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Adrián Vázquez-Ávila

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On strong Skolem starters for $\mathbb{Z}_{pq}$

Adrián Vázquez-Ávila

Subdirección de Ingeniería y Posgrado, Universidad Aeronáutica en Querétaro, Querétaro, México

ABSTRACT

In 1991, N. Shalaby conjectured that any additive group $\mathbb{Z}_n$, where $n \equiv 1$ or 3 (mod 8) and $n \geq 11$, admits a strong Skolem starter and constructed these starters of all admissible orders $11 \leq n \leq 57$. Shalaby et al. [24] was proved if $n = \Pi_{i=1}^{k} p_i^{\alpha_i}$, where p_i is a prime number such that $ord(2)_{p_i} \equiv 2$ (mod 4) and α_i is a non-negative integer, for all $i = 1, \ldots, k$, then $\mathbb{Z}_n$ admits a strong Skolem starter. On the other hand, the author [30] gives different families of strong Skolem starters for $\mathbb{Z}_p$ than Shalaby et al, where $p \equiv 3$ (mod 8) is an odd prime. Recently, the author [31] gives different families of strong Skolem starters of $\mathbb{Z}_{p^n}$ than Shalaby et al, where $p \equiv 3$ (mod 8) and n is an integer greater than 1. In this paper, we give some different families of strong Skolem starters of $\mathbb{Z}_{pq}$, where $p, q \equiv 3$ (mod 8) are prime numbers such that $p < q$ and $(p-1) \nmid (q-1)$.

KEYWORDS

Strong starters; Skolem starters; quadratic residues

1. Introduction

Let G be a finite additive abelian group of odd order $n = 2k+1$, and let $G^* = G \setminus \{0\}$ be the set of non-zero elements of G. A *starter* for G is a set $S = \{\{x_i, y_i\}, i = 1, \ldots, k\}$ such that $\{\{x_i\} \cup \{y_i\} : i = 1, \ldots, k\} = G^*$ and $\{\pm(x_i - y_i) : i = 1, \ldots, k\} = G^*$. Moreover, if all elements $\{x_i + y_i : i = 1, \ldots, k\} \subseteq G^*$ are different, then S is called *strong starter* for G. The reader may consult [8, 9, 13, 14, 18, 20, 22, 23] for works related with.

Strong starters were first introduced by Mullin and Stanton in [28] in constructing Room squares. Starters and strong starters have been useful to construct many combinatorial designs such as Room cubes [12], Howell designs [3, 20], Kirkman triple systems [20, 25], Kirkman squares and cubes [26, 29], and factorizations of complete graphs [2, 4, 10, 11, 13, 16, 17, 21].

Let $n = 2k+1$ and $1 < 2 < \cdots < 2k$ be the order of $\mathbb{Z}_n^*$. A starter for $\mathbb{Z}_n$ is *Skolem* if it can be written as $S = \{\{x_i, y_i\} : i = 1, \ldots, k\}$ such that $y_i > x_i$ and $y_i - x_i = i$ (mod n), for $i = 1, \ldots, k$. In [27], it was proved the Skolem starter for $\mathbb{Z}_n$ exits if and only if $n \equiv 1, 3$ (mod 8). A starter which is both Skolem and strong is called *strong Skolem starter*.

Shalaby in [27] proposed the following:

Conjecture 1.1. *If $n \equiv 1, 3$ (mod 8) and $n \geq 11$, then $\mathbb{Z}_n$ admits a strong Skolem starter.*

In [24], it was proved if $n = \Pi_{i=1}^{k} p_i^{\alpha_i}$, where p_i is a prime such that $ord(2)_{p_i} \equiv 2$ (mod 4) and α_i is a non-negative integer, for all $i = 1, \ldots, k$, then $\mathbb{Z}_n$ admits a strong Skolem starter, where $ord(2)_{p_i}$ is the order of the element 2 in $\mathbb{Z}_{p_i}$. In [30], it was given different families of strong Skolem starters of $\mathbb{Z}_p$, where $p \equiv 3$ (mod 8) is an odd prime, using a different method than in [24]. Recently in [31], it was given different families of strong Skolem starters of $\mathbb{Z}_{p^n}$, where $p \equiv 3$ (mod 8) and n is an integer greater than 1, than in [24]. In this paper, we use the method of Vázquez-Ávila to give different families of strong Skolem starters for $\mathbb{Z}_{pq}$ than in [24], where $p, q \equiv 3$ (mod 8) are prime numbers such that $p < q$ and $(p-1) \nmid (q-1)$.

This paper is organized as follows. In Section 2, we recall some basic properties about quadratic residues and we present the strong Skolem starters of $\mathbb{Z}_p$ given in [30]; this idea is used in the main result of this paper, Theorem 3.3. Finally, in Section 3, we give the main result of this paper, and we present one example. The main theorem states the following:

Theorem 1.2. *Let $p, q \equiv 3$ (mod 8) be odd prime numbers such that $p < q$ and $(p-1) \nmid (q-1)$. If $r \in \mathbb{Z}_p^*$ is a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$, then $\mathbb{Z}_{pq}$ admits a strong Skolem starter.*

2. A family of strong Skolem starters for $\mathbb{Z}_p$

The following definitions and notations are obtained from [30] and [31]: Let p be an odd prime power. An element $x \in \mathbb{Z}_p^*$ is called a *quadratic residue* if there exists an element $y \in \mathbb{Z}_p^*$ such that $y^2 = x$. If there is no such y, then x is called a *non-quadratic residue*. The set of quadratic residues of $\mathbb{Z}_p^*$ is denoted by $QR(p)$ and the set of non-quadratic residues is denoted by $NQR(p)$. It is well known that $QR(p)$ is a (cyclic) subgroup of $\mathbb{Z}_p^*$ of cardinality $\frac{p-1}{2}$ (see for example [15]); also, if either $x, y \in$

CONTACT Adrián Vázquez-Ávila adrian.vazquez@unaq.edu.mx Subdirección de Ingeniería y Posgrado, Universidad Aeronáutica en Querétaro, Parque Aeroespacial de Querétaro, 76278, Querétaro, México.

$QR(p)$ or $x, y \in NQR(p)$, then $xy \in QR(p)$, and if $x \in QR(p)$ and $y \in NQR(p)$, then $xy \in NQR(p)$.

The following theorems are well known results on quadratic residues. For more details of this kind of results the reader may consult [5, 15].

Theorem 2.1 (Euler's criterion). *If p is an odd prime and $x \in \mathbb{Z}_p^*$, then*

1. $x \in QR(p)$ *if and only if* $x^{\frac{p-1}{2}} = 1$.
2. $x \in NQR(p)$ *if and only if* $x^{\frac{pp-1}{2}} = -1$.

Theorem 2.2. *Let p be an odd prime power, then*

1. $-1 \in QR(p)$ *if and only if* $p \equiv 1 \pmod 4$.
2. $-1 \in NQR(p)$ *if and only if* $p \equiv 3 \pmod 4$.

Theorem 2.3. *Let p be an odd prime. If $p \equiv 3 \pmod 4$, then*

1. $x \in QR(p)$ *if and only if* $-x \in NQR(p)$.
2. $x \in NQR(p)$ *if and only if* $-x \in QR(p)$.

In [1], it was proved the following (see also [22]):

Lemma 2.4. [1] *If $p \equiv 3 \pmod 4$ is an odd prime with $p \neq 3$, then the following set*

$$S_\beta = \{\{x, \beta x\} : x \in QR(p)\},$$

is a strong starter for $\mathbb{Z}_p$, for all $\beta \in NQR(p) \setminus \{-1\}$.

In [30], it was proved the following:

Theorem 2.5. [30] *Let $p \equiv 3 \pmod 8$ be an odd prime, then the following strong starter*

$$S_\beta = \{\{x, \beta x\} : x \in QR(p)\}$$

for $\mathbb{Z}_p$ is Skolem, if $\beta = 2$ and $\beta = \frac{1}{2}$.

We will use the strong Skolem starter of Theorem 2.5 for the main result of this paper.

3. A family of strong Skolem starters for $\mathbb{Z}_{pq}$

In this section, we give different families of strong Skolems starters that given in [24]. The following definitions and notations are obtained from [24]: Let G_n be the group of units of the ring $\mathbb{Z}_n$ (elements invertible with respect to multiplication). It is denoted by $\langle x \rangle_n$ the cyclic subgroup of G_n generated by $x \in G_n$. Also, we will use the notation $aB = \{ab : b \in B\}$, where $a \in \mathbb{Z}$ and $B \subseteq \mathbb{Z}$. On the other hand, it is denoted by $ord(x)_n$ the order of the element $x \in G_n$; hence, $ord(x)_n = |\langle x \rangle_n|$. Whenever the group operation is irrelevant, it will consider G_n and its cyclic multiplicative subgroups $\langle x \rangle_n$ in the set-theoretical sense and denote them by $\underline{G}_n$ and $\langle \underline{x} \rangle_n$, respectively.

Let $p, q \equiv 3 \pmod 8$ be odd prime numbers such that $p < q$ and $(p-1) \nmid (q-1)$. We have $\underline{G}_{pq} = \{x \in \mathbb{Z}_{pq}^* : gcd(x, pq) = 1\}$, with $|\underline{G}_{pq}| = (p-1)(q-1)$, see for example [6]. Hence, $p\mathbb{Z}_q^*, q\mathbb{Z}_p^*$ and $\underline{G}_{pq}$ forms a partition of $\mathbb{Z}_{pq}^*$, since every element $x \in \mathbb{Z}_{pq}^*$ lies in one and only one of these sets. Moreover, it is not difficult to prove that if $r \in \mathbb{Z}_p^*$ (or $r \in \mathbb{Z}_q^*$) is a primitive root, then $|\langle r \rangle_{pq}| = lcm(p-1, q-1) = \frac{(p-1)(q-1)}{2}$, since $(p-1) \nmid (q-1)$.

Lemma 3.1. *If $p, q \equiv 3 \pmod 8$ are odd prime numbers with $p < q$, and $r \in \mathbb{Z}_p^*$ is such that $r \in NQR(p)$ and $r \in NQR(q)$, then $r^{\frac{(p-1)(q-1)}{4}} \equiv -1 \pmod{pq}$. Moreover, if r is a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$, then $-1 \in r\langle r^2 \rangle_{pq}$.*

Proof. Recall that G_m is the group of units of $\mathbb{Z}_m$. It is well known that the map $\Psi : G_{pq} \to G_p \times G_q$ defined by $\Psi(k_{pq}) = (k_p, k_q)$, is an isomorphism between G_{pq} and $G_p \times G_q$, see for example [7]. Since p and q are prime numbers then $G_p = \mathbb{Z}_p^*$ and $G_q = \mathbb{Z}_q^*$. Let $r \in \mathbb{Z}_p^*$ such that $r \in NQR(p)$ and $r \in NQR(q)$. Hence $r^{\frac{p-1}{2}} = -1 \pmod p$ and $r^{\frac{q-1}{2}} = -1 \pmod q$, by Theorem 2.1. Then $\Psi(r_{pq}^{\frac{(p-1)(q-1)}{4}}) = (r_p^{\frac{(p-1)(q-1)}{4}}, r_q^{\frac{(p-1)(q-1)}{4}}) = (r_p^{\frac{p-1}{2}\cdot\frac{q-1}{2}}, r_q^{\frac{q-1}{2}\cdot\frac{p-1}{2}}) = (-1_p, -1_q) = \Psi(-1_{pq})$, since $\frac{p-1}{2}$ and $\frac{q-1}{2}$ are odd integers. On the other hand, if r is a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$, then by Theorem 2.2, it follow that $-1_p \in r\langle r^2 \rangle_p$ and $-1_q \in r\langle r^2 \rangle_q$, which implies that $-1_{pq} \in r\langle r^2 \rangle_{pq}$. □

Lemma 3.2. *Let $p, q \equiv 3 \pmod 8$ be odd prime numbers such that $p < q$. If $r \in \mathbb{Z}_p^*$ is a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$, then $2 \notin \langle r^2 \rangle_{pq}$.*

Proof. Let $\Psi : G_{pq} \to G_p \times G_q$ given by $\Psi(k_{pq}) = (k_p, k_q)$, the isomorphism between G_{pq} and $G_p \times G_q$, and let $r \in \mathbb{Z}_p^*$ be a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$. It is well known that $2 \in NQR(p)$ and $2 \in NQR(q)$, where $NQR(p) = r\langle r^2 \rangle_p$ and $NQR(q) = r\langle r^2 \rangle_q$, see for example [19]. Hence, if $2 \in \langle x^2 \rangle_{pq}$, then there exists $j \in \{0, \ldots, \frac{(p-1)(q-1)}{4}\}$ such that $x^{2j} = 2 \pmod{pq}$, which implies that $\Psi(2) = \Psi(x_{pq}^{2j}) = (x_p^{2j}, x_q^{2j}) \neq (2, 2)$, a contradiction. Hence $2 \notin \langle r^2 \rangle_{pq}$. □

Theorem 3.3. *Let $p, q \equiv 3 \pmod 8$ be odd prime numbers such that $p < q$ and $(p-1) \nmid (q-1)$. If $r \in \mathbb{Z}_p^*$ is a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$, then $\mathbb{Z}_{pq}$ admits a strong Skolem starter.*

Proof. Let $r \in \mathbb{Z}_p^*$ be a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$. Hence, r^2 is a generator of $QR(p)$ and $QR(q)$. Since $p\mathbb{Z}_q^*, q\mathbb{Z}_p^*$ and $\underline{G}_{pq}$ forms a partition of $\mathbb{Z}_{pq}^*$, define

$$pS_q = \{\{px, 2px\} : x \in QR(q)\}$$
$$qS_p = \{\{qx, 2qx\} : x \in QR(p)\}$$
$$S_{pq} = \left\{\{x, 2x\} : x \in \langle r^2 \rangle_{pq}\right\} \cup \left\{\{\lambda x, 2\lambda x\} : x \in \langle r^2 \rangle_{pq}\right\},$$

where $\lambda \notin \langle r^2 \rangle_{pq} \cup 2\langle r^2 \rangle_{pq}$. It is easy to see that $\{\pm px : x \in QR(q)\} = p\mathbb{Z}_q^*, \{\pm qx : x \in QR(p)\} = q\mathbb{Z}_p^*$ and $\{\pm x : x \in \langle r^2 \rangle_{pq}\} \cup \{\pm \lambda x : x \in \langle r^2 \rangle_{pq}\} = \underline{G}_{pq}$, by Lemmas 3.1 and 3.2. Hence, the set $S = pS_q \cup qS_p \cup S_{pq}$ is a starter.

Let define

$$qS_p^+ = \{3px : x \in QR(p)\}$$
$$pS_q^+ = \{3qx : x \in QR(q)\}$$
$$S_{pq}^+ = \{3x : x \in \langle r^2 \rangle_{pq}\} \cup \{3\lambda x : x \in \langle r^2 \rangle_{pq}\}$$

Since $l_1 x + l_2 y \neq 0$, for all different $x, y \in QR(p) \cup QR(q) \cup \langle r^2 \rangle_{pq}$, with $l_1, l_2 \in \{1, p, q, \lambda\}$. Then $|qS_p^+| = |QR(p)|$, $|pS_q^+| = |QR(q)|$ and $|S_{pq}^+| = |\langle r^2 \rangle_{pq}|$. By Lemma 3.1, we have $|qS_p^+ \cup pS_q^+ \cup S_pq^+| = |qS_p^+| + |pS_q^+| + |S_{pq}^+| = \frac{p-1}{2} + \frac{q-1}{2} + \frac{(p-1)(q-1)}{2} = \frac{pq-1}{2}$. Hence, the set S is strong.

Finally, we give a proof analogous to the case (i) of the proof of Theorem 2.5 given in [30] to prove S is Skolem. Let $pq = 2t + 1$ and $1 < 2 < \cdots < 2t$ be the order of the non-zero elements of $\mathbb{Z}_{pq}^*$. Define $Q_{\frac{1}{2}} = \{1, 2, ..., t\}$. To prove that S is Skolem, it is sufficient to prove that, if $2lx > lx$ then $lx \in Q_{\frac{1}{2}}$, and if $lx > 2lx$ then $-lx \in Q_{\frac{1}{2}}$, where $l \in \{1, p, q, \lambda\}$. Suppose that $lx \in Q_{\frac{1}{2}}$, for $l \in \{1, p, q, \lambda\}$, then $2lx > lx$, which implies that $2lx - lx = lx \in Q_{\frac{1}{2}}$. On the other hand, if $lx \notin Q_{\frac{1}{2}}$, for $l \in \{1, p, q, \lambda\}$, then $-lx \in Q_{\frac{1}{2}}$. Hence, $2(-lx) > -lx$, which implies that $-2lx + lx = -lx \in Q_{\frac{1}{2}}$. Hence, the set S is Skolem. □

Corollary 3.4. *Let $p, q \equiv 3 \pmod 8$ be odd prime numbers such that $p < q$ and $(p-1) \nmid (q-1)$, and let $r \in \mathbb{Z}_p^*$ be a primitive root of $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$. If $pS_q^- = \{\{px, 2^{-1}px\} : x \in QR(q)\}, S_p^- = \{\{qx, 2^{-1}qx\} : x \in QR(p)\}$ and $S_{pq}^- = \{\{x, 2^{-1}x\} : x \in \langle r^2 \rangle_{pq}\} \cup \{\{\lambda x, 2^{-1}\lambda x\} : x \in \langle r^2 \rangle_{pq}\}$, where $\lambda \notin \langle r^2 \rangle_{pq} \cup 2\langle r^2 \rangle_{pq}$, then set $S^- = pS_q^- \cup qS_p^- \cup S_{pq}^-$ is a strong Skolem starter for $\mathbb{Z}_{pq}$.*

Proof. The proof is analogous to the proof of Theorem 3.3, using the case (ii) of Theorem 2.5 given in [30]. □

Example 1. *Consider $\mathbb{Z}_{11\cdot 19}$. We have $\mathbb{Z}_{11\cdot 19}^* = 19\underline{G}_{11} \cup 11\underline{G}_{19} \cup \underline{G}_{11\cdot 19}$, and $r = 2$ is a primitive root of $\mathbb{Z}_{11}^*$ and $\mathbb{Z}_{19}^*$. Then, taking $\lambda = 3$, the pairs from $19\underline{G}_{11}, 11\underline{G}_{19}$ and $\underline{G}_{11^2}$ are:*

$19\underline{G}_{11}$: $\{19, 38\}, \{76, 152\}, \{95, 190\}, \{171, 133\}, \{57, 114\}$.

$11\underline{G}_{19}$: $\{11, 22\}, \{44, 88\}, \{176, 143\}, \{77, 154\}, \{99, 198\}, \{187, 165\}, \{121, 33\}, \{66, 132\}, \{55, 110\}$.

$\underline{G}_{11\cdot 19}$: $\{1, 2\}, \{4, 8\}, \{16, 32\}, \{64, 128\}, \{47, 94\}, \{188, 167\}, \{125, 41\}, \{82, 164\}, \{119, 29\}, \{58, 116\}, \{23, 46\}, \{92, 184\}, \{159, 109\}, \{9, 18\}, \{36, 72\}, \{144, 79\}, \{158, 107\}, \{5, 10\}, \{20, 40\}, \{80, 160\}, \{111, 13\}, \{26, 52\}, \{104, 208\}, \{207, 205\}, \{201, 193\}, \{177, 145\}, \{81, 162\}, \{115, 21\}, \{42, 84\}, \{168, 127\}, \{45, 90\}, \{180, 151\}, \{93, 186\}, \{163, 117\}, \{25, 50\}, \{100, 200\}, \{191, 173\}, \{137, 65\}, \{130, 51\}, \{102, 204\}, \{199, 189\}, \{169, 129\}, \{49, 98\}, \{196, 183\}, \{157, 105\}, \{3, 6\}$
$\cup \{12, 24\}, \{48, 96\}, \{192, 175\}, \{141, 73\}, \{146, 83\}, \{166, 123\}, \{37, 74\}, \{148, 87\}, \{174, 139\}, \{69, 138\}, \{67, 134\}, \{59, 118\}, \{27, 54\}, \{108, 7\}, \{14, 28\}, \{56, 112\}, \{15, 30\}, \{60, 120\}, \{31, 62\}, \{124, 39\}, \{78, 156\}, \{103, 206\}, \{203, 197\}, \{185, 161\}, \{113, 17\}, \{34, 68\}, \{136, 63\}, \{126, 43\}, \{86, 172\}, \{135, 61\}, \{122, 35\}, \{70, 140\}, \{71, 142\}, \{75, 150\}, \{91, 182\}, \{155, 101\}, \{202, 195\}, \{181, 153\}, \{97, 194\}, \{179, 149\}, \{89, 178\}, \{147, 85\}, \{170, 131\}, \{53, 106\}$.

This strong Skolem starter of $\mathbb{Z}_{11\cdot 19}$ is the same from the Example 4.12 given in [24]. Now, by Corollary 3.4, we have a different strong Skolem starter of $\mathbb{Z}_{11\cdot 19}$, using the same parameters:

$19\underline{G}_{11}$: $\{19, 114\}, \{76, 38\}, \{95, 152\}, \{171, 190\}, \{57, 133\}$.

$11\underline{G}_{19}$: $\{11, 110\}, \{44, 22\}, \{176, 88\}, \{77, 143\}, \{99, 154\}, \{187, 198\}, \{121, 165\}, \{66, 33\}, \{55, 132\}$.

$\underline{G}_{11\cdot 19}$: $\{1, 105\}, \{4, 2\}, \{16, 8\}, \{64, 32\}, \{47, 128\}, \{188, 94\}, \{125, 167\}, \{82, 41\}, \{119, 164\}, \{58, 29\}, \{23, 116\}, \{92, 46\}, \{159, 84\}, \{9, 109\}, \{36, 18\}, \{144, 72\}, \{158, 79\}, \{5, 107\}, \{20, 10\}, \{80, 40\}, \{111, 160\}, \{26, 13\}, \{104, 52\}, \{207, 208\}, \{201, 205\}, \{177, 193\}, \{81, 145\}, \{115, 162\}, \{42, 21\}, \{168, 84\}, \{45, 127\}, \{180, 90\}, \{93, 151\}, \{163, 186\}, \{25, 117\}, \{100, 50\}, \{191, 200\}, \{137, 173\}, \{130, 65\}, \{102, 51\}, \{199, 204\}, \{169, 189\}, \{49, 129\}, \{196, 98\}, \{157, 183\}, \{3, 106\}$
$\cup \{12, 6\}, \{48, 24\}, \{192, 96\}, \{141, 175\}, \{146, 73\}, \{166, 83\}, \{37, 123\}, \{148, 74\}, \{174, 87\}, \{69, 139\}, \{67, 138\}, \{59, 134\}, \{27, 118\}, \{108, 54\}, \{14, 7\}, \{56, 28\}, \{15, 112\}, \{60, 30\}, \{31, 120\}, \{124, 62\}, \{78, 39\}, \{103, 156\}, \{203, 206\}, \{185, 197\}, \{113, 161\}, \{34, 17\}, \{136, 68\}, \{126, 63\}, \{86, 43\}, \{135, 172\}, \{122, 61\}, \{70, 35\}, \{71, 140\}, \{75, 142\}, \{91, 150\}, \{155, 182\}, \{202, 101\}, \{181, 195\}, \{97, 153\}, \{179, 194\}, \{89, 149\}, \{147, 178\}, \{170, 85\}, \{53, 131\}$.

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ORCID

Adrián Vázquez-Ávila http://orcid.org/0000-0003-1602-2872

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