphs* (where every induced subgraph contains an independent set of vertices that "hits" every maximal clique). He defined *stochastic graphs* where every vertex is labelled with a non-negative real number such that the sum of vertex labels of every maximal clique is one. He proved that a graph is strongly perfect if and only if it is perfect and all its induced subgraphs are stochastic. We extend his definition to the following, where maximal cliques are replaced by cycles (such extensions are studied in [7]).

Definition 1. A vertex cycle stochastic function of a graph G is a labeling of vertices $f_V : V(G) \to \mathbb{R}^+ \cup \{0\}$ such that $f_V(C) = \sum_{v \in V(C)} f_V(v) = 1$, for every cycle C of G. Vertex stochastic graphs are those graphs that have vertex cycle stochastic functions.

In [1], the authors considered edge labelings instead of vertex labelings and gave the following definition.

Definition 2. An edge cycle stochastic function of a graph G is a labeling of edges $f_E : E(G) \to \mathbb{R}^+ \cup \{0\}$ such that $f_E(C) = \sum_{e \in E(C)} f_E(e) = 1$, for every cycle C of G. Edge stochastic graphs, denoted $\mathcal{G}_{ECS}$, are those graphs that have edge cycle stochastic functions.

One can combine both the definitions and consider vertex as well as edge labelings $f_{VE} : V(G) \cup E(G) \to \mathbb{R}^+ \cup \{0\}$ such that $f_{VE}(C) = \sum_{v \in V(C)} f_{VE}(v) + \sum_{e \in E(C)} f_{VE}(e) = 1$ holds for every cycle C of G. This class of graphs (where we can define f_{VE}) is equivalent to Edge stochastic graphs: define $f_E(uv) = f_{VE}(uv) + \frac{1}{2}[f_{VE}(u) + f_{VE}(v)]$; so for cycle C, $\sum_{v \in V(C)} f_{VE}(v) + \sum_{e \in E(C)} f_{VE}(e) = 1$ implies $\sum f_E(uv) = 1$. The other direction is trivial.

We denote the class of Vertex stochastic graphs and the class of Edge stochastic graphs as $\mathcal{G}_{VCS}$ and $\mathcal{G}_{ECS}$, respectively. They both are called as cycle stochastic graphs. $\mathcal{G}_{VCS}$ is contained in $\mathcal{G}_{ECS}$ (label edge uv as $f_E(uv) = \frac{1}{2}[f_V(u) + f_V(v)]$). Later we show this containment to be strict.

A straight forward application of graphs in $\mathcal{G}_{VCS}$ is in resource allotment [7]. Stochastic graphs have applications in random walks and matrix theory (see [7]). In this article we give structural as well as forbidden characterization for cycle stochastic graphs.

In the first part of this article, we give structural characterizations of cycle stochastic graphs. We need two classes of biconnected series-parallel graphs $\mathcal{G}_{RSP}$ and $\mathcal{G}_{GRSP}$ for these characterizations.

Definition 3. A biconnected graph G is in $\mathcal{G}_{RSP}$ if G has a cutset S such that

(1) S is an independent set,
(2) S has exactly two vertices of any cycle in G and
(3) $G \setminus S$ is union of two disjoint trees T_1 and T_2.

A graph is in $\mathcal{G}_{GRSP}$ (precise definition is given later) if it is obtained by adding some restricted edges to graphs in $\mathcal{G}_{RSP}$. Now we are ready to present our structural characterizations.

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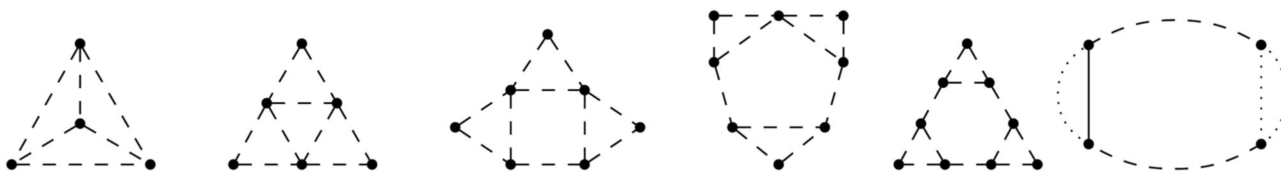

Figure 1. VCS forbidden graph classes (bold lines denote edges, dashed line denotes paths, dotted lines denote paths that are not edges).

Theorem 4. *A biconnected graph G is in $\mathcal{G}_{VCS}$ if and only if $G \in \mathcal{G}_{RSP}$ or there exist a vertex $v \in V(G)$ such that $G \setminus \{v\}$ is a tree.*

Theorem 5. *A graph G is in $\mathcal{G}_{ECS}$ if and only if its blocks B_i belong to $\mathcal{G}_{GRSP}$ or there exist a vertex $v_i \in B_i$ such that $G[V(B_i) \setminus \{v_i\}]$ is a tree.*

The second part of this article deals with forbidden graph characterizations of cycle stochastic graphs. A class of graphs $\mathcal{G}$ is said to be *hereditary* (respectively, *strongly hereditary*) if every induced subgraph (respectively, subgraph) of $G \in \mathcal{G}$ is in $\mathcal{G}$. The folklore result of Greenwell et al. [5] states that any hereditary class of graphs $\mathcal{G}$ can be characterized by a set of minimal forbidden graphs $\mathcal{H}$. We say $G \in \mathcal{G}$ to be $\mathcal{H}$-free.

$\mathcal{G}_{ECS}$ and $\mathcal{G}_{VCS}$ are strongly hereditary. Let $\mathcal{H}$ denote the family of graphs shown in Figure 1. The forbidden graph characterization of $\mathcal{G}_{ECS}$ was found by Balasubramanian et al. [1], as the first five classes of $\mathcal{H}$. The following result gives such a characterization for biconnected graphs in $\mathcal{G}_{VCS}$.

Theorem 6. *A biconnected graph is in $\mathcal{G}_{VCS}$ if and only if it is $\mathcal{H}$–free.*

Using this we give a description of forbidden graph characterization of $\mathcal{G}_{VCS}$.

1.1. Organization

In the rest of this section, we give the necessary definitions. In Section 2, we prove the structural results, namely Theorem 4 and 5. In Section 3, we prove the forbidden graph characterizations, namely Theorem 6 and description of forbidden graph characterization of $\mathcal{G}_{VCS}$.

1.2. Definitions

We follow the notations of West [9]. A *chord* of a cycle is an edge joining two of its non-adjacent vertices. A graph, other than a cycle, is said to be *chordless* if none of the cycles in it contain a chord. To avoid conflicts, we assume cycles are not *chordless*.

A vertex (respectively, edge) is said to be a *cut vertex* (respectively, *cut edge*) of a graph G, if its removal disconnects G. A graph is said to be *biconnected* if it has no cut vertices. A *block* B of a graph is a maximal biconnected graph.

2. Structural characterizations: Proofs of Theorems 4 and 5

The idea of the proof of Theorem 4 is roughly the following. Let G be a biconnected graph in $\mathcal{G}_{ECS}$ (since $\mathcal{G}_{VCS} \subset \mathcal{G}_{ECS}$, it is enough to consider $G \in \mathcal{G}_{ECS}$). Since G is biconnected, it has a cycle. If any of the cycles have a chord, then we prove $G \in \mathcal{G}_{VCS}$ if and only if there exists a vertex whose removal makes G a tree. If G is chordless then we prove $G \in \mathcal{G}_{VCS}$ if and only if $G \in \mathcal{G}_{RSP}$. In order to prove these results we use the forbidden graph characterization of Balasubramanium et al. [1].

Before going into the proof of Theorem 4, we need the following construction.

2.1. Zone decomposition

In this construction, we divide a chordless biconnected graph $G \in \mathcal{G}_{ECS}$ into *zones*, which are composed of 1–*zones*: consisting of one path, and 2–*zones*: consisting of two paths. Let C be a cycle in G. Since G is chordless, it contains vertices α and β that are connected by a path P. Let $G_1 = C \cup P$: we update G_i as we proceed. Now G_1 is a collection of three $\alpha\beta$–paths, each of which is a 1–zone.

Consider a vertex $v \in V(G) \setminus V(G_i)$ with internally disjoint paths, P_{v_1} and P_{v_2}, to G_i. The endpoints of P_{v_1} and P_{v_2} in G_i, say v_1 and v_2 lie in one of the 1–zones, else a K_4–subdivision is induced (and then $G \notin \mathcal{G}_{ECS}$: see Figure 1.1). If v_1 and v_2 are not α and β, then the 1–zone in G_i containing v_1 and v_2 is divided into a 1–zone v_1v_2 and a 2–zone $(\alpha v_1, v_2\beta)$. (Note that a path in a 2–zone can be a vertex.) Also $P_{v_1} \cup P_{v_2} (= v_1vv_2\text{–path})$ is a new 1–zone. If v_1 and v_2 are α and β, then just one new 1–zone $P_{v_1} \cup P_{v_2}$ is introduced. Update G_i as $G_i \cup P_{v_1} \cup P_{v_2}$.

Consider another vertex $u \in V(G) \setminus V(G_i)$ with two internally disjoint paths, P_{u_1} and P_{u_2}, to G_i. Let the endpoints of P_{u_1} and P_{u_2} be u_1 and u_2 in G_i. None of the graphs in Figure 2.1–2.5 can be induced, else one of the graphs in Figure 1.1–1.5 is induced (and then $G \notin \mathcal{G}_{ECS}$). It can be checked that graphs in Figure 1.1–1.5 are not induced if u_1 and u_2 belong to exactly one of the zones of G_i. Update G_i as $G_i \cup P_{u_1} \cup P_{u_2}$. Update the zones of G_i. Keep on adding new vertices and updating the zones till $G_i = G$ (see Figure 3).

We have the following observations.

Observation 2.1. *G is chordless biconnected graph in $\mathcal{G}_{ECS}$ if and only if $G \in \mathcal{G}_{RSP}$.*

Proof. For the "if" part: suppose $G \in \mathcal{G}_{RSP}$, then G is biconnected by definition. To see that G is chordless, suppose for contrary G has a cycle C with chord $\alpha\beta$. Then $C \cup \{\alpha\beta\}$ induces 3 cycles. It can be checked that the only possibility of choosing exactly two vertices per cycle whose removal breaks each cycle into two components is if $\alpha, \beta \in S$, but then S would not be an independent set, a contradiction. Now we give an appropriate edge labeling. For each vertex

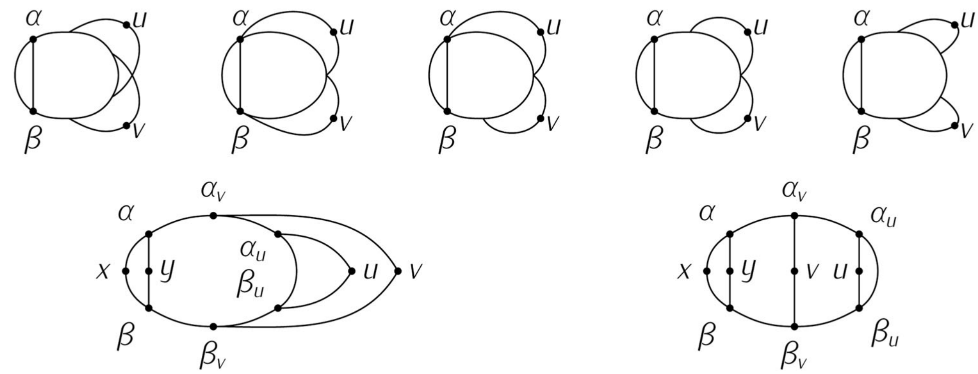

Figure 2. Obstructions and allowed configurations.

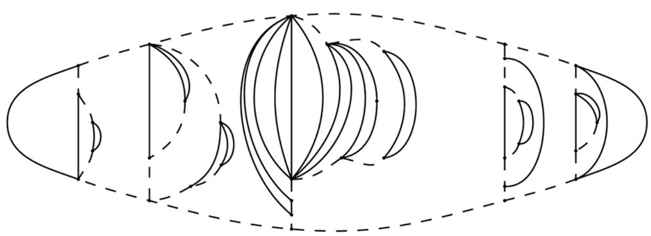

Figure 3. Chordless cycle: The solid (dashed) lines represent the 1–zones (2–zones).

in the cutset, choose an edge and assign label 1/2 to it; to all other edges assign label 0. So G is chordless biconnected graph in $\mathcal{G}_{ECS}$.

For the "only if" part: by zone construction, G is divided into a set of zones (1–zones and 2–zones). By construction every 1–zone in G has at least one internal vertex. Choose one such internal vertex of each 1–zone in set S. Clearly S is a cutset dividing G into exactly two trees. So $G \in \mathcal{G}_{RSP}$. □

Observation 2.2. *The class of chordless biconnected graph in $\mathcal{G}_{ECS}$ is equivalent to the chordless biconnected graph in $\mathcal{G}_{VCS}$.*

Proof. If G is chordless biconnected graph in $\mathcal{G}_{ECS}$ then there is an edge labeling of G where each 1–zone of G has an edge that has label 1/2. For each 1–zone select the end of the edge that is an internal vertex of the 1–zone. By assigning labels 1/2 to these vertices and 0 to the rest, we get $G \in \mathcal{G}_{VCS}$. The other direction follows from the "only if" part of proof of Observation 2.1. □

Definition 7. $\mathcal{G}_{GRSP}$ is the class of graphs obtained after adding edges (i.e. chords) with end points in the same zone to graphs in $\mathcal{G}_{RSP}$.

Now we are ready to prove our results.

Proof of Theorem 4. Let G be a biconnected graph in $\mathcal{G}_{ECS}$. If G is a cycle then $G \in \mathcal{G}_{VCS}$ and removing any vertex results in a path. The rest of the proof follows from these two claims.

Claim 2.1. *If G has a cycle C with a chord $\alpha\beta$, then $G \in \mathcal{G}_{VCS}$ if and only if $G \setminus \{\alpha\}$ or $G \setminus \{\beta\}$ is a tree.*

Proof. The "if" part is obvious. We prove the "only if" part. Assume $G \in \mathcal{G}_{VCS}$; then $f_V(\alpha) + f_V(\beta) = 1$. Any other vertex $v \in V(G) \setminus V(C)$ lies on a cycle containing α and β (since G is biconnected); so $f_V(v) = 0$. So there cannot be any cycle in G which does not contain α or β. Suppose there is a cycle containing (wlog) α and not β, then $f_V(\alpha) = 1$. So every cycle in G contains α. Hence $G \setminus \{\alpha\}$ is a tree. ◀

Claim 2.2. *If G is chordless, then $G \in \mathcal{G}_{VCS}$ if and only if $G \in \mathcal{G}_{RSP}$.*

Proof. For the "if" part, assign label 1/2 to all vertices in the cutset and label 0 to rest of the vertices. The "only if" part follows from Observation 2.1 and 2.2. ◀

This completes the proof of Theorem 4. □

We have the following corollary.

Corollary 8. *Any biconnected VCS graph can have a $\{0, 1/2, 1\}$–labeling.*

Now we prove Theorem 5, which is quite similar to the proof of Theorem 4.

Proof of Theorem 5. If all blocks of a graph G are in $\mathcal{G}_{ECS}$ then $G \in \mathcal{G}_{ECS}$, as each edge belongs to one block and no new cycles are formed. So it suffices to characterize biconnected graphs in $\mathcal{G}_{ECS}$. Let B be such a graph. If B is a cycle then $B \in \mathcal{G}_{ECS}$ and removing any vertex results in a path. The rest of the proof follows from these two claims.

Claim 2.3. *If B is chordless, then $B \in \mathcal{G}_{ECS}$ if and only if $B \in \mathcal{G}_{RSP}$.*

Proof. This is exactly Observation 2.1. ◀

Claim 2.4. *If B has a cycle C with a chord $\alpha\beta$, then $B \in \mathcal{G}_{ECS}$ if and only if $B \in \mathcal{G}_{GRSP}$.*

Proof. For "if" part assign label 1/2 to the chords in $B \in \mathcal{G}_{GRSP}$ and assign label 1/2 to one of the edges adjacent to each vertex in the cutset of the underlying graph of $\mathcal{G}_{RSP}$ and assign label 0 to rest of the edges.

For "only if" part assume B is a biconnected graph in $\mathcal{G}_{ECS}$. The graph obtained after deleting all the chords in B is a chordless biconnected graph in $\mathcal{G}_{ECS}$ (recall $\mathcal{G}_{ECS}$ is strongly hereditary), which is characterized in Claim 2.3. Now add back the chords to get a graph in $\mathcal{G}_{GRSP}$. ◀

This completes the proof of Theorem 5. □

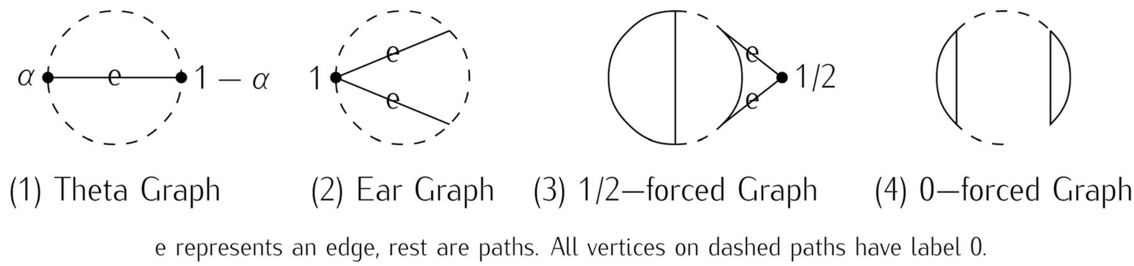

Figure 4. Building minimal forbidden subgraphs for VCS.

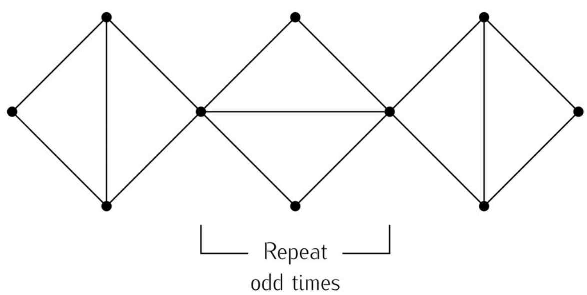

Figure 5. Infinite minimal forbidden subgraphs for VCS.

3. Minimal forbidden graph characterizations

We begin this section with the proof of Theorem 6.

Proof of Theorem 6. If all blocks of a graph G are in $\mathcal{G}_{ECS}$ then $G \in \mathcal{G}_{ECS}$. So the minimal forbidden graph characterization of $\mathcal{G}_{ECS}$ is same as that of biconnected graphs in $\mathcal{G}_{ECS}$. It can be checked that biconnected $\mathcal{G}_{VCS}$ is a subset of biconnected $\mathcal{G}_{ECS}$. So the graphs in Figure 1.1–1.5 are also forbidden in biconnected $\mathcal{G}_{VCS}$.

To see the "only if" part: If a biconnected graph G contains any of graphs in Figures 1.1–1.5, then $G \notin \mathcal{G}_{VCS}$. If G contains the graph in Figure 1.6, then $G \notin \mathcal{G}_{VCS}$ (follows from Theorem 4).

Now we prove the "if" part. Suppose a biconnected graph G is $\mathcal{H}$–free. From Theorem 4 and 5, the only graphs that are in $\mathcal{G}_{ECS}$ and not in $\mathcal{G}_{VCS}$ belong to $\mathcal{G}_{GRSP}$, which are biconnected. The graph in Figure 1.6 is the only minimal forbidden graph of $\mathcal{G}_{GRSP}$. So $G \notin \mathcal{G}_{GRSP}$. Hence $G \notin \mathcal{G}_{VCS}$. □

3.1. Minimal forbidden graphs of $\mathcal{G}_{VCS}$

The above characterization can be extended to $\mathcal{G}_{VCS}$. Recall that if blocks of a graph G belong to $\mathcal{G}_{ECS}$, then $G \in \mathcal{G}_{VCS}$. However this is not true for $\mathcal{G}_{VCS}$, as there might be conflict in the labels of the cut vertices. So apart from the forbidden graphs of biconnected graphs in $\mathcal{G}_{VCS}$, we shall have some forbidden graphs. Due to minimality every cut vertex belongs to exactly two blocks.

From the structural analysis in Section 2 we have the following forced labels. For graphs with a chord $\alpha\beta$ label 0 is forced on all vertices except α and β. If a label x is forced on α, then label $1 - x$ is forced on β. Label 1 is forced on α when there are cycles that pass through α but not β, in which case label 0 is forced on β. For chordless graphs label 0 is fixed on all vertices of 2–zones; label 1/2 is forced on each 1–zone, hence if a 1–zone has just one internal vertex label 1/2 is forced on it. Using these we can form a set of basic blocks as shown in Figure 4.

We can divide the set of minimal forbidden graphs of $\mathcal{G}_{VCS}$ into two types. In type–1 forbidden graphs, there is a conflict in labels of the cutvertex. Such graphs can be obtained by merging two basic block at a vertex which has different forced label in each block (see Figure 5). In type–2 forbidden graphs, one of the restrictions that every cycle has label exactly 1 and every 1–zone has label exactly 1/2 is violated. This can be done by merging each vertex of the cycle or 1–zone by a basic block at a vertex whose label is forced in the basic block.

4. Conclusion

In this article, we explored various types of cycle stochastic graphs and the connections between them. We gave structure theorems for $\mathcal{G}_{ECS}$ and biconnected graphs in $\mathcal{G}_{VCS}$. We also provided an explicit minimal forbidden subgraph characterization for biconnected graphs in $\mathcal{G}_{VCS}$ and then described such a characterization for $\mathcal{G}_{VCS}$.

Regarding graph characteristics, one can observe that cycle stochastic graphs are series-parallel graphs and hence many standard graph problems can be solved in linear time [4, 6, 8]. Graphs in both $\mathcal{G}_{VCS}$ and $\mathcal{G}_{ECS}$ have edge chromatic number Δ except when the graph is a odd cycle where it is 3. Apart from the chordless cycle case, this is easy to see. For the chordless case, it follows from an inductive argument on the number of 1–zones.

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