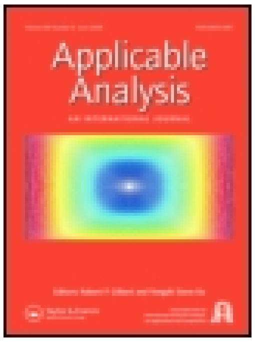

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Adaptive BEM for elliptic PDE systems, part I: abstract framework, for weakly-singular integral equations

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ABSTRACT
In the present work, we consider weakly-singular integral equations arising from linear second-order elliptic PDE systems with constant coefficients, including, e.g. linear elasticity. We introduce a general framework for optimal convergence of adaptive Galerkin BEM. We identify certain abstract conditions for the underlying meshes, the corresponding mesh-refinement strategy, and the ansatz spaces that guarantee that the weighted-residual error estimator is reliable and converges at optimal algebraic rate if used within an adaptive algorithm. These conditions are satisfied, e.g. for discontinuous piecewise polynomials on simplicial meshes as well as certain ansatz spaces used for isogeometric analysis. Technical contributions include the localization of (non-local) fractional Sobolev norms and local inverse estimates for the (non-local) boundary integral operators associated to the PDE system.

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1. Introduction

1.1. State of the art

For the Laplace model problem, adaptive boundary element methods (BEM) using (dis)continuous piecewise polynomials on triangulations have been intensively studied in the literature. In particular, optimal convergence of adaptive mesh-refining algorithms has been proved for polyhedral boundaries [1–3] as well as smooth boundaries [4]. The work [5] allows to transfer these results to piecewise smooth boundaries; see also the discussion in the review article [6]. In [7], these results have been generalized to the Helmholtz problem. In recent years, we have also shown optimal convergence of adaptive isogeometric BEM (IGABEM) using one-dimensional splines for the 2D Laplace problem [8,9]. However, the important case of 3D IGABEM remained open. Moreover, a generalization to other PDE operators is highly non-trivial (see (6) below), but especially linear elasticity is of great interest in the context of isogeometric analysis.

In [10], we have considered isogeometric finite element methods (IGAFEM). We have derived an abstract framework which guarantees that, first, the classical residual FEM error estimator is reliable, and second, the related adaptive algorithm yields optimal convergence; see [10, Section 2 and 4]. We then showed that, besides standard FEM with piecewise polynomials, this abstract framework covers IGAFEM with hierarchical splines (see [10, Section 3 and 5]) as well as IGAFEM with analysis suitable T-splines (see the recent work [11]).

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The aim of the present work is to develop such an abstract framework also for BEM, which is mathematically much more demanding than FEM. In ongoing research [12], we aim to show that this framework covers, besides standard discretizations with piecewise polynomials, also IGABEM with hierarchical splines resp. T-splines.

To this end, the present work focusses on weakly-singular integral equations. For a given Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ with compact boundary $\Gamma := \partial\Omega$ and right-hand side $f : \Gamma \to \mathbb{C}$, we consider

$$(\mathfrak{V}\phi)(x) := \int_\Gamma G(x-y)\phi(y)\,\mathrm{d}y = f(x) \quad \text{for almost all } x \in \Gamma. \tag{1}$$

Here, the fundamental solution G stems from an elliptic PDE operator

$$\mathfrak{P}u := -\sum_{i=1}^{d}\sum_{i'=1}^{d} \partial_i(A_{ii'}\partial_{i'}u) + \sum_{i=1}^{d} b_i\partial_i u + cu, \tag{2}$$

where the coefficients $A_{ii'} = \overline{A_{i'i}}^\top, b_i, c \in \mathbb{C}^{D\times D}$ are constant for some fixed dimension $D \geq 1$.

1.2. Outline & contributions

In Section 2, we fix some general notation, recall Sobolev spaces on the boundary, and precisely state the considered problem. Section 3 can be paraphrased as follows. We formulate an adaptive algorithm (Algorithm 3.3) of the form

$$\boxed{\text{SOLVE}} \longrightarrow \boxed{\text{ESTIMATE}} \longrightarrow \boxed{\text{MARK}} \longrightarrow \boxed{\text{REFINE}} \tag{3}$$

driven by some weighted-residual a posteriori error estimator (see (4) below) in the frame of conforming Galerkin BEM. The algorithm particularly generates meshes $\mathcal{T}_\ell$, BEM solutions Φ_ℓ in associated nested ansatz spaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D$, and error estimators η_ℓ for all $\ell \in \mathbb{N}_0$. We formulate five Assumptions (M1)–(M5) on the underlying meshes (Section 3.1), five Assumptions (R1)–(R5) on the mesh refinement (Section 3.2), and six Assumptions (S1)–(S6) on the BEM spaces (Section 3.3). First, these assumptions are sufficient to guarantee that the a posteriori error estimator η_ℓ associated with the BEM solution Φ_ℓ is reliable, i.e. there exists a constant $C_{\rm rel} > 0$ such that

$$C_{\rm rel}^{-1}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \leq \eta_\ell := \|h_\ell^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_\ell)\|_{L^2(\Gamma)} \quad \text{for all } \ell \in \mathbb{N}_0, \tag{4}$$

where $h_\ell \in L^\infty(\Gamma)$ denotes the local mesh-size function and ∇_Γ is the surface gradient. Second, Theorem 3.3 states that Algorithm 3.3 leads to linear convergence at optimal algebraic rate with respect to the number of mesh elements. In Theorem 3.7, we briefly note that the introduced conditions have already been implicitly proved for standard discretizations with piecewise polynomials on conforming triangulations. Moreover, we mention expected applications to adaptive IGABEM on quadrilateral meshes in Remark 3.8.

Section 4 is devoted to the proof of Theorem 3.3. To prove reliability (4), we use a localization argument (Proposition 4.2), which generalizes earlier works [13,14] for standard discretizations. More precisely, we prove that

$$\|v\|_{H^{1/2}(\Gamma)}^2 \leq C_{\rm split} \sum_{T\in\mathcal{T}_\ell} \sum_{T'\in\Pi_\ell(T)} |v|_{H^{1/2}(T\cup T')}^2 \tag{5}$$

for all $v \in H^{1/2}(\Gamma)^D$ that are L^2-orthogonal onto the ansatz space $\mathcal{X}_\ell$ corresponding to some mesh $\mathcal{T}_\ell$, where $C_{\rm split} > 0$ is independent of v and $\Pi_\ell(T)$ denotes the patch of $T \in \mathcal{T}_\ell$. Finally, the proof of reliability (4) requires the derivation of a local Poincaré-type inequality (Proposition 4.9). In Remark 4.10, we note that one obtains at least plain convergence $\lim_{\ell\to\infty} \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$ if Algorithm 3.3 is steered by the so-called *Faermann estimator*

$$\widetilde{C}_{\rm rel}^{-1}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \le F_\ell := \Bigg(\sum_{T\in\mathcal{T}_\ell}\sum_{T'\in\Pi_\ell(T)} |f - \mathfrak{V}\Phi_\ell|^2_{H^{1/2}(T\cup T')}\Bigg)^{1/2} \le \widetilde{C}_{\rm eff}^{-1}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)},$$

which is reliable and efficient. To prove linear convergence at optimal rate for the weighted-residual estimator (4), we show that the assumptions of Section 3 imply the *axioms of adaptivity* [6]. The latter are properties for abstract mesh-refinements and abstract error estimators, which automatically yield the desired convergence result. In contrast to [1,3] which (implicitly) verify the axioms of adaptivity only for the Laplace problem, our analysis allows for general PDE operators (2). The crucial step is the generalization (Proposition 4.13) of the non-trivial *local* inverse inequality for the *non-local* boundary integral operator $\mathfrak{V}$: With the help of a Caccioppoli-type inequality (Lemma 4.12), we prove that there exists a constant $C_{\rm inv} > 0$ such that

$$\|h_\ell^{1/2}\nabla_\Gamma\mathfrak{V}\psi\|_{L^2(\Gamma)} \le C_{\rm inv}\Big(\|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}\Big) \quad \text{for all } \psi \in L^2(\Gamma)^D; \tag{6}$$

see [5] for standard BEM discretizations of the Laplacian. Similar estimates hold also for the other boundary integral operators related to (2), namely the double-layer integral operator $\mathfrak{K}$, its adjoint $\mathfrak{K}'$, and the hypersingular integral operator $\mathfrak{W}$. These are stated and proved in Appendix B; again we refer to [5] for standard BEM discretizations of the Laplacian. In each case, ellipticity of the PDE operator is not required for the inverse inequalities.

While the present work focusses on the numerical analysis aspects only, we refer to the literature (see, e.g. [7,15–18]) for numerical experiments for the Laplace problem, the Helmholtz problem, and linear elasticity.

2. Preliminaries

In this section, we fix some general notation, recall Sobolev spaces on the boundary, and precisely state the considered problem. Throughout the work, let $\Omega \subset \mathbb{R}^d$ for $d \ge 2$ be a bounded Lipschitz domain as in [19, Definition 3.28] and $\Gamma := \partial\Omega$ its boundary.

2.1. General notation

Throughout and without any ambiguity, $|\cdot|$ denotes the absolute value of scalars, the Euclidean norm of vectors in $\mathbb{R}^n$, as well as the Hausdorff measure of any d-dimensional set in $\mathbb{R}^n$. Let $B_\varepsilon(x) := \{y \in \mathbb{R}^n : |x-y| < \varepsilon\}$ denote the open ball around x with radius $\varepsilon > 0$. For $\emptyset \ne \omega_1, \omega_2 \subseteq \mathbb{R}^n$, let $B_\varepsilon(\omega_1) := \bigcup_{x\in\omega_1} B_\varepsilon(x)$. Moreover, let $\operatorname{diam}(\omega_1) := \sup\{|x-y| : x, y \in \omega_1\}$, and $\operatorname{dist}(\omega_1, \omega_2) := \inf\{|x-y| : x \in \omega_1, y \in \omega_2\}$. We write $A \lesssim B$ to abbreviate $A \le CB$ with some generic constant $C > 0$, which is clear from the context. Moreover, $A \simeq B$ abbreviates $A \lesssim B \lesssim A$. Throughout, mesh-related quantities have the same index, e.g. $\mathcal{X}_\bullet$ is the ansatz space corresponding to the mesh $\mathcal{T}_\bullet$. The analogous notation is used for meshes $\mathcal{T}_\circ$, $\mathcal{T}_\star$, $\mathcal{T}_\ell$, etc.

2.2. Sobolev spaces

For $\sigma \in [0, 1]$, we define the Hilbert spaces $H^{\pm\sigma}(\Gamma)$ as in [19, page 99] by use of Bessel potentials on $\mathbb{R}^{d-1}$ and liftings via bi-Lipschitz mappings[1] that describe Γ. For $\sigma = 0$, it holds that $H^0(\Gamma) = L^2(\Gamma)$ with equivalent norms. We thus may define $\|\cdot\|_{H^0(\Gamma)} := \|\cdot\|_{L^2(\Gamma)}$.

For $\sigma \in (0,1]$, any measurable subset $\omega \subseteq \Gamma$, and all $v \in H^\sigma(\Gamma)$, we define the associated Sobolev–Slobodeckij norm

$$\|v\|_{H^\sigma(\omega)}^2 := \|v\|_{L^2(\omega)}^2 + |v|_{H^\sigma(\omega)}^2 \text{ with } |v|_{H^\sigma(\omega)}^2 := \begin{cases} \displaystyle\int_\omega \int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}}\,\mathrm{d}x\mathrm{d}y & \text{if } \sigma \in (0,1), \\ \|\nabla_\Gamma v\|_{L^2(\omega)}^2 & \text{if } \sigma = 1. \end{cases} \tag{7}$$

It is well known that $\|\cdot\|_{H^\sigma(\Gamma)}$ provides an equivalent norm on $H^\sigma(\Gamma)$; see, e.g. [20, Lemma 2.19] and [19, Theorem 3.30 and page 99] for $\sigma \in (0,1)$ and [18, Theorem 2.28] for $\sigma = 1$. Here, $\nabla_\Gamma(\cdot)$ denotes the usual (weak) surface gradient which can be defined for almost all $x \in \Gamma$ as follows. Since Γ is a Lipschitz boundary, there exist an open cover $(O_j)_{j=1}^J$ in $\mathbb{R}^d$ of Γ such that each $\omega_j := O_j \cap \Gamma$ can be parametrized by a bi-Lipschitz mapping $\gamma_{\omega_j} : \widehat{\omega}_j \to \omega_j$, where $\widehat{\omega}_j \subset \mathbb{R}^{d-1}$ is an open set. By Rademacher's theorem, γ_{ω_j} is almost everywhere differentiable. The corresponding Gram determinant $\det(D\gamma_{\omega_j}^\top D\gamma_{\omega_j})$ is almost everywhere positive. Moreover, by definition of the space $H^1(\Gamma)$, $v \in H^1(\Gamma)$ implies that $v \circ \gamma_{\omega_j} \in H^1(\widehat{\omega}_j)$. With the weak derivative $\nabla(v \circ \gamma_{\omega_j}) \in L^2(\widehat{\omega}_j)^d$, we can hence define

$$(\nabla_\Gamma v)|_{\omega_j} := \Big(D\gamma_{\omega_j} (D\gamma_{\omega_j}^\top D\gamma_{\omega_j})^{-1} \nabla(v \circ \gamma_{\omega_j}) \Big) \circ \gamma_{\omega_j}^{-1} \quad \text{for all } v \in H^1(\Gamma). \tag{8}$$

This definition does not depend on the particular choice of the open sets $(O_j)_{j=1}^J$ and the corresponding parametrizations $(\gamma_{\omega_j})_{j=1}^J$; see, e.g. [18, Theorem 2.28]. With (8), we immediately obtain the chain rule

$$\nabla(v \circ \gamma_{\omega_j}) = D\gamma_{\omega_j}^\top ((\nabla_\Gamma v) \circ \gamma_{\omega_j}(\cdot)) \quad \text{for all } v \in H^1(\Gamma). \tag{9}$$

For $\sigma \in (0,1]$, $H^{-\sigma}(\Gamma)$ is a realization of the dual space of $H^\sigma(\Gamma)$; see [19, Theorem 3.30 and page 99]. With the duality bracket $\langle \cdot\,;\,\cdot \rangle$, we define an equivalent norm

$$\|\psi\|_{H^{-\sigma}(\Gamma)} := \sup\big\{ \langle v\,;\,\psi \rangle \,:\, v \in H^\sigma(\Gamma) \wedge \|v\|_{H^\sigma(\Gamma)} = 1 \big\} \quad \text{for all } \psi \in H^{-\sigma}(\Gamma). \tag{10}$$

Moreover, we abbreviate

$$(v\,;\,\psi) := \langle \overline{v}\,;\,\psi \rangle \quad \text{for all } v \in H^\sigma(\Gamma), \quad \psi \in H^{-\sigma}(\Gamma). \tag{11}$$

[19, page 76] states that the inclusion $H^{\sigma_1}(\Gamma) \subseteq H^{\sigma_2}(\Gamma)$ for $-1 \le \sigma_1 \le \sigma_2 \le 1$ is continuous and dense. In particular, $H^\sigma(\Gamma) \subset L^2(\Gamma) \subset H^{-\sigma}(\Gamma)$ form a Gelfand triple in the sense of [21, Section 2.1.2.4] for all $\sigma \in (0,1]$, where $\psi \in L^2(\Gamma)$ is interpreted as function in $H^{-\sigma}(\Gamma)$ via

$$\langle v\,;\,\psi \rangle := (\overline{v}\,;\,\psi)_{L^2(\Gamma)} = \int_\Gamma v\,\psi\,\mathrm{d}x \quad \text{for all } v \in H^\sigma(\Gamma), \quad \psi \in L^2(\Gamma). \tag{12}$$

Here, $(\cdot\,;\,\cdot)_{L^2(\Gamma)}$ is the usual complex scalar product on $L^2(\Gamma)$.

So far, we have only dealt with scalar-valued functions. For $D \ge 1$, $\sigma \in [0,1]$, $v = (v_1, \dots, v_D) \in H^\sigma(\Gamma)^D$, we define $\|v\|_{H^{\pm\sigma}(\Gamma)}^2 := \sum_{j=1}^D \|v_j\|_{H^{\pm\sigma}(\Gamma)}^2$. If $\sigma > 0$, and $\omega \subseteq \Gamma$ is an arbitrary measurable set, we define $\|v\|_{H^\sigma(\omega)}$ and $|v|_{H^\sigma(\omega)}$ analogously. With the definition

$$\nabla_\Gamma v := \begin{pmatrix} \nabla_\Gamma v_1 \\ \vdots \\ \nabla_\Gamma v_D \end{pmatrix} \in L^2(\Gamma)^{D^2} \quad \text{for all } v \in H^1(\Gamma)^D, \tag{13}$$

it holds that $|v|_{H^1(\omega)} = \|\nabla_\Gamma v\|_{L^2(\omega)}$. Note that $H^{-\sigma}(\Gamma)^D$ with $\sigma \in (0,1]$ can be identified with the dual space of $H^\sigma(\Gamma)^D$, where we set

$$\langle v\,;\,\psi\rangle := \sum_{j=1}^{D} \langle v_j\,;\,\psi_j\rangle \quad \text{for all } v \in H^\sigma(\Gamma)^D, \quad \psi \in H^{-\sigma}(\Gamma)^D. \tag{14}$$

As before, we abbreviate

$$(v\,;\,\psi) := \langle \overline{v}\,;\,\psi\rangle \quad \text{for all } \ v \in H^\sigma(\Gamma)^D, \quad \psi \in H^{-\sigma}(\Gamma)^D \tag{15}$$

and set

$$\langle v\,;\,\psi\rangle := (\overline{v}\,;\,\psi)_{L^2(\Gamma)} = \sum_{j=1}^{D} \int_\Gamma v_j \psi_j \,\mathrm{d}x \quad \text{for all } v \in H^\sigma(\Gamma)^D, \quad \psi \in L^2(\Gamma)^D. \tag{16}$$

The spaces $H^\sigma(\Gamma)$ can also be defined as trace spaces or via interpolation, where the resulting norms are always equivalent with constants depending only on the dimension d and the boundary Γ. More details and proofs are found, e.g. in the monographs [19–21].

2.3. Continuous problem

We consider a general second-order linear system of PDEs

$$\mathfrak{P}u := -\sum_{i=1}^{d}\sum_{i'=1}^{d} \partial_i(A_{ii'}\partial_{i'}u) + \sum_{i=1}^{d} b_i\partial_i u + cu, \tag{17}$$

where the coefficients $A_{ii'}, b_i, c \in \mathbb{C}^{D\times D}$ are constant for some fixed dimension $D \geq 1$. We suppose that $A_{ii'}^\top = \overline{A_{i'i}}$. Moreover, we assume that $\mathfrak{P}$ is elliptic on $H_0^1(\Omega)^D$ in the sense of the Lax–Milgram lemma, i.e. the sesquilinear form

$$(u\,;\,v)_{\mathfrak{P}} := \int_\Omega \sum_{i=1}^{d}\sum_{i'=1}^{d} (A_{ii'}\partial_{i'}u)\cdot\partial_i v + \sum_{i=1}^{d}(b_i\partial_i u)\cdot v + (cu)\cdot v \,\mathrm{d}x \tag{18}$$

satisfies that

$$(u\,;\,u)_{\mathfrak{P}} \geq C_{\mathrm{ell}}\|u\|_{H^1(\Omega)}^2 \quad \text{for all } u \in H_0^1(\Omega)^D. \tag{19}$$

of the matrices $A_{ii'}$ in the sense of [19, page 119]. Here, the standard complex scalar product on $\mathbb{C}^D$ is denoted by $w\cdot z = \sum_{j=1}^{D} \overline{w}_j z_j$.

Let $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}^{D\times D}$ be a corresponding (matrix-valued) fundamental solution in the sense of [19, page 198], i.e. a distributional solution of $\mathfrak{P}G = \delta$, where δ denotes the Dirac delta distribution.

For $\psi \in L^\infty(\Gamma)^D$, we define the *single-layer operator* as

$$(\mathfrak{V}\psi)(x) := \int_\Gamma G(x-y)\psi(y)\,dy \quad \text{for all } x \in \Gamma. \tag{20}$$

According to [19, page 209 and 219–220] and [22, Corollary 3.38], this operator can be extended for arbitrary $\sigma \in (-1/2, 1/2]$ to a bounded linear operator

$$\mathfrak{V} : H^{-1/2+\sigma}(\Gamma)^D \to H^{1/2+\sigma}(\Gamma)^D. \tag{21}$$

[19, Theorem 7.6] states that $\mathfrak{V}$ is always coercive, i.e. elliptic up to some compact perturbation. We assume that it is elliptic even without perturbation, i.e.

$$\mathrm{Re}\,(\mathfrak{V}\psi\,;\,\psi) \geq C_{\rm ell}\|\psi\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } \psi \in H^{-1/2}(\Gamma)^D. \tag{22}$$

This is particularly satisfied for the Laplace problem or for the Lamé problem, where the case $d = 2$ requires an additional scaling of the geometry Ω; see, e.g. [20, Chapter 6]. Moreover, the sesquilinear form $(\mathfrak{V}\,\cdot\,;\,\cdot)$ is continuous due to (21), i.e. it holds with $C_{\rm cont} := \|\mathfrak{V}\|_{H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D}$ that

$$|(\mathfrak{V}\psi\,;\,\xi)| \leq C_{\rm cont}\|\psi\|_{H^{-1/2}(\Gamma)}\|\xi\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \psi, \quad \xi \in H^{-1/2}(\Gamma)^D. \tag{23}$$

Given a right-hand side $f \in H^1(\Gamma)^D$, we consider the boundary integral equation

$$\mathfrak{V}\phi = f. \tag{24}$$

Such equations arise from (and are even equivalent to) the solution of Dirichlet problems of the form $\mathfrak{P}u = 0$ in Ω with $u = g$ on Γ for some $g \in H^{1/2}(\Gamma)^D$; see, e.g. [19, page 226–229] for more details. The Lax–Milgram lemma provides existence and uniqueness of the solution $\phi \in H^{-1/2}(\Gamma)^D$ of the equivalent variational formulation of (24)

$$(\mathfrak{V}\phi\,;\,\psi) = (f\,;\,\psi) \quad \text{for all } \psi \in H^{-1/2}(\Gamma)^D. \tag{25}$$

In particular, we see that $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ is an isomorphism. In the Galerkin boundary element method, the test space $H^{-1/2}(\Gamma)^D$ is replaced by some discrete subspace $\mathcal{X}_\bullet \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D$. Again, the Lax–Milgram lemma guarantees existence and uniqueness of the solution $\Phi_\bullet \in \mathcal{X}_\bullet$ of the discrete variational formulation

$$(\mathfrak{V}\Phi_\bullet\,;\,\Psi_\bullet) = (f\,;\,\Psi_\bullet) \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet. \tag{26}$$

Moreover, $\Phi_\bullet$ can in fact be computed by solving a linear system of equations. Note that (21) implies that $\mathfrak{V}\Psi_\bullet \in H^1(\Gamma)^D$ for arbitrary $\Psi_\bullet \in \mathcal{X}_\bullet$. The additional regularity $f \in H^1(\Gamma)^D$ instead of $f \in H^{1/2}(\Gamma)^D$ is only needed to define the residual error estimator (37) below. For a more detailed introduction to boundary integral equations, the reader is referred to the monographs [19–21].

3. Axioms of adaptivity (revisited)

The aim of this section is to formulate an adaptive algorithm (Algorithm 3.3) for conforming BEM discretizations of our model problem (24), where adaptivity is driven by the *residual a posteriori error estimator* (see (37) below). We identify conditions for the underlying meshes, the mesh-refinement, as well as the boundary element spaces which ensure that the residual error estimator is reliable and fits into the general framework of [6] and which hence guarantee optimal convergence behavior of the adaptive algorithm. We mention that we have already identified similar (but not identical) conditions for the finite element method in [10, Section 3]. The main result of this work is Theorem 3.3 which is proved in Section 4.

3.1. Meshes

Throughout, $\mathcal{T}_\bullet$ is a *mesh* of the boundary $\Gamma = \partial\Omega$ of the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ in the following sense:

(i) $\mathcal{T}_\bullet$ is a finite set of compact Lipschitz domains on Γ, i.e. each element T has the form $T = \gamma_T(\widehat{T})$, where $\widehat{T}$ is a compact[2] Lipschitz domain in $\mathbb{R}^{d-1}$ and $\gamma_T : \widehat{T} \to T$ is bi-Lipschitz.
(ii) $\mathcal{T}_\bullet$ covers Γ, i.e. $\Gamma = \bigcup_{T\in\mathcal{T}_\bullet} T$.
(iii) For all $T, T' \in \mathcal{T}_\bullet$ with $T \neq T'$, the intersection $T \cap T'$ has $(d-1)$-dimensional Hausdorff measure zero.

We suppose that there is a countably infinite set $\mathbb{T}$ of *admissible* meshes. In order to ease notation, we introduce for $\mathcal{T}_\bullet \in \mathbb{T}$ the corresponding *mesh-width function*

$$h_\bullet \in L^\infty(\Gamma) \quad \text{with } h_\bullet|_T = h_T := |T|^{1/(d-1)} \quad \text{for all } T \in \mathcal{T}_\bullet. \tag{27}$$

For $\omega \subseteq \Gamma$, we define the patches of order $q \in \mathbb{N}_0$ inductively by

$$\pi_\bullet^0(\omega) := \omega, \quad \pi_\bullet^q(\omega) := \bigcup \big\{T \in \mathcal{T}_\bullet \,:\, T \cap \pi_\bullet^{q-1}(\omega) \neq \emptyset\big\}. \tag{28}$$

The corresponding set of elements is

$$\Pi_\bullet^q(\omega) := \big\{T \in \mathcal{T}_\bullet \,:\, T \subseteq \pi_\bullet^q(\omega)\big\}, \quad \text{i.e., } \pi_\bullet^q(\omega) = \bigcup \Pi_\bullet^q(\omega). \tag{29}$$

If $\omega = \{z\}$ for some $z \in \Gamma$, we simply write $\pi_\bullet^q(z) := \pi_\bullet^q(\{z\})$ and $\Pi_\bullet^q(z) := \Pi_\bullet^q(\{z\})$. For $\mathcal{S} \subseteq \mathcal{T}_\bullet$, we define $\pi_\bullet^q(\mathcal{S}) := \pi_\bullet^q(\bigcup \mathcal{S})$ and $\Pi_\bullet^q(\mathcal{S}) := \Pi_\bullet^q(\bigcup \mathcal{S})$. To abbreviate notation, the index $q = 1$ is omitted, e.g. $\pi_\bullet(\omega) := \pi_\bullet^1(\omega)$ and $\Pi_\bullet(\omega) := \Pi_\bullet^1(\omega)$.

We assume the existence of constants $C_{\rm patch}, C_{\rm locuni}, C_{\rm shape}, C_{\rm cent}, C_{\rm semi} > 0$ such that the following assumptions are satisfied for all $\mathcal{T}_\bullet \in \mathbb{T}$:

(M1) **Bounded element patch:** The number of elements in a patch is uniformly bounded, i.e.

$$\#\Pi_\bullet(T) \leq C_{\rm patch} \quad \text{for all } T \in \mathcal{T}_\bullet.$$

(M2) **Local quasi-uniformity:** Neighboring elements have comparable diameter, i.e.

$$\mathrm{diam}(T)/\mathrm{diam}(T') \leq C_{\rm locuni} \quad \text{for all } T \in \mathcal{T}_\bullet \text{ and all } T' \in \Pi_\bullet(T).$$

(M3) **Shape-regularity:** It holds that

$$C_{\rm shape}^{-1} \leq \mathrm{diam}(T)/h_T \leq C_{\rm shape} \quad \text{for all } T \in \mathcal{T}_\bullet.$$

(M4) **Elements lie in the center of their patches:** It holds[3] that

$$\mathrm{diam}(T) \leq C_{\rm cent}\, \mathrm{dist}(T, \Gamma \setminus \pi_\bullet(T)) \quad \text{for all } T \in \mathcal{T}_\bullet.$$

(M5) **Local seminorm estimate:** For all $v \in H^1(\Gamma)$, it holds that

$$|v|_{H^{1/2}(\pi_\bullet(z))} \leq C_{\rm semi}\, \mathrm{diam}(\pi_\bullet(z))^{1/2} |v|_{H^1(\pi_\bullet(z))} \quad \text{for all } z \in \Gamma.$$

The following proposition shows that (M5) is actually always satisfied. However, in general the multiplicative constant depends on the shape of the point patches. The proof is inspired by [23, Proposition 2.2], where an analogous assertion for norms instead of seminorms is found. For $\sigma = 1/2$ and

$d = 2$, we have already shown the assertion in the recent own work [16, Lemma 4.5]. For polyhedral domains Ω with triangular meshes, it is proved in [24, Proposition 3.3] via interpolation techniques. A detailed proof for our setting is found in [17, Proposition 5.2.2], where we essentially follow the proof of [16, Lemma 4.5].

Proposition 3.1: *Let $\widehat{\omega} \subset \mathbb{R}^{d-1}$ be a bounded and connected Lipschitz domain and $\gamma_\omega : \widehat{\omega} \to \omega \subseteq \Gamma$ be bi-Lipschitz, i.e. there exists a constant $C_{\rm lipref} > 0$ such that*

$$C_{\rm lipref}^{-1}|s-t| \le \frac{|\gamma_\omega(s)-\gamma_\omega(t)|}{\mathrm{diam}(\omega)} \le C_{\rm lipref}|s-t| \quad \text{for all } s,\ t \in \widehat{\omega}. \tag{30}$$

Then, for arbitrary $\sigma \in (0,1)$ there exists a constant $C_{\rm semi}(\widehat{\omega}) > 0$ such that

$$|v|_{H^\sigma(\omega)} \le C_{\rm semi}(\widehat{\omega})\,\mathrm{diam}(\omega)^{1-\sigma}|v|_{H^1(\omega)} \quad \text{for all } v \in H^1(\Gamma). \tag{31}$$

The constant $C_{\rm semi}(\widehat{\omega}) > 0$ depends only on the dimension d, σ, the set $\widehat{\omega}$, and $C_{\rm lipref}$.

3.2. Mesh refinement

For $\mathcal{T}_\bullet \in \mathbb{T}$ and an arbitrary set of marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$, we associate a corresponding *refinement* $\mathcal{T}_\circ := \mathtt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) \in \mathbb{T}$ with $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$, i.e. at least the marked elements are refined. Moreover, we suppose for the cardinalities that $\#\mathcal{T}_\bullet < \#\mathcal{T}_\circ$ if $\mathcal{M}_\bullet \neq \emptyset$ and $\mathcal{T}_\circ = \mathcal{T}_\bullet$ else. Let $\mathtt{refine}(\mathcal{T}_\bullet) \subseteq \mathbb{T}$ be the set of all $\mathcal{T}_\circ$ such that there exist meshes $\mathcal{T}_{(0)}, \dots, \mathcal{T}_{(J)}$ and marked elements $\mathcal{M}_{(0)}, \dots, \mathcal{M}_{(J-1)}$ with $\mathcal{T}_\circ = \mathcal{T}_{(J)} = \mathtt{refine}(\mathcal{T}_{(J-1)}, \mathcal{M}_{(J-1)}), \dots, \mathcal{T}_{(1)} = \mathtt{refine}(\mathcal{T}_{(0)}, \mathcal{M}_{(0)})$ and $\mathcal{T}_{(0)} = \mathcal{T}_\bullet$. We assume that there exists a fixed initial mesh $\mathcal{T}_0 \in \mathbb{T}$ with $\mathbb{T} = \mathtt{refine}(\mathcal{T}_0)$.

We suppose that there exist $C_{\rm son} \ge 2$ and $0 < \rho_{\rm son} < 1$ such that all meshes $\mathcal{T}_\bullet \in \mathbb{T}$ satisfy for arbitrary marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$ with corresponding refinement $\mathcal{T}_\circ := \mathtt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet)$, the following elementary properties (R1)–(R3):

(R1) **Son estimate:** One step of refinement leads to a bounded increase of elements, i.e.

$$\#\mathcal{T}_\circ \le C_{\rm son}\,\#\mathcal{T}_\bullet,$$

(R2) **Father is union of sons:** Each element is the union of its successors, i.e.

$$T = \bigcup\left\{T' \in \mathcal{T}_\circ \,:\, T' \subseteq T\right\} \quad \text{for all } T \in \mathcal{T}_\bullet.$$

(R3) **Reduction of sons:** Successors are uniformly smaller than their father, i.e.

$$|T'| \le \rho_{\rm son}\,|T| \quad \text{for all } T \in \mathcal{T}_\bullet \text{ and all } T' \in \mathcal{T}_\circ \text{ with } T' \subsetneqq T.$$

By induction and the definition of $\mathtt{refine}(\mathcal{T}_\bullet)$, one easily sees that (R2)–(R3) remain valid if $\mathcal{T}_\circ$ is an arbitrary mesh in $\mathtt{refine}(\mathcal{T}_\bullet)$. In particular, (R2)–(R3) imply that each refined element $T \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ is split into at least two sons, wherefore

$$\#(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \le \#\mathcal{T}_\circ - \#\mathcal{T}_\bullet \quad \text{for all } \mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet). \tag{32}$$

Besides (R1)–(R3), we suppose the following less trivial requirements (R2)–(R3) with generic constants $C_{\rm clos}, C_{\rm over} > 0$:

(R4) **Closure estimate:** Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be a sequence in $\mathbb{T}$ such that $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ with some $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. Then, it holds that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\mathrm{clos}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \quad \text{for all } \ell \in \mathbb{N}_0.$$

(R5) **Overlay property:** For all $\mathcal{T}_\bullet, \mathcal{T}_\star \in \mathbb{T}$, there exists a common refinement $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet) \cap \texttt{refine}(\mathcal{T}_\star)$ which satisfies the overlay estimate

$$\#\mathcal{T}_\circ \le C_{\mathrm{over}}(\#\mathcal{T}_\star - \#\mathcal{T}_0) + \#\mathcal{T}_\bullet.$$

3.3. Boundary element space

With each $\mathcal{T}_\bullet \in \mathbb{T}$, we associate a finite dimensional space of vector valued functions

$$\mathcal{X}_\bullet \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D. \tag{33}$$

Let $\Phi_\bullet \in \mathcal{X}_\bullet$ be the corresponding Galerkin approximation of $\phi \in H^{-1/2}(\Gamma)^D$ from (24), i.e.

$$(\mathfrak{V}\Phi_\bullet\,;\,\Psi_\bullet) = (f\,;\,\Psi_\bullet) \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet. \tag{34}$$

We note the Galerkin orthogonality

$$(f - \mathfrak{V}\Phi_\bullet\,;\,\Psi_\bullet) = 0 \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet, \tag{35}$$

as well as the resulting Céa type quasi-optimality

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\text{Céa}} \min_{\Psi_\bullet\in\mathcal{X}_\bullet} \|\phi - \Psi_\bullet\|_{H^{-1/2}(\Gamma)} \quad \text{with } C_{\text{Céa}} := C_{\mathrm{cont}}/C_{\mathrm{ell}}. \tag{36}$$

We assume the existence of constants $C_{\mathrm{inv}} > 0$, $q_{\mathrm{loc}}, q_{\mathrm{proj}}, q_{\mathrm{supp}} \in \mathbb{N}_0$, and $0 < \rho_{\mathrm{unity}} < 1$ such that the following properties (S1)–(S4) hold for all $\mathcal{T}_\bullet \in \mathbb{T}$:

(S1) **Inverse inequality:** For all $\Psi_\bullet \in \mathcal{X}_\bullet$, it holds that

$$\|h_\bullet^{1/2}\Psi_\bullet\|_{L^2(\Gamma)} \le C_{\mathrm{inv}}\,\|\Psi_\bullet\|_{H^{-1/2}(\Gamma)}.$$

(S2) **Nestedness:** For all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ.$$

(S3) **Local domain of definition:** For all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\mathrm{loc}}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ$, and $\Psi_\circ \in \mathcal{X}_\circ$, it holds that

$$\Psi_\circ|_{\pi_\bullet^{q_{\mathrm{proj}}}(T)} \in \big\{\Psi_\bullet|_{\pi_\bullet^{q_{\mathrm{proj}}}(T)} \;:\; \Psi_\bullet \in \mathcal{X}_\bullet\big\}.$$

(S4) **Componentwise local approximation of unity:** For all $T \in \mathcal{T}_\bullet$ and all $j \in \{1,\dots,D\}$, there exists some $\Psi_{\bullet,T,j} \in \mathcal{X}_\bullet$ with

$$T \subseteq \mathrm{supp}(\Psi_{\bullet,T,j}) \subseteq \pi_\bullet^{q_{\mathrm{supp}}}(T),$$

such that only the jth component does not vanish, i.e.

$$(\Psi_{\bullet,T,j})_{j'} = 0 \quad \text{for } j' \neq j$$

and

$$\|1 - (\Psi_{\bullet,T,j})_j\|_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \le \rho_{\mathrm{unity}}|\mathrm{supp}(\Psi_{\bullet,T,j})_j|^{1/2}.$$

Remark 3.2: Clearly, (S4) is in particular satisfied if $\mathcal{X}_\bullet$ is a product space, i.e. $\mathcal{X}_\bullet = \prod_{j=1}^D (\mathcal{X}_\bullet)_j$, and each component $(\mathcal{X}_\bullet)_j \subset L^2(\Gamma)$ satisfies (S4).

Besides (S1)–(S4), we suppose that there exist constants $C_{\rm sz} > 0$ as well as $q_{\rm sz} \in \mathbb{N}_0$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{S} \subseteq \mathcal{T}_\bullet$, there exists a linear Scott–Zhang-type operator $J_{\bullet,\mathcal{S}} : L^2(\Gamma)^D \to \{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet \setminus \mathcal{S})} = 0\}$ with the following properties (S5)–(S6):

(S5) **Local projection property.** Let $q_{\rm loc}, q_{\rm proj} \in \mathbb{N}_0$ from (S3). For all $\psi \in L^2(\Gamma)^D$ and $T \in \mathcal{T}_\bullet$ with $\Pi_\bullet^{q_{\rm loc}}(T) \subseteq \mathcal{S}$, it holds that

$$(J_{\bullet,\mathcal{S}}\psi)|_T = \psi|_T \quad \text{if } \psi|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \big\{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \Psi_\bullet \in \mathcal{X}_\bullet\big\}.$$

(S6) **Local L^2-stability.** For all $\psi \in L^2(\Gamma)^D$ and $T \in \mathcal{T}_\bullet$, it holds that

$$\|J_{\bullet,\mathcal{S}}\psi\|_{L^2(T)} \le C_{\rm sz}\|\psi\|_{L^2(\pi_\bullet^{q_{\rm sz}}(T))}.$$

3.4. Error estimator

Let $\mathcal{T}_\bullet \in \mathbb{T}$. Due to the regularity assumption $f \in H^1(\Gamma)^D$, the mapping property (21), and $\mathcal{X}_\bullet \subset L^2(\Gamma)^D$, it holds that $f - \mathfrak{V}\Psi_\bullet \in H^1(\Gamma)^D$ for all $\Psi_\bullet \in \mathcal{X}_\bullet$. This allows to employ the weighted-residual a posteriori error estimator

$$\eta_\bullet := \eta_\bullet(\mathcal{T}_\bullet) \quad \text{with } \eta_\bullet(\mathcal{S})^2 := \sum_{T\in\mathcal{S}} \eta_\bullet(T)^2 \quad \text{for all } \mathcal{S} \subseteq \mathcal{T}_\bullet, \tag{37a}$$

where the local refinement indicators read

$$\eta_\bullet(T)^2 := h_T|f - \mathfrak{V}\Phi_\bullet|^2_{H^1(T)} \quad \text{for all } T \in \mathcal{T}_\bullet. \tag{37b}$$

The latter estimator goes back to the works [25,26], where reliability (42) is proved for standard 2D BEM with piecewise polynomials on polygonal geometries, while the corresponding result for 3D BEM is found in [15].

3.5. Adaptive algorithm

We consider the following concrete realization of the abstract algorithm from [6, Algorithm 2.2].

Algorithm 3.3: **Input:** Dörfler parameter $\theta \in (0,1]$ and marking constant $C_{\min} \in [1,\infty]$.
Loop: For each $\ell = 0,1,2,\dots,$ iterate the following steps:

(i) Compute Galerkin approximation $\Phi_\ell \in \mathcal{X}_\ell$.
(ii) Compute refinement indicators $\eta_\ell(T)$ for all elements $T \in \mathcal{T}_\ell$.
(iii) Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ which has up to the multiplicative constant $C_{\min}$ minimal cardinality, such that the following Dörfler marking is satisfied

$$\theta\, \eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2. \tag{38}$$

(iv) Generate refined mesh $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.

Output: Refined meshes $\mathcal{T}_\ell$ and corresponding Galerkin approximations Φ_ℓ with error estimators η_ℓ for all $\ell \in \mathbb{N}_0$.

3.6. Optimal convergence

Define

$$\mathbb{T}(N) := \{\mathcal{T}_\bullet \in \mathbb{T} : \#\mathcal{T}_\bullet - \#\mathcal{T}_0 \le N\} \quad \text{for all } N \in \mathbb{N}_0 \tag{39}$$

and for all $s > 0$

$$C_{\mathrm{approx}}(s) := \sup_{N\in\mathbb{N}_0} \min_{\mathcal{T}_\bullet\in\mathbb{T}(N)} (N+1)^s\, \eta_\bullet \in [0,\infty]. \tag{40}$$

We say that the solution $\phi \in H^{-1/2}(\Gamma)^D$ lies in the *approximation class s with respect to the estimator* if

$$\|\phi\|_{\mathbb{A}_s^{\mathrm{est}}} := C_{\mathrm{approx}}(s) < \infty. \tag{41}$$

By definition, $\|\phi\|_{\mathbb{A}_s^{\mathrm{est}}} < \infty$ implies that the error estimator $\eta_\bullet$ on the optimal meshes $\mathcal{T}_\bullet$ decays at least with rate $\mathcal{O}((\#\mathcal{T}_\bullet)^{-s})$. The following main theorem states that each possible rate $s > 0$ is in fact realized by Algorithm 3.1. We stress that the range of possible $s > 0$ depends, in particular, on the chosen admissible meshes $\mathbb{T}$. The proof is given in Section 4. It essentially follows by verifying the *axioms of adaptivity* from [6]. Such an optimality result was first proved in [3] for the Laplace operator $\mathfrak{P} = -\Delta$ on a polyhedral domain Ω. As ansatz space, they considered piecewise constants on shape-regular triangulations. [1] in combination with [5] extends the assertion to piecewise polynomials on shape-regular curvilinear triangulations of some piecewise smooth boundary Γ. Independently, [4] proved the same result for globally smooth Γ and general self-adjoint and elliptic boundary integral operators.

Theorem 3.4: *Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be the sequence of meshes generated by Algorithm 3.3. Then, there hold the following assertions* (i)–(iii):

(i) *Suppose* (M1)–(M5), *and* (S4). *Then, the residual error estimator satisfies reliability, i.e. there exists a constant $C_{\mathrm{rel}} > 0$ such that*

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\mathrm{rel}}\eta_\bullet \quad \textit{for all } \mathcal{T}_\bullet \in \mathbb{T}. \tag{42}$$

(ii) *Suppose* (M1)–(M5), (R2)–(R3), (S1)–(S2) *and* (S4). *Then, for arbitrary* $0 < \theta \le 1$ *and* $C_{\min} \in [1,\infty]$, *the estimator converges linearly, i.e. there exist constants* $0 < \rho_{\mathrm{lin}} < 1$ *and* $C_{\mathrm{lin}} \ge 1$ *such that*

$$\eta_{\ell+j}^2 \le C_{\mathrm{lin}}\rho_{\mathrm{lin}}^j \eta_\ell^2 \quad \textit{for all } j,\ell \in \mathbb{N}_0. \tag{43}$$

(iii) *Suppose* (M1)–(M5), (R1)–(R5), *and* (S1)–(S6). *Then, there exists a constant* $0 < \theta_{\mathrm{opt}} \le 1$ *such that for all* $0 < \theta < \theta_{\mathrm{opt}}$ *and* $C_{\min} \in [1,\infty)$, *the estimator converges at optimal rate, i.e. for all $s > 0$ there exist constants c_{opt},$C_{\mathrm{opt}} > 0$ such that*

$$c_{\mathrm{opt}}\|\phi\|_{\mathbb{A}_s^{\mathrm{est}}} \le \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s\, \eta_\ell \le C_{\mathrm{opt}}\|\phi\|_{\mathbb{A}_s^{\mathrm{est}}}, \tag{44}$$

where the lower bound requires only (R1) to hold.

All involved constants C_{rel}, C_{lin}, q_{lin},θ_{opt}, and C_{opt} depend only on the assumptions made as well as the dimensions d, D, the coefficients of the differential operator $\mathfrak{P}$, and Γ, while C_{lin}, ρ_{lin} depend additionally on θ and the sequence $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$, and C_{opt} depends furthermore on $C_{\min}$, and $s > 0$. The constant c_{opt} depends only on C_{son}, $\#\mathcal{T}_0$, s, and if there exists ℓ_0 with $\eta_{\ell_0} = 0$, then also on ℓ_0 and η_0.

Remark 3.5: If the sesquilinear form $(\mathfrak{V}\cdot\,;\,\cdot)$ is Hermitian, then C_{lin}, ρ_{lin}, and C_{opt} are independent of $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$; see Remark 4.14 below.

Remark 3.6: Let $\Gamma_0 \subsetneqq \Gamma$ be an open subset of $\Gamma = \partial\Omega$ and let $\mathfrak{E}_0 : L^2(\Gamma_0)^D \to L^2(\Gamma)^D$ denote the operator that extends a function defined on Γ_0 to a function on Γ by zero. We define the space of restrictions $H^{1/2}(\Gamma_0) := \{v|_{\Gamma_0} : v \in H^{1/2}(\Gamma)\}$ endowed with the quotient norm $v_0 \mapsto \inf\{\|v\|_{H^{1/2}(\Gamma)} : v|_{\Gamma_0} = v_0\}$ and its dual space $\widetilde{H}^{-1/2}(\Gamma_0) := H^{1/2}(\Gamma_0)^*$. According to [5, Section 2.1], $\mathfrak{E}_0$ can be extended to an isometric operator $\mathfrak{E}_0 : \widetilde{H}^{-1/2}(\Gamma_0)^D \to H^{-1/2}(\Gamma)^D$. Then, one can consider the integral equation

$$(\mathfrak{V}\mathfrak{E}_0\phi)|_{\Gamma_0} = f|_{\Gamma_0}, \tag{45}$$

where $(\mathfrak{V}\mathfrak{E}_0(\cdot))|_{\Gamma_0} : \widetilde{H}^{-1/2}(\Gamma_0)^D \to H^{1/2}(\Gamma_0)^D$. In the literature, such problems are known as *screen problems*; see, e.g. [21, Section 3.5.3]. Theorem 3.3 holds analogously for the screen problem (45). Indeed, the works [1,3–5] cover this case as well. However, to ease the presentation, we focus on closed boundaries $\Gamma_0 = \Gamma = \partial\Omega$.

Remark 3.7: (a) Let us additionally assume that $\mathcal{X}_\bullet$ contains all componentwise constant functions, i.e.

$$x \in \mathcal{X}_\bullet \quad \text{for all } x \in \mathbb{C}^D. \tag{46}$$

Then, under the assumption that $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ as $\ell \to \infty$, one can show that $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{X}_\ell} = H^{-1/2}(\Gamma)^D$. To see this, recall that $H^{1/2}(\Gamma)^D$ is continuously and densely embedded in $L^2(\Gamma)^D$ which is itself continuously and densely embedded in $H^{-1/2}(\Gamma)^D$. For $\psi \in H^{-1/2}(\Gamma)^D$ and arbitrary $\varepsilon > 0$, let $\psi_\varepsilon \in H^{1/2}(\Gamma)^D$ with $\|\psi - \psi_\varepsilon\|_{H^{-1/2}(\Gamma)} \le \varepsilon$. We abbreviate the projection operator $J_\ell := J_{\ell,\mathcal{T}_\ell}$ for all $\ell \in \mathbb{N}_0$. For all $T \in \mathcal{T}_\ell$, the projection property (S5) in combination with our additional assumption (46), the triangle inequality, and the local L^2-stability (S6) show that

$$\begin{aligned}\|(1-J_\ell)\psi_\varepsilon\|_{L^2(T)} &\overset{(S5)}{=} \left\|(1-J_\ell)\left(\psi_\varepsilon - \frac{1}{|\pi_\bullet^{q_{sz}}(T)|}\int_{\pi_\bullet^{q_{sz}}(T)} \psi_\varepsilon \,dx\right)\right\|_{L^2(T)} \\ &\overset{(S6)}{\le} (1+C_{sz})\left\|\psi_\varepsilon - \frac{1}{|\pi_\bullet^{q_{sz}}(T)|}\int_{\pi_\bullet^{q_{sz}}(T)} \psi_\varepsilon \,dx\right\|_{L^2(\pi_\bullet^{q_{sz}}(T))}.\end{aligned}$$

With this, the Poincaré-type inequality from Lemma 4.6 below, and (M1)–(M3), we see that

$$\|(1-J_\ell)\psi_\varepsilon\|_{L^2(T)} \lesssim h_T^{1/2}|\psi_\varepsilon|_{H^{1/2}(\pi_\bullet^{q_{sz}}(T))} \le \|h_\ell\|_{L^\infty(\Gamma)}^{1/2}|\psi_\varepsilon|_{H^{1/2}(\pi_\bullet^{q_{sz}}(T))}.$$

Summing over all elements, we obtain that

$$\|(1-J_\ell)\psi_\varepsilon\|_{H^{-1/2}(\Gamma)}^2 \lesssim \|(1-J_\ell)\psi_\varepsilon\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell\|_{L^\infty(\Gamma)} \sum_{T\in\mathcal{T}_\bullet} |\psi_\varepsilon|_{H^{1/2}(\pi_\bullet^{q_{sz}}(T))}^2.$$

With (M1)–(M4), Proposition 4.1 and Lemma 4.8 from below prove that $\sum_{T\in\mathcal{T}_\bullet} |\psi_\varepsilon|_{H^{1/2}(\pi_\bullet^{q_{sz}}(T))}^2 \lesssim |\psi_\varepsilon|_{H^{1/2}(\Gamma)}^2$. Overall, this shows that

$$\min_{\psi_\ell\in\mathcal{X}_\ell} \|\psi - \psi_\ell\|_{H^{-1/2}(\Gamma)} \le \|\psi - \psi_\varepsilon\|_{H^{-1/2}(\Gamma)} + \|(1-J_\ell)\psi_\varepsilon\|_{H^{-1/2}(\Gamma)} \lesssim \varepsilon + \|h_\ell\|_{L^\infty(\Gamma)}^{1/2}|\psi_\varepsilon|_{H^{1/2}(\Gamma)}.$$

Since $\lim_{\ell\to\infty} \|h_\ell\|_{L^\infty(\Gamma)} = 0$ and ε was arbitrary, this concludes the proof.

(b) The latter observation allows to follow the ideas of [27] and to show that the adaptive algorithm yields convergence provided that the sesquilinear forms $(\cdot\,;\,\cdot)_{\mathfrak{P}}$ as well as $(\mathfrak{V}\,\cdot\,;\,\cdot)$ are only coercive, i.e. elliptic up to some compact perturbation and that the continuous problem is well-posed; see also the introductory text of Section 4.4. This includes, e.g. adaptive BEM for the Helmholtz equation; see [20, Section 6.9]. For details, the reader is referred to [7,27].

3.7. Application to BEM with piecewise polynomials on triangulations

For $d = 2, 3$, we fix the reference simplex T_{ref} as the closed convex hull of the d vertices $\{0, e_1, \dots, e_{d-1}\}$. The convex hull of any $d-1$ vertices is called *facet*. A set $\mathcal{T}_\bullet$ of subsets of Γ is called κ*-shape regular triangulation* if the following properties (i)–(v) are satisfied:

(i) $\mathcal{T}_\bullet$ is a finite set of elements T of the form $T = \gamma_T(\widehat{T})$, where $\gamma_T : T_{\mathrm{ref}} \to T$ is a bi-Lipschitz mapping whose Lipschitz constants are bounded from above by κ.
(ii) $\mathcal{T}_\bullet$ covers Γ, i.e. $\Gamma = \bigcup_{T \in \mathcal{T}_\bullet} T$.
(iii) There are no hanging nodes in the sense that the intersection $T \cap T'$ of any $T, T' \in \mathcal{T}_\bullet$ with $T \neq T'$ is either empty or a common facet, i.e. $T \cap T' = \gamma_T(f) = \gamma_{T'}(f')$ for some facets f and f' of T_{ref}.
(iv) The parametrizations of neighboring elements are compatible in the sense that for all nodes z (i.e. images of the $\{0, e_1, \dots, e_{d-1}\}$ under an element map γ_T), there exists an interval $\widetilde{\pi}_\bullet(z)$ for $d = 2$ and a convex polygonal $\widetilde{\pi}_\bullet(z)$ for $d = 3$ respectively as well as a bijective and bi-Lipschitz continuous mapping $\gamma_z : \widetilde{\pi}_\bullet(z) \to \pi_\bullet(z)$ such that $\gamma_z^{-1} \circ \gamma_T$ is affine for all $T \in \Pi_\bullet(z)$.
(v) If $d = 2$, $\mathcal{T}_\bullet$ is locally-quasi uniform in the sense that $\mathrm{diam}(T) \leq \kappa\, \mathrm{diam}(T')$ for all $T, T' \in \mathcal{T}_\bullet$ with $T \cap T' \neq \emptyset$.

Up to the fact that we allow γ_T to be bi-Lipschitz instead of C^1, this definition is slightly stronger than [5, Definition 2.4]. The property (iii) stems from [21, Assumption 4.3.25] and is stronger than the corresponding assumption [5, Definition 2.4 (iii)]. Further, (i) implies [5, Definition 2.4 (v)], i.e. for all $T \in \mathcal{T}_\bullet$, there holds with the extremal eigenvalues $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ that

$$\sup_{t \in T_{\mathrm{ref}}} \left(\frac{\mathrm{diam}(T)^2}{\lambda_{\min}(D\gamma_T^\top(t) D\gamma_T(t))} + \frac{\lambda_{\max}(D\gamma_T^\top(t) D\gamma_T(t))}{\mathrm{diam}(T)^2} \right) \lesssim 1; \tag{47}$$

see, e.g. [24, (3.26)–(3.27)] or [17, Lemma 5.2.1].

Let $\mathcal{T}_0$ be a κ_0-shape regular triangulation. For $d = 2$, we define $\mathtt{refine}(\cdot)$ as in [28] via 1D-bisection in the parameter domain. For $d = 3$, we define $\mathtt{refine}(\cdot)$ as in [29] via newest vertex bisection in the parameter domain. In particular, all corresponding refinements $\mathcal{T}_\bullet \in \mathbb{T} = \mathtt{refine}(\mathcal{T}_0)$ are again κ-shape regular triangulations with some fixed κ depending on κ_0. We also note that the number of different $\widetilde{\pi}_\bullet(z)$ in (iv) is uniformly bounded, i.e. there exist only finitely many reference node patches. Finally, let $p \in \mathbb{N}_0$ be a fixed polynomial order. For each $\mathcal{T}_\bullet$, we associate the space of (transformed) piecewise polynomials

$$\mathcal{X}_\bullet := \mathcal{P}^p(\mathcal{T}_\bullet) := \left\{ \Psi_\bullet \in L^2(\Gamma) \,:\, \Psi_\bullet \circ \gamma_T \text{ is a polynomial of degree } p \text{ for all } T \in \mathcal{T}_\bullet \right\}. \tag{48}$$

For this concrete setting, we already pointed out that [1] in combination with [5] proved linear convergence (43) at optimal rate (44) if $\mathfrak{P} = -\Delta$ is the Laplace operator. The following theorem generalizes this result to arbitrary $\mathfrak{P}$ as in Section 2.3.

Theorem 3.8: *Piecewise polynomials on κ-shape regular triangulations satisfy the abstract properties* (M1)–(M5), (R1)–(R5), *and* (S1)–(S6), *where the constants depend only on the dimension D, the regularity constant κ, the initial mesh $\mathcal{T}_\bullet$, and the polynomial order p. By Theorem 3.4, this implies reliability* (42) *of the error estimator and linear convergence* (43) *at optimal rate* (44) *for the adaptive strategy from Algorithm* 3.3.

Proof: The elementary mesh properties (M1)–(M3) are verified in [5, Section 2.3]. (M4) is stated in [5, Section 4.1]. (M5) follows from Proposition 3.1 together with the fact that there are only finitely many reference node patches.

For $d = 2$, the son estimate (R1) is clearly satisfied with $C_{\rm son} = 2$. For $d = 3$, it is well known that NVB satisfies (R1) with $C_{\rm son} = 4$. Further, (R1) holds true by definition. Reduction of sons (R3) is obviously satisfied in the parameter domain, i.e. $|\gamma_T^{-1}(T')| \le |\gamma_T^{-1}(T)|/2$ for all $T' \in \mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$ with $T' \subsetneqq T$. Since γ_T is bi-Lipschitz, this property transfers to the physical domain, i.e. $|\gamma_T^{-1}(T')| \le \rho_{\rm son}|\gamma_T^{-1}(T)|$, where $0 < \rho_{\rm son} < 1$ depends only on κ; see, e.g. [17, Section 4.5.3] for details. For $d = 2$, (R4)–(R5) are found in [28, Theorem 2.3]. For $d = 3$, the closure estimate (R4) is proved in [30, Theorem 2.4], [31, Theorem 6.1], or [32, Theorem 2], where the latter result avoids any additional assumption on $\mathcal{T}_0$. The overlay property is proved in [31, Proof of Lemma 5.2] or [33, Section 2.2].

The inverse inequality (S1) for piecewise polynomials on the boundary is proved, e.g. in [5, Lemma A.1]. Nestedness (S2) is trivially satisfied. Also (S3) is trivially satisfied with $q_{\rm loc}, q_{\rm proj} = 0$. Clearly, (S4) holds with $(\Psi_{\bullet,T,j})_{j'} := 0$ for $j' \neq j$ and $(\Psi_{\bullet,T,j})_j := \chi_T$, where χ_T denotes the indicator function on T. Finally, for $\mathcal{S} \subseteq \mathcal{T}_\bullet \in \mathbb{T}$, we define with the elementwise $L^2(T)$-orthogonal projection $P_{\bullet,T} : L^2(T)^D \to \{\Psi_\bullet|_T : \Psi_\bullet \in \mathcal{X}_\bullet\}$

$$J_{\bullet,\mathcal{S}} : L^2(\Gamma) \to \mathcal{X}_\bullet, \quad \psi \mapsto J_{\bullet,\mathcal{S}} := \begin{cases} P_{\bullet,T}\psi & \text{on all } T \in \mathcal{S}, \\ 0 & \text{on all } T \in \mathcal{T}_\bullet \setminus \mathcal{S}. \end{cases} \tag{49}$$

This definition immediately yields (S5)–(S6) with $q_{\rm sz} = 0$. ∎

Remark 3.9: We mention that Theorem 3.8 is also valid if $d = 2$ and $\mathcal{X}_\bullet$ is chosen as set of (transformed) splines which are piecewise polynomials with certain differentiability conditions at the break points. The required properties are (implicitly) verified in [16]. As in [10] (resp. [11]), where we have verified the abstract FEM framework of [10] for IGAFEM with hierarchical splines [34] and the mesh-refinement from [10] (resp. T-splines with the mesh-refinement from [35]), the verification of the present abstract BEM framework for 3D IGABEM will be addressed in the future work [12].

4. Proof of Theorem 3.4

In the following sections, we prove Theorem 3.4. Reliability (42) is treated explicitly in Section 4.2. It follows immediately from an auxiliary result on the localization of the Sobolev–Slobodeckij norm which is investigated in Section 4.1. To prove Theorem 3.4 (ii)–(iii), we verify the following abstract properties (E1)–(E4) for the error estimator. Together with (R1), the closure estimate (R4), and the overlay property (R5), these already imply linear convergence of the estimator at optimal algebraic rate; see [6].

There exist $C_\rho, C_{\rm qo}, C_{\rm ref}, C_{\rm drel}, C_{\rm son}, C_{\rm clos}, C_{\rm over} \ge 1$, and $0 \le \rho_{\rm red}, \varepsilon_{\rm qo}, \varepsilon_{\rm drel} < 1$ such that there hold:

(E1) **Stability on non-refined elements:** For all $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| \le \varrho_{\bullet,\circ} := C_\rho \|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}.$$

(E2) **Reduction on refined elements:** For all $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \le \rho_{\rm red}\eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)^2 + \varrho_{\bullet,\circ}^2.$$

(E3) **General quasi-orthogonality:** It holds that

$$0 \le \varepsilon_{\rm qo} < \sup_{\delta > 0} \frac{1 - (1+\delta)(1 - (1 - \rho_{\rm red})\theta)}{2 + \delta^{-1}},$$

and for all $\ell, N \in \mathbb{N}_0$, the sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 3.3 satisfies that

$$\sum_{j=\ell}^{\ell+N} (\varrho_{j,j+1}^2 - \varepsilon_{\mathrm{qo}}\eta_j^2) \le C_{\mathrm{qo}}\eta_\ell^2.$$

(E4) **Discrete reliability:** For all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet)$, there exists $\mathcal{T}_\bullet \setminus \mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet,\circ} \subseteq \mathcal{T}_\bullet$ with $\#\mathcal{R}_{\bullet,\circ} \le C_{\mathrm{ref}}(\#\mathcal{T}_\circ - \#\mathcal{T}_\bullet)$ such that

$$\varrho_{\bullet,\circ}^2 \le \varepsilon_{\mathrm{drel}}\eta_\bullet^2 + C_{\mathrm{drel}}^2\eta_\bullet(\mathcal{R}_{\bullet,\circ})^2.$$

4.1. Localization of the Sobolev–Slobodeckij norm

Let $\mathcal{T}_\bullet \in \mathbb{T}$. In contrast to the integer case, for $\sigma \in (0,1)$, the norm $\|\cdot\|_{H^\sigma(\Gamma)}$ is not additive in the sense that

$$\|v\|_{H^\sigma(\Gamma)}^2 \simeq \sum_{T\in\mathcal{T}_\bullet} \|v\|_{H^\sigma(T)}^2 \quad \text{for all } v \in H^\sigma(\Gamma)^D.$$

Although the upper bound '$\lesssim$' is in general false (see [36, Section 3]), the lower bound '$\gtrsim$' can be proved elementarily for arbitrary $v \in H^\sigma(\Gamma)^D$.

Proposition 4.1: *Let $0 < \sigma < 1$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then,* (M1) *implies the existence of a constant $C'_{\mathrm{split}} > 0$ such that for any $v \in H^\sigma(\Gamma)^D$, there holds that*

$$\sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|_{H^\sigma(T\cup T')}^2 \le C'_{\mathrm{split}}|v|_{H^\sigma(\Gamma)}^2. \tag{50}$$

The constant C'_{split} depends only on the constant from (M1)

Proof: With the abbreviation

$$V(x,y) := \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}} \quad \text{for all } x,y \in \Gamma \text{ with } x \ne y, \tag{51}$$

(M1) shows that

$$\sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|_{H^\sigma(T\cup T')}^2 = \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} \left(|v|_{H^\sigma(T)}^2 + 2\int_T\int_{T'} V(x,y)\,\mathrm{d}x\,\mathrm{d}y + |v|_{H^\sigma(T')}^2 \right)$$
$$= 2\sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} \left(\int_T\int_T V(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_T\int_{T'} V(x,y)\,\mathrm{d}x\,\mathrm{d}y \right) \le 2(C_{\mathrm{patch}}+1)|v|_{H^\sigma(\Gamma)}^2.$$

This concludes the proof. ■

However, if one replaces the elements T by some overlapping patches, then also the converse inequality is satisfied for functions $v \in H^\sigma(\Gamma)^D$ which are L^2-orthogonal to the ansatz space $\mathcal{X}_\bullet$.

Proposition 4.2: *Let $0 < \sigma < 1$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then,* (M1)–(M4) *and* (S4) *imply the existence of a constant $C_{\mathrm{split}} > 0$ such that for any $v \in H^\sigma(\Gamma)^D$ which satisfies that $(v\,;\,(\Psi_{\bullet,T,j})_j)_{L^2(\Gamma)} = 0$ for all $T \in \mathcal{T}_\bullet$ and all $j \in \{1,\dots,D\}$, where $\Psi_{\bullet,T,j}$ are the functions from* (S4)*, it holds that*

$$\|v\|_{H^\sigma(\Gamma)}^2 \le C_{\mathrm{split}} \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|_{H^\sigma(T\cup T')}^2. \tag{52}$$

The constant C_{split} depends only on the dimension d, σ, Γ, and the constants from (M1)–(M4) *and* (S4).

With this result, one can immediately construct a reliable and efficient error estimator, namely the so-called *Faermann estimator*; see Remark 4.10. For $d = 2$, the result of the proposition goes back to [13], where $\mathcal{X}_\bullet$ is chosen as space of splines transformed via the arclength parametrization $\gamma : [a, b] \to \Gamma$ onto the one-dimensional boundary. In the recent own works [37], we generalized the assertion to rational splines, where we could also drop the restriction that γ is the arclength parametrization. For $d = 3$, [14] proved the result for discrete spaces which contain certain (transformed) polynomials of degree $p \in \{0, 1, 5, 6\}$ on a curvilinear triangulation of Γ. Our proof of Proposition 4.2 is inspired by [14]. The key ingredient is the assumption (S4) which is exploited in Lemma 4.7. Before proving Proposition 4.2, we provide an easy corollary which is the key ingredient for the proof of reliability (42).

Corollary 4.3: *Let $\mathcal{T}_\bullet \in \mathbb{T}$. Then,* (M1)–(M5) *and* (S4) *imply the existence of a constant $C'_{\rm rel} > 0$ such that for any $v \in H^1(\Gamma)^D$ which satisfies that $(v\,;\,\Psi_{\bullet,T,j})_{L^2(\Gamma)} = 0$ for all $T \in \mathcal{T}_\bullet$ and all $j \in \{1, \dots, D\}$, where $\Psi_{\bullet,T,j}$ are the functions from* (S4)*, it holds that*

$$\|v\|_{H^{1/2}(\Gamma)} \le C'_{\rm rel}\|h_\bullet^{1/2}\nabla_\Gamma v\|_{L^2(\Gamma)}. \tag{53}$$

The constant $C'_{\rm rel}$ depends only on the dimension d, Γ, as well as the constants from (M1)–(M5) *and* (S4).

To prove Proposition 4.2, we start with the following basic estimate, which is proved in [38, Lemma 8.2.4] or in [17, Lemma 5.3.1].

Lemma 4.4: *For all $\lambda > 0$, there is a constant $C(\lambda) > 0$ such that for all $x \in \mathbb{R}^d$ and all $\varepsilon > 0$, there holds that*

$$\int_{\Gamma\setminus B_\varepsilon(x)} |x - y|^{-d+1-\lambda}\,\mathrm{d}y \le C(\lambda)\varepsilon^{-\lambda}. \tag{54}$$

The constant $C(\lambda)$ depends only on the parameter λ, the dimension d, and Γ.

The following lemma is the first step towards the localization of the norm $\|v\|_{H^\sigma(\Gamma)}$ for certain functions $v \in H^\sigma(\Gamma)^D$. In [14, Lemma 3.1], this result is stated for triangular meshes. The elementary proof extends to our situation; see also [17, Lemma 5.3.2] for details.

Lemma 4.5: *Let $0 < \sigma < 1$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then,* (M4) *implies the existence of a constant $C > 0$ such that for all $v \in H^\sigma(\Gamma)^D$, it holds that*

$$\|v\|^2_{H^\sigma(\Gamma)} \le \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |v|^2_{H^\sigma(T\cup T')} + C\sum_{T\in\mathcal{T}_\bullet} \mathrm{diam}(T)^{-2\sigma}\,\|v\|^2_{L^2(T)}. \tag{55}$$

The constant C depends only on the dimension d, σ, Γ, and the constant from (M4).

It remains to control the second summand in (55). To this end, we need the following elementary Poincaré-type inequality of [13, Lemma 2.5].

Lemma 4.6: *For any $\sigma \in (0, 1)$ and any measurable $\omega \subseteq \Gamma$, there holds for all $v \in H^\sigma(\omega)$ that*

$$\|v\|^2_{L^2(\omega)} \le \frac{\mathrm{diam}(\omega)^{d-1+2\sigma}}{2|\omega|}|v|^2_{H^\sigma(\omega)} + \frac{1}{|\omega|}\left|\int_\omega v(x)\,\mathrm{d}x\right|^2. \tag{56}$$

We start to estimate the second summand in (55).

Lemma 4.7: *Let* $\sigma \in (0,1)$, $\mathcal{T}_\bullet \in \mathbb{T}$ *and* $T \in \mathcal{T}_\bullet$. *Then,* (M1)–(M3) *and* (S4) *imply the existence of a constant* $C > 0$ *such that for all* $v \in H^\sigma(\Gamma)^D$ *with* $(v_j\,;\,\Psi_{\bullet,T,j})_{L^2(\Gamma)} = 0$ *for all* $j \in \{1,\dots,D\}$, *where* $\Psi_{\bullet,T,j}$ *are the functions from* (S4), *it holds that*

$$\|h_\bullet^{-\sigma} v\|_{L^2(T)} \le C|v|_{H^\sigma(\pi_\bullet^{q_{\rm supp}}(T))}, \tag{57}$$

where $q_{\rm supp}$ *is the constant from* (S4). *The constant* C *depends only on the dimension* d, σ, Γ, *and the constants from* (M1)–(M3) *and* (S4).

Proof: We prove (57) for each component v_j of v, where $j \in \{1,\dots,D\}$. Then, squaring and summing up all components, we conclude the proof. (S4) and Lemma 4.6 show that

$$\begin{aligned}\|v_j\|_{L^2(T)}^2 &\le \|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))}\\ &\le \frac{\mathrm{diam}(\mathrm{supp}(\Psi_{\bullet,T,j}))^{d-1+2\sigma}}{2|\mathrm{supp}(\Psi_{\bullet,T,j})|}|v_j|^2_{H^\sigma(\mathrm{supp}(\Psi_{\bullet,T,j}))} + \frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\left|\int_{\mathrm{supp}(\Psi_{\bullet,T,j})} v_j(x)\,\mathrm{d}x\right|^2.\end{aligned} \tag{58}$$

Now, we apply the orthogonality and (S4) to get for the second summand that

$$\begin{aligned}&\frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\left|\int_{\mathrm{supp}(\Psi_{\bullet,T,j})} v_j(x)\,\mathrm{d}x\right|^2\\ &= \frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\left|\int_{\mathrm{supp}(\Psi_{\bullet,T,j})} \overline{v}_j(x)(1-\Psi_{\bullet,T,j}(x))\,\mathrm{d}x\right|^2\\ &\le \frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))}\|1-(\Psi_{\bullet,T,j})_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \le \rho_{\rm unity}^2\|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))}.\end{aligned}$$

Inserting this in (58) gives that

$$(1-\rho_{\rm unity}^2)\|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \le \frac{\mathrm{diam}(\mathrm{supp}(\Psi_{\bullet,T,j}))^{d-1+2\sigma}}{2|\mathrm{supp}(\Psi_{\bullet,T,j})|}|v_j|^2_{H^\sigma(\mathrm{supp}(\Psi_{\bullet,T,j}))}. \tag{59}$$

With (S4) and (M1)–(M3), we see that $\mathrm{diam}(\mathrm{supp}(\Psi_{\bullet,T,j})) \le \mathrm{diam}(\pi_\bullet^{q_{\rm supp}}(T)) \lesssim \mathrm{diam}(T) \simeq h_T$. Further, (S4) implies that $|\mathrm{supp}(\Psi_{\bullet,T,j})| \ge |T| = h_T^{d-1}$. Inserting this in (59) and using again (S4), we derive that

$$\|v_j\|^2_{L^2(T)} \le \|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \lesssim h_T^{2\sigma}|v_j|^2_{H^\sigma(\mathrm{supp}(\Psi_{\bullet,T,j}))} \le h_T^{2\sigma}|v_j|^2_{H^\sigma(\pi_\bullet^{q_{\rm supp}}))}.$$

Altogether, this concludes the proof. ∎

The following lemma allows us to further estimate the term $|v|_{H^\sigma(\pi_\bullet^{q_{\rm supp}}(T))}$ of (57).

Lemma 4.8: *Let* $q \in \mathbb{N}_0$ *and* $\mathcal{T}_\bullet \in \mathbb{T}$. *Then,* (M1)–(M4) *imply the existence of a constant* $C(q) > 0$ *such that for all* $v \in H^\sigma(\Gamma)^D$ *and all* $T \in \mathcal{T}_\bullet$ *there holds that*

$$|v|^2_{H^\sigma(\pi_\bullet^q(T))} \le C(q) \sum_{\substack{T',T''\in\Pi_\bullet^q(T)\\ T'\cap T''\neq\emptyset}} |v|^2_{H^\sigma(T'\cup T'')}. \tag{60}$$

The constant depends only on the dimension d, σ, q, *and the constants from* (M1)–(M4).

Proof: Without loss of generality, we may assume that $D = 1$. We prove the assertion in two steps.

Step 1: Let $T_0, T_1, \dots, T_m$ be a *chain* of elements in $\Pi^q_\bullet(T)$ with $T_i \cap T_j = \emptyset$ for $|i-j| > 1$ and $T_i \cap T_j \neq \emptyset$ if $|i-j| = 1$, where $1 \le m \le q$. We set $T_i^j := \bigcup_{k=i}^j T_\ell$ for $i \le j$ and prove by induction on m that there exists a constant $C_1(m) > 0$ which depends only on d, σ, q, m, and (M2)–(M4), such that

$$|v|^2_{H^\sigma(T_0^m)} \le C_1(m) \sum_{i=0}^{m-1} |v|^2_{H^\sigma(T_i \cup T_{i+1})}. \tag{61}$$

For $m = 1$, (61) with $C_1(1) = 1$ even holds with equality. Thus the induction hypothesis reads: for all $1 \le m-1 < q$ and for any chain $T_0, \dots, T_{m-1}$ of elements in $\Pi^q_\bullet(T)$, it holds that

$$|v|^2_{H^\sigma(T_0^{m-1})} \le C_1(m-1) \sum_{i=0}^{m-2} |v|^2_{H^\sigma(T_i \cup T_{i+1})}. \tag{62}$$

Let $T_m \in \Pi^q_\bullet(T)$ with $T_m \cap T_i = \emptyset$ for $i \le m-2$ and $T_m \cap T_i \neq \emptyset$ for $i = m-1$. For all $x, y \in \Gamma, x \neq y$, we abbreviate $V(x,y) := \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}}$. The definition (7) of the Sobolev–Slobodeckij seminorm shows that

$$\begin{aligned}
|v|^2_{H^\sigma(T_0^m)} &= \int_{T_0^m} \int_{T_0^m} V(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{T_0^{m-1}} \int_{T_0^{m-1}} V(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{T_m} \int_{T_m} V(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\int_{T_m} \int_{T_0^{m-1}} V(x,y)d\,\mathrm{d}x\,\mathrm{d}y \\
&= |v|^2_{H^\sigma(T_0^{m-1})} + |v|^2_{H^\sigma(T_m)} + 2\int_{T_m} \int_{T_0^{m-2}} V(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\int_{T_m} \int_{T_{m-1}} V(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\le |v|^2_{H^\sigma(T_0^{m-1})} + |v|^2_{H^\sigma(T_{m-1} \cup T_m)} + 2\int_{T_m} \int_{T_0^{m-2}} V(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

With the induction hypothesis (62), it remains to estimate $\int_{T_m} \int_{T_0^{m-2}} V(x,y)\mathrm{d}x\mathrm{d}y$. First, we note that for $x \in T_0^{m-2}, y \in T_m, z \in T_{m-1}$, it holds that

$$V(x,y) = \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}} \le 2\frac{|v(x)-v(z)|^2}{|x-y|^{d-1+2\sigma}} + 2\frac{|v(z)-v(y)|^2}{|x-y|^{d-1+2\sigma}}. \tag{63}$$

Moreover, (M4) shows that $|x-y| \ge \mathrm{dist}(T_m, \Gamma \setminus \pi_\bullet(T_m)) \gtrsim \mathrm{diam}(T_m)$. Since $x, y, z \in T_0^m$, (M2) shows $\max\{|x-z|, |y-z|\} \lesssim \mathrm{diam}(T_m)$. Hence, we can proceed the estimate of (63)

$$V(x,y) \lesssim V(x,z) + V(z,y).$$

This implies that

$$\begin{aligned}
&\int_{T_m} \int_{T_0^{m-2}} V(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{|T_{m-1}|} \int_{T_{m-1}} \int_{T_m} \int_{T_0^{m-2}} V(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad \lesssim \frac{1}{|T_{m-1}|} \int_{T_{m-1}} \int_{T_m} \int_{T_0^{m-2}} V(x,z) + V(y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

$$= \frac{1}{|T_{m-1}|}\left(\int_{T_{m-1}}\int_{T_0^{m-2}} |T_m| V(x,z)\,\mathrm{d}x\,\mathrm{d}z + \int_{T_{m-1}}\int_{T_{m-1}} |T_0^{m-2}| V(y,z)\,\mathrm{d}y\,\mathrm{d}z\right)$$

$$\leq \frac{\max\{|T_m|, |T_0^{m-2}|\}}{|T_{m-1}|}\left(|v|^2_{H^\sigma(T_0^{m-1})} + |v|^2_{H^\sigma(T_{m-1}\cup T_m)}\right).$$

Note that $\max\{|T_m|, |T_0^{m-2}|\}/|T_{m-1}| \lesssim 1$ by (M2)–(M3). Together with the induction hypothesis (62), this concludes the induction step.

Step 2: We come to the assertion itself. By definition, we have that

$$|v|^2_{H^{1/2}(\pi_\bullet^q(T))} = \sum_{\widetilde{T}', \widetilde{T}'' \in \Pi_\bullet^q(T)} \int_{\widetilde{T}'}\int_{\widetilde{T}''} V(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Let $\widetilde{T}', \widetilde{T}'' \in \Pi_\bullet^q(T)$. First, we suppose that $\widetilde{T}' \neq \widetilde{T}'' = \emptyset$. Then, there exists a chain as in Step 1 with $\widetilde{T}' = T_0$ and $\widetilde{T}'' = T_m$. Step 1 proves that

$$\int_{\widetilde{T}'}\int_{\widetilde{T}''} V(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq |v|^2_{H^\sigma(T_0^m)} \lesssim \sum_{\substack{T', T'' \in \Pi_\bullet^q(T) \\ T' \cap T'' \neq \emptyset}} |v|^2_{H^\sigma(T' \cup T'')}.$$

If $\widetilde{T}' = \widetilde{T}''$, the same estimate holds true. Since the number of $\widetilde{T}', \widetilde{T}'' \in \Pi_\bullet^q(T)$ is uniformly bounded by a constant, which depends only on the constant of (M1) and q, this estimate concludes the proof. ∎

With the property (M5), one immediately derives the following Poincaré inequality.

Proposition 4.9: *Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $T \in \mathcal{T}_\bullet$. Then,* (M1)–(M5) *and* (S4) *imply the existence of a constant $C_{\mathrm{poinc}} > 0$ such that for all $v \in H^1(\Gamma)^D$ which satisfy that $(v\,;\,\Psi_{\bullet,T,j})_{L^2(\Gamma)} = 0$ for all $j \in \{1,\ldots,D\}$, where $\Psi_{\bullet,T,j}$ are the functions from* (S4)*, it holds that*

$$\|h_\bullet^{-1} v\|_{L^2(T)} \leq C_{\mathrm{poinc}} |v|_{H^1\left(\pi_\bullet^{q_{\mathrm{supp}}+1}(T)\right)}, \tag{64}$$

where q_{supp} is the constant from (S4). *The constant C_{poinc} depends only on the dimension d, Γ, and the constants from* (M1)–(M5) *and* (S4).

Proof: We apply Lemma 4.7 and Lemma 4.8 to see that

$$\|h_\bullet^{-1/2} v\|^2_{L^2(T)} \lesssim |v|^2_{H^{1/2}(\pi_\bullet^{q_{\mathrm{supp}}}(T))} \lesssim \sum_{\substack{T', T'' \in \Pi_\bullet^{q_{\mathrm{supp}}}(T) \\ T' \cap T'' \neq \emptyset}} |v|^2_{H^{1/2}(T' \cup T'')}.$$

For $T', T'' \in \mathcal{T}_\bullet$ with $T' \cap T'' \neq \emptyset$, we fix some point $z(T', T'') \in T' \cap T''$. With (M5), we continue our estimate

$$\|h_\bullet^{-1/2} v\|^2_{L^2(T)} \lesssim |v|^2_{H^{1/2}(\pi_\bullet^{q_{\mathrm{supp}}}(T))} \lesssim \sum_{\substack{T', T'' \in \Pi_\bullet^{q_{\mathrm{supp}}}(T) \\ T' \cap T'' \neq \emptyset}} |v|^2_{H^{1/2}(\pi_\bullet(z(T',T'')))}$$

$$\lesssim \sum_{\substack{T', T'' \in \Pi_\bullet^{q_{\mathrm{supp}}}(T) \\ T' \cap T'' \neq \emptyset}} \operatorname{diam}\left(\pi_\bullet(z(T', T''))\right) \|\nabla_\Gamma v\|^2_{L^2(\pi_\bullet(T'))}.$$

(M1)–(M3) imply that $h_T \simeq h_\bullet$ on $\pi_\bullet^{q_{\mathrm{supp}}+1}(T)$, and that the last term of the latter estimate can be bounded from above (up to a multiplicative constant) by $\|h_\bullet^{1/2} \nabla_\Gamma v\|^2_{L^2(\pi_\bullet^{q_{\mathrm{supp}}+1}(T))}$. This concludes the proof. ∎

With all the preparations, we can finally prove the main result of this section.

Proof of Proposition 4.2: Together with (M3), Lemma 4.5 proves that

$$\|v\|_{H^\sigma(\Gamma)}^2 \lesssim \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|_{H^\sigma(T\cup T')}^2 + \sum_{T\in\mathcal{T}_\bullet} h_T^{-2\sigma} \|v\|_{L^2(T)}^2.$$

It remains to estimate the second sum. With Lemma 4.7 and Lemma 4.8, we see that

$$\sum_{T\in\mathcal{T}_\bullet} h_T^{-2\sigma} \|v\|_{L^2(T)}^2 \lesssim \sum_{T\in\mathcal{T}_\bullet} |v|_{H^\sigma(\pi_\bullet^{q_{\rm supp}}(T))}^2 \lesssim \sum_{T\in\mathcal{T}_\bullet} \sum_{\substack{T',T''\in\Pi_\bullet^{q_{\rm supp}}(T)\\ T'\cap T''\neq\emptyset}} |v|_{H^\sigma(T'\cup T'')}^2. \tag{65}$$

If $T\in\mathcal{T}_\bullet$ and $T',T''\in\Pi_\bullet^{q_{\rm supp}}(T)$ with $T'\cap T''\neq\emptyset$, then $T\in\Pi_\bullet^{q_{\rm supp}}(T')$ and $T''\in\Pi_\bullet(T')$. Plugging this into (65) shows that

$$\sum_{T\in\mathcal{T}_\bullet} h_T^{-2\sigma} \|v\|_{L^2(T)}^2 \lesssim \sum_{T'\in\mathcal{T}_\bullet} \sum_{T\in\Pi_\bullet^{q_{\rm supp}}(T')} \sum_{T''\in\Pi_\bullet(T')} |v|_{H^\sigma(T'\cup T'')}^2,$$

and $\#\Pi_\bullet^{q_{\rm supp}}(T')\lesssim 1$ (see (M1)) concludes the proof. ∎

4.2. Reliability (42)

Let $\mathcal{T}_\bullet\in\mathbb{T}$. Recall that $\mathfrak{V}: H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ is an isomorphism. Due to Galerkin orthogonality (35), Corollary 4.3 leads to

$$\|\phi-\Phi_\bullet\|_{H^{-1/2}(\Gamma)} \simeq \|f-\mathfrak{V}\Phi_\bullet\|_{H^{1/2}(\Gamma)} \lesssim \|h_\bullet^{1/2}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|_{L^2(\Gamma)} = \eta_\bullet. \tag{66}$$

Remark 4.10: Proposition 4.1 and Proposition 4.2 show that

$$\|\phi-\Phi_\bullet\|_{H^{-1/2}(\Gamma)}^2 \simeq \|f-\mathfrak{V}\Phi_\bullet\|_{H^{1/2}(\Gamma)}^2 \simeq \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |f-\mathfrak{V}\Phi_\bullet|_{H^{1/2}(T\cup T')}^2. \tag{67}$$

This is even true for arbitrary $f\in H^{1/2}(\Gamma)^D$ without the additional restriction $f\in H^1(\Gamma)^D$. In particular,

$$\mathcal{F}_\bullet(T)^2 := \sum_{T'\in\Pi_\bullet(T)} |f-\mathfrak{V}\Phi_\bullet|_{H^{1/2}(T\cup T')}^2 \quad \text{for all } T\in\mathcal{T}_\bullet \tag{68}$$

provides a local error indicator. The corresponding error estimator $\mathcal{F}_\bullet$ is often referred to as Faermann estimator. In BEM, it is the only known estimator which is reliable and efficient (without further assumptions as, e.g. the saturation assumption [39, Section 1]). Obviously, one could replace the residual estimator η_ℓ in Algorithm 3.3 by $\mathcal{F}_\ell$. However, due to the lack of an h-weighting factor, it is unclear whether the reduction property (E2) of Section 4.2 is satisfied. [24, Theorem 7] proves at least plain convergence of $\mathcal{F}_\ell$ even for $f\in H^{1/2}(\Gamma)^D$ if one uses piecewise constants on affine triangulations of Γ as ansatz space. The proof immediately extends to our current situation, where the assumptions (M1)–(M5), (R2)–(R3), and (S1)–(S2) are employed. The key ingredient is the construction of an equivalent mesh-size function $\widetilde{h}_\bullet\in L^\infty(\Gamma)$ which is contractive on each element which touches a refined element, i.e. there exists a uniform constant $0<\rho_{\rm ctr}<1$ such that

$$\widetilde{h}_\circ|_T \le \rho_{\rm ctr}\widetilde{h}_\bullet|_T \quad \text{for all } \mathcal{T}_\circ\in\texttt{refine}(\mathcal{T}_\bullet) \text{ and all } T\in\Pi_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ). \tag{69}$$

The existence of such a mesh-size function is proved in [6, Section 8.7] for shape-regular triangular meshes. The proof works verbatim for the present setting.

4.3. Convergence of $\|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)}$

Nestedness (S2) ensures that $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{X}_\ell}$ is a closed subspace of $H^{-1/2}(\Gamma)^D$ and hence admits a unique Galerkin solution $\Phi_\infty \in \mathcal{X}_\infty$. Note that Φ_ℓ is also a Galerkin approximation of Φ_∞. Hence, the Céa lemma (36) with ϕ replaced by Φ_∞ proves that $\|\Phi_\infty - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \to 0$ as $\ell \to \infty$. In particular, we obtain that $\lim_{\ell\to\infty} \|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$.

4.4. An inverse inequality for $\mathfrak{V}$

In Proposition 4.13, we establish an inverse inequality for the single-layer operator $\mathfrak{V}$. Throughout this section, neither the ellipticity of $\mathfrak{P}$ nor the ellipticity of $\mathfrak{V}$ are exploited (and we can drop these assumption here). Indeed, it is sufficient to assume that $\mathfrak{P}$ is only coercive. Then, the definitions and properties presented in Section 2.3 remain valid; see [19, page 119]. For the Laplace operator $\mathfrak{P} = -\Delta$, an inverse estimate as in Proposition 4.13 was already proved in [3, Theorem 3.1] for shape-regular triangulations of a polyhedral boundary Γ. Independently, [4] derived a similar result for globally smooth Γ and arbitrary self-adjoint and elliptic boundary integral operators. In [5, Theorem 3.1], [3, Theorem 3.1] is generalized to piecewise polynomial ansatz functions on shape-regular curvilinear triangulations. In particular, our Proposition 4.13 does not only extend these results to arbitrary general meshes as in Section 3.1, but is also completely novel for, e.g. linear elasticity. The proof follows the lines of [5, Section 4]. We start with the following lemma, which was proved in [40, Theorem 4.1] on shape-regular triangulations. With Lemma 4.6, the proof immediately extends to our situation; see also [17, Lemma 5.3.11].

Lemma 4.11: *For $\mathcal{T}_\bullet \in \mathbb{T}$, let $\mathcal{P}^0(\mathcal{T}_\bullet)^D \subset L^2(\Gamma)^D$ be the set of all functions whose D components are $\mathcal{T}_\bullet$-piecewise constant functions on Γ. Let $P_\bullet : L^2(\Gamma)^D \to \mathcal{P}^0(\mathcal{T}_\bullet)^D$ be the corresponding L^2-projection. Then,* (M1) *and* (M3) *imply for arbitrary $0 < \sigma < 1$ the existence of a constant $C > 0$ such that*

$$\|(1 - P_\bullet)\psi\|_{H^{-\sigma}(\Gamma)} \le C\|h_\bullet^\sigma \psi\|_{L^2(\Gamma)} \quad \textit{for all } \psi \in L^2(\Gamma). \tag{70}$$

The constant C depends only on the dimension D, the boundary Γ, σ, and the constants from (M3).

In contrast to [5], we cannot use the Caccioppoli-type inequality from [41, Lemma 5.7.1] which is only shown for the Poisson problem there. Therefore, we prove the following generalization. For an open set $O \subset \mathbb{R}^d$ and an arbitrary $u \in H^2(O)$, we abbreviate $|u|_{H^1(O)} := \|\nabla u\|_{L^2(O)}$ and $|u|_{H^2(O)} := \left(\sum_{i=1}^d |\partial_i u|^2_{H^1(O)}\right)^{1/2}$.

Lemma 4.12: *Let $r > 0$, $x \in \mathbb{R}^d$, and $u \in H^1(B_{2r}(x))^D$ be a weak solution of $\mathfrak{P}u = 0$. Then, $u|_{B_r(x)} \in C^\infty(B_r(x))^D$ and there exists a constant $C > 0$ such that*

$$|u|_{H^2(B_r(x))} \le C\left(\|u\|_{L^2(B_{2r}(x))} + \frac{1 + r + r^2}{r}|u|_{H^1(B_{2r}(x))}\right). \tag{71}$$

The constant C depends only on the dimensions d, D, and the coefficients of the partial differential operator $\mathfrak{P}$.

Proof: By [19, Theorem 4.16], there holds that $u|_{B_{3r/2}(x)} \in H^k(B_{3r/2}(x))^D$ for all $k \in \mathbb{N}_0$, and the Sobolev embedding theorem proves that $u|_{B_{3r/2}(x)} \in C^\infty(B_{3r/2}(x))^D$. In particular, u is a strong solution of $\mathfrak{P}u = 0$ on $B_{3r/2}(x)$. To prove (71), let $\lambda \in \mathbb{R}^D$ be an arbitrary constant vector, and define $\widetilde{u} := u \circ \varphi$ with the affine bijection $\varphi : B_{3/2}(0) \to B_{3r/2}(x)$, $\varphi(\widetilde{y}) = r\widetilde{y} + x$ for $\widetilde{y} \in B_{3/2}(0)$. Since the coefficients of $\mathfrak{P}$ are constant and u is a strong solution, there holds for all $\widetilde{y} \in B_{3/2}(0)$ with $y := \varphi(\widetilde{y})$ that

$$
\begin{aligned}
-\sum_{i=1}^{d}\sum_{i'=1}^{d} \partial_i(A_{ii'}\partial_{i'}(\widetilde{u}-\lambda))(\widetilde{y}) &= -\sum_{i=1}^{d}\sum_{i'=1}^{d} \partial_i(A_{ii'}\partial_{i'}(u-\lambda))(y)\, r^2 \\
&= -r^2\left(\sum_{i=1}^{d} b_i\partial_i(u-\lambda)(y) + c(u-\lambda)(y) + c\,\lambda\right).
\end{aligned} \tag{72}
$$

We define the right-hand side as $\widetilde{f} \in C^\infty(B_{3/2}(0))$, i.e.

$$
\widetilde{f}(\widetilde{y}) := -r^2\left(\sum_{i=1}^{d} b_i\partial_i(u-\lambda)(\varphi(\widetilde{y})) + c(u-\lambda)(\varphi(\widetilde{y})) + c\,\lambda\right).
$$

This shows that $\widetilde{u}-\lambda$ is a strong (and thus weak) solution of a coercive system of second-order PDEs with smooth coefficients and smooth right-hand side. The application of [19, Theorem 4.16] yields the existence of a constant $C_1 > 0$, which depends only on d, D, and the coefficients of the matrices $A_{ii'}$, such that

$$
|\widetilde{u}-\lambda|_{H^2(B_1(0))} \le C_1\left(\|\widetilde{u}-\lambda\|_{H^1(B_{3/2}(0))} + \|\widetilde{f}\|_{L^2(B_{3/2}(0))}\right). \tag{73}
$$

Standard scaling arguments prove that

$$
\begin{aligned}
|\widetilde{u}-\lambda|_{H^2(B_1(0))} &\simeq \frac{r^2}{r^{d/2}}\,|u|_{H^2(B_r(x))}, \\
\|\widetilde{u}-\lambda\|_{L^2(B_{3/2}(0))} &\simeq \frac{1}{r^{d/2}}\,\|u-\lambda\|_{L^2(B_{3r/2}(x))}, \\
|\widetilde{u}-\lambda|_{H^1(B_{3/2}(0))} &\simeq \frac{r}{r^{d/2}}|u|_{H^1(B_{3r/2}(x))}, \\
\|\widetilde{f}\|_{L^2(B_{3/2}(0))} &\lesssim \frac{r^2}{r^{d/2}}|u|_{H^1(B_{3r/2}(x))} + \frac{r^2}{r^{d/2}}\|u-\lambda\|_{L^2(B_{3r/2}(x))} + r^2|\lambda|.
\end{aligned}
$$

Plugging this into (73), we obtain that

$$
|u|_{H^2(B_r(x))} \lesssim \left(\frac{1+r^2}{r^2}\|u-\lambda\|_{L^2(B_{3r/2}(x))} + \frac{1+r}{r}|u|_{H^1(B_{3r/2}(x))} + r^{d/2}|\lambda|\right). \tag{74}
$$

We choose λ as the integral mean $\lambda := \int_{B_{3r/2}(x)} u(y)dy/|B_{3r/2}(x)|$. The Cauchy–Schwarz inequality implies that

$$
|\lambda| \lesssim \|u\|_{L^1(B_{3r/2}(x))}/|B_{3r/2}(x)| \le \|u\|_{L^2(B_{3r/2}(x))}/|B_{3r/2}(x)|^{1/2} \simeq r^{-d/2}\|u\|_{L^2(B_{3r/2}(x))}.
$$

Using this and the Poincaré inequality in (74), we see that

$$
|u|_{H^2(B_r(x))} \lesssim \left(\|u\|_{L^2(B_{3r/2}(x))} + \frac{1+r+r^2}{r}|u|_{H^1(B_{3r/2}(x))}\right).
$$

Together with the fact that $B_{3r/2}(x) \subset B_{2r}(x)$, this concludes the proof. ■

For the proof of the next proposition, we need the linear and continuous *single-layer potential* from [19, Theorem 6.11]

$$\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(U)^D, \tag{75}$$

where U is an arbitrary bounded domain with $\Gamma \subset U$. The single-layer operator $\mathfrak{V}$ is just the trace of $\widetilde{\mathfrak{V}}$, i.e.

$$\mathfrak{V} = \widetilde{\mathfrak{V}}(\cdot)|_\Gamma : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D; \tag{76}$$

see [19, page 219–220]. Indeed, for $\psi \in L^\infty(\Gamma)$, [19, page 201–202] states the following integral representation:

$$(\widetilde{\mathfrak{V}}\psi)(x) = \int_\Gamma G(x-y)\psi(y)\,\mathrm{d}y \quad \text{for all } x \in U. \tag{77}$$

Proposition 4.13: *Suppose* (M1)–(M5). *For $\mathcal{T}_\bullet \in \mathbb{T}$, let $w_\bullet \in L^\infty(\Gamma)$ be a weight function which satisfies for some $\alpha > 0$ and all $T \in \mathcal{T}_\bullet$ that*

$$\|w_\bullet\|_{L^\infty(T)} \le \alpha w_\bullet(x) \quad \textit{for almost all } x \in \pi_\bullet(T). \tag{78}$$

Then, there exists a constant $C_{\mathrm{inv},\mathfrak{V}} > 0$ such that for all $\psi \in L^2(\Gamma)^D$, it holds that

$$\|w_\bullet \nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} \le C_{\mathrm{inv},\mathfrak{V}}\left(\|w_\bullet/h_\bullet^{1/2}\|_{L^\infty(\Gamma)}\|\psi\|_{H^{-1/2}(\Gamma)} + \|w_\bullet\psi\|_{L^2(\Gamma)}\right). \tag{79}$$

The constant $C_{\mathrm{inv},\mathfrak{V}}$ depends only on (M1)–(M5), Γ, *the coefficients of $\mathfrak{P}$, and the admissibility constant α. The particular choice $w_\bullet = h_\bullet^{1/2}$ shows that*

$$\|h_\bullet^{1/2}\nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} \le C_{\mathrm{inv},\mathfrak{V}}\left(\|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)}\right). \tag{80}$$

Proof: The proof works essentially as in [5, Section 4]. Therefore, we mainly emphasize the differences and refer to [17, Proposition 5.3.15] for further details.

By (M4) and with the abbreviation $\delta_1(T) := \mathrm{diam}(T)/(2C_{\mathrm{cent}})$ and $U_T := B_{\delta_1(T)}(T)$, there holds for all $T \in \mathcal{T}_\bullet$ that $U_T \cap \Gamma \subset B_{2\delta_1(T)}(T) \cap \Gamma \subset \pi_\bullet(T)$. This provides us with an open covering of $\Gamma \subset \bigcup_{T\in\mathcal{T}_\bullet} U_T$. We show that this is even locally finite in the sense that there exists a constant $C > 0$ with $\#\{T \in \mathcal{T}_\bullet : x \in U_T\} \le C$ for all $x \in \mathbb{R}^d$: Let $x \in \mathbb{R}^d$. Clearly, we may assume that $x \in \bigcup_{T\in\mathcal{T}_\bullet} U_T$. Choose $T_0 \in \mathcal{T}_\bullet$ with $x \in U_{T_0}$ such that $\delta_1(T_0)$ is minimal, and let $x_0 \in T_0$ with $|x - x_0| < \delta_1(T_0)$. If $T \in \mathcal{T}_\bullet$ with $x \in U_T$, the triangle inequality yields that $\mathrm{dist}(\{x_0\}, T) < 2\delta_1(T)$. By choice of $\delta_1(T)$, (M4) hence yields that $x_0 \in \pi_\bullet(T)$. Thus, $\{T \in \mathcal{T}_\bullet : x \in U_T\} \subseteq \{T \in \mathcal{T}_\bullet : x_0 \in \pi_\bullet(T)\}$, and (M1) implies that

$$\#\{T \in \mathcal{T}_\bullet : x \in U_T\} \le \#\{T \in \mathcal{T}_\bullet : x_0 \in \pi_\bullet(T)\} \le C_{\mathrm{patch}}^2. \tag{81}$$

We fix (independently of $\mathcal{T}_\bullet$) a bounded domain $U \subset \mathbb{R}^d$ with $U_T \subset U$ for all $T \in \mathcal{T}_\bullet$. We define for $T \in \mathcal{T}_\bullet$ the near-field and the far-field of $u_{\mathfrak{V}} := \widetilde{\mathfrak{V}}\psi$ by

$$u_{\mathfrak{V},T}^{\mathrm{near}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\cap U_T}) \quad \text{and} \quad u_{\mathfrak{V},T}^{\mathrm{far}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\setminus U_T}). \tag{82}$$

In the following five steps, the near-field and the far-field are estimated separately. The first two steps deal with the near-field, whereas the last three steps deal with the far-field.

Step 1: As in [5, Lemma 4.1], one shows that for all $T \in \mathcal{T}_\bullet$, all $\mathcal{T}_\bullet$-piecewise (componentwise) constant functions $\Psi_\bullet^T \in \mathcal{P}^0(\mathcal{T}_\bullet)^D$ with $\mathrm{supp}(\Psi_\bullet^T) \subseteq \pi_\bullet(T)$ satisfy that

$$\|\widetilde{\mathfrak{V}}\Psi_\bullet^T\|_{H^1(U_T)} \lesssim \|h_\bullet^{1/2}\Psi_\bullet^T\|_{L^2(\pi_\bullet(T))}. \tag{83}$$

The proof of [5] uses only (81) and the fact that $|\nabla_x G(x-y)| \lesssim |x-y|^{-d+1}$ $(x,y) \in U \times \Gamma$ with $x \neq y$ for the Laplacian fundamental solution G. However, according to [19, Theorem 6.3 and Corollary

6.5] this fact is also valid for general coercive second-order PDEs with C^∞ coefficients. Moreover, [5] bounds only the H^1-seminorm in (83), but the L^2-norm can be bounded similarly due to $|G(x-y)| \lesssim \max\{|\log|x-y||, |x-y|^{-d+2}\} \lesssim |x-y|^{-d+1}$ (see again [19, Theorem 6.3 and Corollary 6.5]).

Step 2: With Step 1, one shows as in [5, Proposition 4.2] that $u^{\rm near}_{\mathfrak{V},T} \in H^1(U)$ and $u^{\rm near}_{\mathfrak{V},T}|_\Gamma \in H^1(\Gamma)$ with

$$\sum_{T\in\mathcal{T}_\bullet} \|w_\bullet \nabla_\Gamma u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|u^{\rm near}_{\mathfrak{V},T}\|^2_{H^1(U_T)} \lesssim \|w_\bullet \psi\|^2_{L^2(\Gamma)}. \tag{84}$$

In the proof, one applies the stability of $\mathfrak{V} : L^2(\Gamma)^D \to H^1(\Gamma)^D$ (see (21)) and $\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(U)^D$ (see (75)). Moreover, the approximation property (70) is exploited by splitting $\psi\chi_{\Gamma\cap U_T} = P_\bullet(\psi\chi_{\Gamma\cap U_T}) + (1-P_\bullet)(\psi\chi_{\Gamma\cap U_T})$ and choosing $\Psi^T_\bullet := P_\bullet(\psi\chi_{\gamma\cap U_T})$. Note that [5] only proves (84) with $|u^{\rm near}_{\mathfrak{V},T}|^2_{H^1(U_T)}$ instead of $\|u^{\rm near}_{\mathfrak{V},T}\|^2_{H^1(U_T)}$.

Step 3: We consider the far-field. We set $\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$. According to [19, Theorem 6.11], for all $T \in \mathcal{T}_\bullet$, the potential $u^{\rm far}_{\mathfrak{V},T}$ is a solution of the transmission problem

$$\begin{aligned} \mathfrak{P} u^{\rm far}_{\mathfrak{V},T} &= 0 \quad \text{on } \Omega \cup \Omega^{\rm ext}, \\ [u^{\rm far}_{\mathfrak{V},T}]_\Gamma &= 0 \quad \text{in } H^{1/2}(\Gamma)^D, \\ [\mathfrak{D}_\nu u^{\rm far}_{\mathfrak{V},T}]_\Gamma &= -\psi\chi_{\Gamma\setminus U_T} \quad \text{in } H^{-1/2}(\Gamma)^D, \end{aligned}$$

where $[\cdot]_\Gamma$ and $[\mathfrak{D}_\nu(\cdot)]_\Gamma$ denote the jump of the traces and the conormal derivatives respectively (see [19, page 117] for a precise definition) across the boundary Γ. Twofold integration by parts that uses these jump conditions shows that $\mathfrak{P} u^{\rm far}_{\mathfrak{V},T} = 0$ weakly on U_T. Since $B_{\delta_1(T)}(x) \subseteq U_T$ for all $x \in T$, Lemma 4.12 shows that $u^{\rm far}_{\mathfrak{V},T} \in C^\infty(B_{\delta_1(T)/2}(x))$ with

$$|u^{\rm far}_{\mathfrak{V},T}|_{H^2(B_{\delta_1(T)/2}(x))} \lesssim \|u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{\delta_1(T)}(x))} + \operatorname{diam}(T)^{-1} |u^{\rm far}_{\mathfrak{V},T}|_{H^1(B_{\delta_1(T)}(x))}. \tag{85}$$

Note that [5] proves (85) even without the term $\|u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{\delta_1(T)}(x))}$. Indeed, since the kernel of the Laplace operator contains all constants, [5] employs a Poincaré inequality to bound $\|u^{\rm far}_{\mathfrak{V},T}\|_{L^2(B_{\delta_1(T)}(x))}$ by $\operatorname{diam}(T)^{-1}|u^{\rm far}_{\mathfrak{V},T}|_{H^1(B_{\delta_1(T)}(x))}$.

Step 4: With inequality (85) at hand, one can prove the following local far-field bound for the single-layer potential $\widetilde{\mathfrak{V}}$

$$\|h_\bullet^{1/2} \nabla_\Gamma u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)} \le \|h_\bullet^{1/2} \nabla u^{\rm far}_{\mathfrak{V},T}\|_{L^2(T)} \lesssim \|u^{\rm far}_{\mathfrak{V},T}\|_{H^1(U_T)}. \tag{86}$$

The proof works as in [5, Lemma 4.4] and relies on a standard trace inequality on Ω, the Caccioppoli inequality (85) as well as the Besicovitch covering theorem. Note that in [5], the estimates (85) and thus (86) even hold without the L^2-norm of $u^{\rm far}_{\mathfrak{V},T}$, since the Laplace problem is considered.

Step 5: Finally, (81), (82), (84), (86), and the stability of $\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(U)^D$ (see (75)) easily lead to the far-field bound for $\widetilde{\mathfrak{V}}$

$$\begin{aligned} \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet \nabla_\Gamma u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} &\le \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet \nabla u^{\rm far}_{\mathfrak{V},T}\|^2_{L^2(T)} \\ &\lesssim \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{H^{-1/2}(\Gamma)} + \|w_\bullet\psi\|^2_{L^2(\Gamma)}. \end{aligned} \tag{87}$$

For the simple proof, we refer to [5, Proposition 4.5]. By definition (82), (87) together with (84) from Step 2 concludes the proof. ∎

[5] does not only treat the single-layer operator $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ but also derives similar inverse estimates as in (79) for the double-layer operator, the adjoint double-layer operator, and

the hyper-singular operator. With similar techniques as in Proposition 4.13, we will also generalize this result in Appendix B. However, we will indeed only need the inverse estimate (80) for the single-layer operator in the remainder of the paper.

4.5. Stability on non-refined elements (E1)

We show that the assumptions (M1)–(M5) and (S1)–(S2) imply stability (E1), i.e. the existence of $C_{\rm stab} \geq 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$, and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| \leq C_{\rm stab}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{88}$$

In Section 4.6, we will fix the constant C_ϱ for $\varrho_{\bullet,\circ}$ defined in (E1) such that $C_{\rm stab} \leq C_\varrho$. The reverse triangle inequality and the fact that $h_\circ = h_\bullet$ on $\omega := \bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$ prove that

$$\begin{aligned}|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| &= \big|\|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\circ)\|_{L^2(\omega)} - \|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\omega)}\big| \\ &\leq \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\omega)} \\ &\leq \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}.\end{aligned}$$

(S2) shows that $\Phi_\circ - \Phi_\bullet \in \mathcal{X}_\circ$. Therefore, the inverse inequalities from (S1) and (80) are applicable, which implies (88). The constant $C_{\rm stab}$ depends only on d, D, Γ, the coefficients of $\mathfrak{P}$, and the constants from (M1)–(M5) and (S1).

4.6. Reduction on refined elements (E2)

We show that the assumptions (M1)–(M5), (R2)–(R3), and (S1)–(S2) imply reduction on refined elements (E2), i.e. the existence of $C_{\rm red} \geq 1$ and $0 < \rho_{\rm red} < 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \leq \rho_{\rm red}\,\eta_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)^2 + C_{\rm red}\|\Phi_\circ - \Phi_\bullet\|^2_{H^{-1/2}(\Gamma)}. \tag{89}$$

With this, we can fix the constant for $\varrho_{\bullet,\circ}$ defined in (E1) as

$$C_\varrho := \max\{C_{\rm stab}, C_{\rm red}^{1/2}\}. \tag{90}$$

Let $\omega := \bigcup(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)$. First, we apply the triangle inequality

$$\begin{aligned}\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) &= \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\circ)\|_{L^2(\omega)} \\ &\leq \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\omega)} + \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\omega)}.\end{aligned}$$

Clearly, (R2)–(R3) show that $\omega = \bigcup(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) = \bigcup(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$ and $h_\circ \leq \rho_{\rm son}^{1/(d-1)} h_\bullet$ on ω. Thus we can proceed the estimate as follows:

$$\begin{aligned}\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) &\leq \rho_{\rm son}^{1/(2d-2)}\|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\omega)} + \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\omega)} \\ &= \rho_{\rm son}^{1/(2d-2)}\eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) + \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\omega)}.\end{aligned}$$

Since $\Phi_\bullet \in \mathcal{X}_\bullet \subseteq \mathcal{X}_\circ$ according to (S2), we can apply the inverse estimates (S1) and (80). Together with the Young inequality, we derive for arbitrary $\delta > 0$ that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \leq (1+\delta)\rho_{\rm son}^{1/(d-1)}\eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)^2 + (1+\delta^{-1})C^2_{\mathrm{inv},\mathfrak{V}}(1 + C_{\rm inv})^2\|\Phi_\circ - \Phi_\bullet\|^2_{H^{-1/2}(\Gamma)}.$$

Choosing $\delta > 0$ sufficiently small, we obtain (89). The constant $C_{\rm red}$ depends only on d, D, Γ, the coefficients of $\mathfrak{P}$, and the constants from (M1)–(M5), (R2)–(R3), and (S1).

4.7. General quasi-orthogonality (E3)

According to [6, Section 4.3], Section 4.3, Section 4.5, and Section 4.6 already imply estimator convergence $\lim_{\ell\to\infty}\eta_\ell = 0$. Therefore, reliability (42) implies error convergence $\lim_{\ell\to\infty}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$. In particular, we obtain that $\phi \in \mathcal{X}_\infty = \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell}$. Recall that we have already fixed the constant C_ϱ in (90). We introduce the *principal part* of $\mathfrak{P}$ as the corresponding partial differential operator without lower-order terms

$$\mathfrak{P}_0 v := -\sum_{i=1}^{d}\sum_{i'=1}^{d}\partial_i(A_{ii'}\partial_{i'}v). \tag{91}$$

According to [19, Lemma 4.5], the principal part is coercive on $H_0^1(\Omega)^D$. We denote its corresponding single-layer operator which can be defined as in Section 2.3 by $\mathfrak{V}_0 : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$. Our assumption $A_{ii'}^\top = \overline{A_{i'i}}$ easily implies that $\mathfrak{V}_0$ is self-adjoint; see, e.g. [19, page 218]. With (76) and (A14) below, we particularly see the stability of $\mathfrak{V} - \mathfrak{V}_0 : H^{-1/2}(\Gamma)^D \to H^{1-\varepsilon}(\Gamma)^D$ for all $\varepsilon > 0$. Thus the Rellich compactness theorem [19, Theorem 3.27] implies that $\mathfrak{V} - \mathfrak{V}_0 : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ is compact. This yields that $\mathfrak{V}$ is an elliptic operator which can be written as the sum of a self-adjoint operator $\mathfrak{V}_0$ plus a compact operator $\mathfrak{V} - \mathfrak{V}_0$. From [27,42], we thus derive the general quasi-orthogonality (E3) (see also [17, Section 4.4.3] for all details), i.e. the existence of

$$0 \le \varepsilon_{\rm qo} < \sup_{\delta>0}\frac{1-(1+\delta)(1-(1-\rho_{\rm red})\theta)}{2+\delta^{-1}} \tag{92}$$

and $C_{\rm qo} \ge 1$ such that

$$\sum_{j=\ell}^{\ell+N}(C_\varrho\|\Phi_{j+1}-\Phi_j\|_{H^{-1/2}(\Gamma)}^2 - \varepsilon_{\rm qo}\eta_j^2) \le C_{\rm qo}\eta_\ell^2 \quad \text{for all } \ell, N \in \mathbb{N}_0. \tag{93}$$

Remark 4.14: If the sesquilinear form $(\mathfrak{V}\,\cdot\,;\,\cdot)$ is Hermitian, (93) follows from the Pythagoras theorem $\|\phi - \Phi_j\|_{\mathfrak{V}}^2 + \|\Phi_{j+1} - \Phi_j\|_{\mathfrak{V}}^2 = \|\phi - \Phi_j\|_{\mathfrak{V}}^2$ and norm equivalence

$$\sum_{j=\ell}^{\ell+N}\|\Phi_{j+1}-\Phi_j\|_{H^{-1/2}(\Gamma)}^2 \simeq \sum_{j=\ell}^{\ell+N}\|\Phi_{j+1}-\Phi_j\|_{\mathfrak{V}}^2 = \|\phi-\Phi_\ell\|_{\mathfrak{V}}^2 - \|\phi-\Phi_{\ell+N}\|_{\mathfrak{V}}^2$$
$$\lesssim \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}^2.$$

Together with reliability (42), this proves (93) even for $\varepsilon_{\rm qo} = 0$, and $C_{\rm qo}$ is independent of the sequence $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$.

4.8. Discrete reliability (E4)

The proof of (E4) is inspired by [3, Proposition 5.3] which considers piecewise constants on shape-regular triangulations as ansatz space. Under the assumptions (M1)–(M5), (32), and (S1)–(S6), we show that there exist $C_{\rm drel}, C_{\rm ref} \ge 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, the subset

$$\mathcal{R}_{\bullet,\circ} := \Pi_\bullet^{q_{\rm supp}+q_{\rm loc}+2}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \tag{94}$$

satisfies that

$$C_\varrho\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\rm drel}\,\eta_\bullet(\mathcal{R}_{\bullet,\circ}), \quad \mathcal{T}_\bullet \setminus \mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet,\circ}, \text{ and } \#\mathcal{R}_{\bullet,\circ} \le C_{\rm ref}(\#\mathcal{T}_\circ - \#\mathcal{T}_\bullet).$$

The last two properties are obvious with $C_{\rm ref} = C_{\rm patch}^{q_{\rm supp}+q_{\rm loc}+2}$ by validity of (M1) and (32). The first estimate is proved in three steps:

Step 1: For $\mathcal{S}_1 := \mathcal{T}_\bullet \cap \mathcal{T}_\circ$, let $J_{\bullet,\mathcal{S}_1}$ be the corresponding projection operator from (S5)–(S6). Ellipticity (22), nestedness (S2) of the ansatz spaces, and the definition (34) of the Galerkin approximations yield that

$$\begin{aligned}\|\Phi_\circ - \Phi_\bullet\|^2_{H^{-1/2}(\Gamma)} &\lesssim \mathrm{Re}\,(\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\,;\,\Phi_\circ - \Phi_\bullet)_{L^2(\Gamma)}\\ &= \mathrm{Re}\,(\mathfrak{V}(\phi - \Phi_\bullet)\,;\,(1 - J_{\bullet,\mathcal{S}_1})(\Phi_\circ - \Phi_\bullet))_{L^2(\Gamma)}.\end{aligned}$$

(S3) shows that $(\Phi_\circ - \Phi_\bullet)|_{\pi_\bullet^{\mathrm{proj}}(T)} \in \left\{\Psi_\bullet|_{\pi_\bullet^{\mathrm{proj}}(T)}\,:\,\Psi_\bullet \in \mathcal{X}_\bullet\right\}$ for any $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\mathrm{loc}}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$. Moreover, one easily sees that

$$\Pi_\bullet^{q_{\mathrm{loc}}}(T) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ = \mathcal{S}_1 \quad \text{for all } T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\mathrm{loc}}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ). \tag{95}$$

Hence, the local projection property (S5) of $J_{\bullet,\mathcal{S}_1}$ is applicable and proves that $J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet) = \Phi_\circ - \Phi_\bullet$ on $\Gamma \setminus \pi_\bullet^{q_{\mathrm{loc}}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$. With $\mathcal{S}_2 := \Pi_\bullet^{q_{\mathrm{loc}}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$, Lemma A.1 provides a smooth cut-off function $\chi := \widetilde{\chi}_{\mathcal{S}_2} \in H^1(\Gamma)$ with $0 \le \chi \le 1$ such that $\chi = 1$ on $\bigcup \mathcal{S}_2$, $\chi = 0$ on $\Gamma \setminus \pi_\bullet(\mathcal{S}_2)$, and $|\nabla_\Gamma \chi| \lesssim h_\bullet^{-1}$, where the hidden constant depends only on d and (M1)–(M4). This leads to

$$\|\Phi_\circ - \Phi_\bullet\|^2_{H^{-1/2}(\Gamma)} \lesssim \mathrm{Re}\,(\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\,;\,(1 - J_{\bullet,\mathcal{S}_1})(\Phi_\circ - \Phi_\bullet))_{L^2(\Gamma)}. \tag{96}$$

We bound the two terms $\mathrm{I} := \mathrm{Re}\,(\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\,;\,\Phi_\circ - \Phi_\bullet)_{L^2(\Gamma)}$ and $\mathrm{II} := \mathrm{Re}\,(\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\,;\,J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet))_{L^2(\Gamma)}$ separately. Since $H^{-1/2}(\Gamma)^D$ is the dual space of $H^{1/2}(\Gamma)^D$, it holds that

$$\mathrm{I} \le \|\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{97}$$

The Cauchy–Schwarz inequality shows that

$$\mathrm{II} \le \|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|h_\bullet^{1/2}J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}.$$

Since $J_{\bullet,\mathcal{S}_1} : L^2(\Gamma)^D \to \left\{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet \setminus \mathcal{S}_1)} = 0\right\}$, it holds that $\mathrm{supp}(J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)) \subseteq \bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$. This together with the fact that $h_\bullet = h_\circ$ on $\bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$, the local L^2-stability (S6) and (M1)–(M3) implies that

$$\begin{aligned}\mathrm{II} &= \|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\,\left\|h_\circ^{1/2}J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\right\|_{L^2(\bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ))}\\ &\lesssim \|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|h_\circ^{1/2}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}.\end{aligned}$$

With the inverse inequality (S1) applied to $\Phi_\circ - \Phi_\bullet \in \mathcal{X}_\circ$ (see (S2)), the latter estimate implies that

$$\mathrm{II} \lesssim \|h_\bullet^{-1/2}\chi\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{98}$$

Plugging (97)–(98) into (96) shows that

$$\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \lesssim \|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)} + \|\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}. \tag{99}$$

Step 2: Next, we deal with the first summand of (99). With $\mathrm{supp}(\chi) \subseteq \pi_\bullet^{q_{\mathrm{loc}}+1}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$ and $0 \le \chi \le 1$, this implies that

$$\|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)} \le \|h_\bullet^{-1/2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2\left(\pi_\bullet^{q_{\mathrm{loc}}+1}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)\right)}. \tag{100}$$

By Galerkin orthogonality (35), we see that $\mathfrak{V}(\phi - \Phi_\bullet)$ is L^2-orthogonal to all functions of $\mathcal{X}_\bullet$ which includes in particular the functions $\Psi_{\bullet,T,j}$ from (S4). Hence, we can apply Proposition 4.9. Together with (M1)–(M3) and recalling (94), (100) proves that

$$\|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi-\Phi_\bullet)\|_{L^2(\Gamma)} \lesssim \|h_\bullet^{1/2}\nabla_\Gamma \mathfrak{V}(\phi-\Phi_\bullet)\|_{L^2\left(\pi_\bullet^{q_{\rm supp}+q_{\rm loc}+2}(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)\right)} = \eta_\bullet(\mathcal{R}_{\bullet,\circ}).$$

Step 3: It remains to consider the second summand of (99). Lemma 4.5 in conjunction with shape-regularity (M3) implies that

$$\|\chi\,\mathfrak{V}(\phi-\Phi_\bullet)\|_{H^{1/2}(\Gamma)}^2 \lesssim \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |\chi\,\mathfrak{V}(\phi-\Phi_\bullet)|_{H^{1/2}(T\cup T')}^2 + \|h_\bullet^{-1/2}\chi\,\mathfrak{V}(\phi-\Phi_\bullet)\|_{L^2(\Gamma)}.$$

We have already dealt with the second summand in Step 2 (see (100)). For the first one, we fix again some $z(T,T')\in T\cap T'$ for any $T\in\mathcal{T}_\bullet, T'\in\Pi_\bullet(T)$. (M1)–(M3) and (M5) show that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |\chi\,\mathfrak{V}(\phi-\Phi_\bullet)|_{H^{1/2}(T\cup T')}^2 \\
&\quad\le \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |\chi\,\mathfrak{V}(\phi-\Phi_\bullet)|_{H^{1/2}(\pi_\bullet(z(T,T')))}^2 \\
&\quad\le \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} \|h_\bullet^{1/2}\nabla_\Gamma(\chi\,\mathfrak{V}(\phi-\Phi_\bullet))\|_{L^2(\pi_\bullet(z(T,T')))}^2 \lesssim \|h_\bullet^{1/2}\nabla_\Gamma(\chi\,\mathfrak{V}(\phi-\Phi_\bullet))\|_{L^2(\Gamma)}^2.
\end{aligned}$$

With the product rule and (A2), we continue our estimate

$$\|\chi\,\mathfrak{V}(\phi-\Phi_\bullet)\|_{H^{1/2}(\Gamma)}^2 \lesssim \|h_\bullet^{-1/2}\mathfrak{V}(\phi-\Phi_\bullet))\|_{L^2(\mathrm{supp}(\chi))}^2 + \|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi-\Phi_\bullet))\|_{L^2(\mathrm{supp}(\chi))}^2.$$

Note that we have already dealt with the first summand in Step 2 (see (100)). Finally, $\mathrm{supp}(\chi)\subseteq \pi_\bullet^{q_{\rm loc}+1}(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)$ and $\Pi_\bullet^{q_{\rm loc}+1}(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)\subseteq\mathcal{R}_{\bullet,\circ}$ (see (94)) prove for the second one that

$$\|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi-\Phi_\bullet))\|_{L^2(\mathrm{supp}(\chi))}^2 \le \eta_\bullet(\mathcal{R}_{\bullet,\circ})^2.$$

With this, we conclude the proof of discrete reliability (E4). The constant $C_{\rm drel}$ depends only on C_ϱ, d, D, Γ, and the constants from (M1)–(M5) and (S1)–(S6).

Notes

1. For $\widehat{\omega}\subseteq\mathbb{R}^{d-1}$ and $\omega\subseteq\mathbb{R}^d$, a mapping $\gamma:\widehat{\omega}\to\omega$ is bi-Lipschitz if it is bijective and γ as well as its inverse γ^{-1} are Lipschitz continuous.
2. A compact Lipschitz domain is the closure of a bounded Lipschitz domain. For $d=2$, it is the finite union of compact intervals with non-empty interior.
3. We use the convention $\mathrm{dist}(T,\emptyset):=\mathrm{diam}(\Gamma)$.

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Appendices

Appendix 1. Smooth characteristic functions

The following lemma provides a smooth cut-off function as used in the proof of discrete reliability (E4); see Section 4.8. For regular simplicial meshes as in Section 3.7, such functions can easily be constructed by means of hat functions; see [5, Section 5.3]. For the present abstract setting, the construction is more technical.

Lemma A.1: *Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{S} \subseteq \mathcal{T}_\bullet$. Suppose* (M1)–(M4). *Then there exists a function $\widetilde{\chi}_{\mathcal{S}} \in H^1(\Gamma)$ such that for almost all $x \in \Gamma$*

$$\widetilde{\chi}_{\mathcal{S}}(x) = 1 \quad \text{if } x \in \bigcup \mathcal{S}, \tag{A.1a}$$

$$0 \le \widetilde{\chi}_{\mathcal{S}}(x) \le 1 \quad \text{if } x \in \pi_\bullet(\mathcal{S}), \tag{A.1b}$$

$$\widetilde{\chi}_{\mathcal{S}}(x) = 0 \quad \text{if } x \notin \pi_\bullet(\mathcal{S}). \tag{A.1c}$$

Further, there exists a constant $C > 0$ such that

$$|\nabla_\Gamma \widetilde{\chi}_{\mathcal{S}}(x)| \le C h_\bullet(x)^{-1} \quad \text{for almost all } x \in \Gamma. \tag{A2}$$

The constant C depends only on the dimension d and the constants from (M1)–(M4).

Proof: In the following three steps, we will even prove the existence of a function $\widetilde{\chi}_{\mathcal{S}} \in C^\infty(O)$ with an open superset $O \supset \Gamma$ such that the restriction to Γ has the desired properties. With the constants from (M1)–(M2) and (M4), we define for all $T \in \mathcal{T}_\bullet$,

$$\delta_1(T) := \frac{\mathrm{diam}(T)}{2C_{\mathrm{patch}}C_{\mathrm{locuni}}C_{\mathrm{cent}}}, \quad \delta_2(T) := \frac{\mathrm{diam}(T)}{C_{\mathrm{cent}}}, \quad \delta_3(T) := \frac{\mathrm{diam}(T)}{2C_{\mathrm{cent}}}. \tag{A3}$$

Step 1: First, we construct an equivalent smooth mesh-size function $\delta_\bullet \in C^\infty(\mathbb{R}^d)$ with uniformly bounded gradient on Γ. Let $K_1 \in C^\infty(\mathbb{R}^d)$ be a standard mollifier with $0 \le K_1 \le 1$ on $B_1(0)$, $K_1 = 0$ on $\mathbb{R}^d \setminus B_1(0)$, and $\int_{\mathbb{R}^d} K_1\, dx = 1$. For $s > 0$, we set $K_s(\cdot) := K_1(\cdot/s)s^{-d}$. By convolution, we define

$$\delta_\bullet := \sum_{T \in \mathcal{T}_\bullet} \delta_1(T)\, \chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}. \tag{A4}$$

Note that $\mathrm{supp}(\chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}) \subseteq B_{2\delta_2(T)}(T)$ for $T \in \mathcal{T}_\bullet$. Thus (M4) and the choice (A3) of $\delta_2(T)$ yield that $\mathrm{supp}(\chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}) \cap \Gamma \subseteq \pi_\bullet(T)$. Together with (M1)–(M2) and $0 \le \chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)} \le 1$, this implies for the interior $\operatorname{int}(T')$ of any $T' \in \mathcal{T}_\bullet$ that

$$\delta_\bullet|_{\operatorname{int}(T')} \le \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\chi_{\pi_\bullet(T)}|_{\operatorname{int}(T')} = \sum_{T\in\Pi_\bullet(T')} \delta_1(T) \le C_{\mathrm{patch}}C_{\mathrm{locuni}}\,\delta_1(T').$$

Note that the restriction to the interior is indeed necessary since the second term is discontinuous across faces of $\mathcal{T}_\bullet$. However, by continuity of $\delta_\bullet$, this estimate is also satisfied if $\operatorname{int}(T')$ is replaced by T', i.e. $\delta_\bullet|_{T'} \le C_{\mathrm{patch}}C_{\mathrm{locuni}}\delta_1(T')$. The fact that $\chi_{B_{\delta_2(T')}(T')} * K_{\delta_2(T')} = 1$ on T' shows that also the converse estimate is valid. This leads to

$$\frac{\operatorname{diam}(T')}{2C_{\mathrm{patch}}C_{\mathrm{locuni}}C_{\mathrm{cent}}} = \delta_1(T') \le \delta_\bullet|_{T'} \le C_{\mathrm{patch}}C_{\mathrm{locuni}}\,\delta_1(T') = \delta_3(T') \quad \text{for all } T' \in \mathcal{T}_\bullet. \tag{A5}$$

In particular, this proves the existence of an open set $\mathbb{R}^d \supset O \supset \Gamma$ such that $\delta_\bullet > 0$ on O. Finally, we consider the gradient of $\delta_\bullet$ for $x \in \Gamma$. Recall that $\operatorname{supp}(\chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}) \subseteq \pi_\bullet(T)$. Together with the Hölder inequality, $\|\nabla K_s\|_{L^1(\mathbb{R}^d)} \lesssim s^{-1}$, and (M1)–(M2), this proves that

$$\begin{aligned}|\nabla\delta_\bullet(x)| &= \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\,\chi_{\pi_\bullet(T)}(x)\,|\chi_{B_{\delta_2(T)}(T)} * \nabla K_{\delta_2(T)}(x)| \\ &\lesssim \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\,\chi_{\pi_\bullet(T)}(x)\delta_2(T)^{-1} \lesssim 1.\end{aligned} \tag{A6}$$

Step 2: In this step, we construct $\widetilde{\chi}_{\mathcal{S}}$ and prove (A.1a)–(A.1c). For $x \in O$, we define the quasi-convolution

$$\widetilde{\chi}_{\mathcal{S}}(x) := \int_{\mathbb{R}^d} \chi_{\widetilde{S}}(y)\,K_{\delta_\bullet(x)}(x-y)\,\mathrm{d}y, \quad \text{where } \widetilde{S} := \bigcup\big\{B_{\delta_3(T)}(T) \;:\; T \in \mathcal{S}\big\}.$$

Since $\delta_\bullet > 0$ on O, the chain rule shows that any derivative with respect to x of the term $K_{\delta_\bullet(x)}(x-y)$ exists and is continuous at all $(x,y) \in O \times \widetilde{S}$. This yields that $\widetilde{\chi}_{\mathcal{S}} \in C^\infty(O)$ and $\widetilde{\chi}_{\mathcal{S}}|_\Gamma \in H^1(\Gamma)$; see, e.g. [19, page 98–99]. Since $\operatorname{supp}(K_s) = B_s(0)$, it holds that

$$\widetilde{\chi}_{\mathcal{S}}(x) = \int_{B_{\delta_\bullet(x)}(x)} \chi_{\widetilde{S}}(y)\,K_{\delta_\bullet(x)}(x-y)\,\mathrm{d}y. \tag{A7}$$

If $B_{\delta_\bullet(x)}(x) \subseteq \widetilde{S}$ and thus $\chi_{\widetilde{S}}(y) = 1$, the properties of K_1 show that $\widetilde{\chi}_{\mathcal{S}}(x) = 1$. Due to $\delta_\bullet|_{T'} \le \delta_3(T')$ for all $T' \in \mathcal{T}_\bullet$ (which follows from (A5)), this is particularly satisfied if $x \in \bigcup \mathcal{S}$. This proves (A.1a). Moreover, (A7) shows that $0 \le \widetilde{\chi}_{\mathcal{S}}(x) \le 1$ for all $x \in \mathbb{R}^d$ (and hence verifies (A.1b)), and $\widetilde{\chi}_{\mathcal{S}}(x) = 0$ if $B_{\delta_\bullet(x)}(x) \cap \widetilde{S} = \emptyset$. For (A.1c), it thus remains to prove that $x \in \Gamma \setminus \pi_\bullet(\mathcal{S})$ implies that $B_{\delta_\bullet(x)}(x) \cap \widetilde{S} = \emptyset$. We prove the contraposition. Let $x \in \Gamma$ and suppose that $B_{\delta_\bullet(x)}(x) \cap \widetilde{S} \neq \emptyset$. Then, there exists $T \in \mathcal{S}$ and $y \in \mathbb{R}^d$ such that $|x-y| < \delta_\bullet(x)$ and $\operatorname{dist}(\{y\},T) < \delta_3(T)$. The triangle inequality yields that

$$\operatorname{dist}(\{x\},T) \le |x-y| + \operatorname{dist}(\{y\},T) < \delta_\bullet(x) + \delta_3(T) \le 2\max\big\{\delta_\bullet(x), \delta_3(T)\big\}. \tag{A8}$$

Now, we differ two different cases. If $\delta_\bullet(x) \le \delta_3(T)$, then we have that $\operatorname{dist}(\{x\},T) < 2\delta_3(T)$. The choice (A3) of $\delta_3(T)$ together with (M4) shows that $x \in \pi_\bullet(T) \subseteq \pi_\bullet(\mathcal{S})$. If $\delta_\bullet(x) > \delta_3(T)$, then we have that $\operatorname{dist}(\{x\},T) < 2\delta_\bullet(x)$. Let $T' \in \mathcal{T}_\bullet$ with $x \in T'$ and $z \in T$ with $|x-z| = \operatorname{dist}(\{x\},T)$. Together with (A5) and (A8), this yields that

$$\operatorname{dist}(\{z\},T') \le |x-z| = \operatorname{dist}(\{x\},T) < 2\delta_\bullet(x) \le 2C_{\mathrm{patch}}C_{\mathrm{locuni}}\,\delta_1(T') = \frac{\operatorname{diam}(T')}{C_{\mathrm{cent}}}.$$

Hence, (M4) implies that $z \notin \Gamma \setminus \pi_\bullet(T')$ and thus $z \in \pi_\bullet(T')$. Any $z' \in \Gamma$ that is sufficiently close to z also satisfies that $\operatorname{dist}(\{z'\},T') < \operatorname{diam}(T')/C_{\mathrm{cent}}$ and thus $z' \in \pi_\bullet(T')$. Since $z \in T = \overline{\operatorname{int}(T)}$, z' can be particularly chosen in $\operatorname{int}(T)$. Hence, we see that $\operatorname{int}(T) \cap \pi_\bullet(T') \neq \emptyset$. This is equivalent to $T' \in \Pi_\bullet(T)$, which concludes that $x \in T' \subseteq \pi_\bullet(T) \subseteq \pi_\bullet(\mathcal{S})$.

Step 3: Finally, we prove (A2). We recall that $\delta_\bullet > 0$ on O; see Step 1. With the identity matrix $I \in \mathbb{R}^{d\times d}$ and the matrix $(x-y)(\nabla\delta_\bullet(x))^\top \in \mathbb{R}^{d\times d}$, elementary calculations prove for all $x \in O \supset \Gamma$ and all $y \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\nabla_x\big[K_{\delta_\bullet(x)}(x-y)\big]\big)^\top &= \left[\nabla K_1\left(\frac{x-y}{\delta_\bullet(x)}\right)\right]^\top \frac{\delta_\bullet(x)I - (x-y)(\nabla\delta_\bullet(x))^\top}{\delta_\bullet(x)^2}\,\delta_\bullet(x)^{-d} \\ &\quad + K_1\left(\frac{x-y}{\delta_\bullet(x)}\right)\delta_\bullet(x)^{-d-1}(-d)(\nabla\delta_\bullet(x))^\top.\end{aligned}$$

Considering the norm, we see that

$$\big|\nabla_x\big(K_{\delta_\bullet(x)}(x-y)\big)\big| \lesssim \delta_\bullet(x)^{-d-1} + |x-y||\nabla\delta_\bullet(x)|\,\delta_\bullet(x)^{-d-2} + \delta_\bullet(x)^{-d-1}|\nabla\delta_\bullet(x)|.$$

Together with $\mathrm{supp}(K_s) = B_s(0)$, this shows for all $x \in \Gamma$ that

$$|\nabla\widetilde{\chi}_{\mathcal{S}}(x)| = \Big|\int_{\mathbb{R}^d} \chi_{\mathcal{S}}(y)\nabla_x \left(K_{\delta_\bullet(x)}(x-y)\right)\,\mathrm{d}y\Big|$$
$$\lesssim \int_{B_{\delta_\bullet(x)}(x)} \delta_\bullet(x)^{-d-1} + |x-y||\nabla\delta_\bullet(x)|\,\delta_\bullet(x)^{-d-2} + \delta_\bullet(x)^{-d-1}|\nabla\delta_\bullet(x)|\,\mathrm{d}y$$
$$\lesssim \delta_\bullet(x)^{-1}(1+\|\nabla\delta_\bullet\|_{L^\infty(\Gamma)}).$$

Thus (A5)–(A6) and (M3) prove that $|\nabla\widetilde{\chi}_{\mathcal{S}}(x)| \lesssim h_\bullet(x)^{-1}$ for almost all $x \in \Gamma$. Moreover, for smooth functions, the surface gradient ∇_Γ is the orthogonal projection of the gradient ∇ onto the tangent plane; see, e.g. [18, Lemma 2.22]). With the outer normal vector ν, this implies that $\nabla_\Gamma\widetilde{\chi}_{\mathcal{S}} = \nabla\widetilde{\chi}_{\mathcal{S}} - (\nabla\widetilde{\chi}_{\mathcal{S}}\cdot\nu)\nu$ almost everywhere on Γ and concludes the proof with the previous estimate. ■

Appendix 2. Inverse inequalities for other integral operators

In Proposition 4.13, we have generalized an inverse estimate from [5] for the single-layer operator $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$. [5] additionally derived similar estimates for the *double-layer operator* $\mathfrak{K}' : H^{1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$, the *adjoint double-layer operator* $\mathfrak{K}' : H^{-1/2}(\Gamma)^D \to H^{-1/2}(\Gamma)^D$, and the *hyper-singular operator* $\mathfrak{W} : H^{1/2}(\Gamma)^D \to H^{-1/2}(\Gamma)^D$; see, e.g. [19, page 218] for a precise definition (where these operators are denoted by $\frac{1}{2}T$, $\frac{1}{2}\widetilde{T}^*$, and R, respectively). Although [5] considered only the Laplace problem, the techniques of the proof of Proposition 4.13 extend the result at least to partial differential operators without lowest-order term cu. With some further effort, one can even prove it for arbitrary PDE operators $\mathfrak{P}$ with constant coefficients as in Section 2.3, where, as in Section 4.4, ellipticity of $\mathfrak{P}$ can be replaced by ellipticity up to some compact perturbation. To this end, one requires additional regularity of the trace operator $(\cdot)|_\Gamma : H^{3/2}(\Omega)^D \to H^1(\Gamma)^D$, which is satisfied for piecewise smooth boundaries Γ; see, e.g. [21, Remark 3.1.18].

For the proof, we will frequently use [19, Theorem 4.24], which reads as follows: Let $u \in H^1(\Omega)$ be arbitrary with $\mathfrak{P}u \in L^2(\Omega)^D$ in the weak sense and $u|_\Gamma \in H^1(\Gamma)$. Then, $\mathfrak{D}_\nu u \in L^2(\Gamma)^D$ and

$$\|\mathfrak{D}_\nu u\|_{L^2(\Gamma)} \lesssim \|u|_\Gamma\|_{H^1(\Gamma)} + \|u\|_{H^1(\Omega)} + \|\mathfrak{P}u\|_{L^2(\Omega)}. \tag{B.1}$$

Proposition B.1: *Suppose* (M1)–(M5). *For* $\mathcal{T}_\bullet \in \mathbb{T}$, *let* $w_\bullet \in L^\infty(\Gamma)$ *be a weight function which satisfies for some* $\alpha > 0$ *and all* $T \in \mathcal{T}_\bullet$ *that*

$$\|w_\bullet\|_{L^\infty(T)} \le \alpha w_\bullet(x) \quad \textit{for almost all } x \in \pi_\bullet(T). \tag{B.2}$$

Then, there exists a constant $C_1 > 0$ *such that for all* $\psi \in L^2(\Gamma)^D$ *and* $v \in H^1(\Gamma)^D$,

$$\|w_\bullet\nabla_\Gamma\mathfrak{V}\psi\|_{L^2(\Gamma)} + \|w_\bullet\mathfrak{K}'\psi\|_{L^2(\Gamma)} \le C_1\left(\|w_\bullet/h_\bullet^{1/2}\|_{L^\infty(\Gamma)}\|\psi\|_{H^{-1/2}(\Gamma)} + \|w_\bullet\psi\|_{L^2(\Gamma)}\right), \tag{B.3}$$

If we additionally suppose that the trace operator satisfies the stability $(\cdot)|_\Gamma : H^{3/2}(\Omega)^D \to H^1(\Gamma)^D$, *there exists a constant* $C_2 > 0$ *such that for all* $\psi \in L^2(\Gamma)^D$ *and* $v \in H^1(\Gamma)^D$,

$$\|w_\bullet\nabla_\Gamma\mathfrak{K}v\|_{L^2(\Gamma)} + \|w_\bullet\mathfrak{W}v\|_{L^2(\Gamma)} \le C_2\left(\|w_\bullet/h_\bullet^{1/2}\|_{L^\infty(\Gamma)}\|v\|_{H^{1/2}(\Gamma)} + \|w_\bullet\nabla_\Gamma v\|_{L^2(\Gamma)}\right). \tag{B.4}$$

The constants C_1 *and* C_2 *depend only on* (M1)–(M5), Γ, *the coefficients of* $\mathfrak{P}$, *and the admissibility constant* α.

Proof of (B.3): The bound for $\|w_\bullet\nabla_\Gamma\mathfrak{V}\psi\|_{L^2(\Gamma)}$ is just the assertion of Proposition 4.13. With the results from the proof of Proposition 4.13, one can easily estimate the second summand $\|w_\bullet\mathfrak{K}'\psi\|_{L^2(\Gamma)}$ as in [5, Section 6]. In particular, the stability of $\mathfrak{K}' : L^2(\Gamma) \to L^2(\Gamma)$ is exploited, which is stated in [19, page 209]. Further, one uses the fact from [19, page 218] that $\mathfrak{K}' = \widetilde{\mathfrak{D}}_\nu^{\mathrm{int}}\widetilde{\mathfrak{V}} - 1/2$, where $\widetilde{\mathfrak{D}}_\nu^{\mathrm{int}}(\cdot)$ denotes the interior modified conormal derivative from [19, page 117–118]. ■

Proof of (B.4): As in the proof of Proposition 4.13, we abbreviate $\delta_1(T) := \mathrm{diam}(T)/(2C_{\mathrm{cent}})$ and $U_T := B_{\delta_1(T)}(T)$ for all $T \in \mathcal{T}_\bullet$. Let $v_T := |T|^{-1}\int_T v\,dx$. Further, let $\widetilde{\chi}_T := \widetilde{\chi}_{\{T\}}$ be the smooth quasi-indicator function of T from Lemma A.1. With the Poincaré inequality (56), the localization properties (55) as well as (60), and (M5), one can easily verify that

$$h_T^{-1}\|v - v_T\|_{L^2(\pi_\bullet(T))} + h_T^{-1/2}\|(v - v_T)\widetilde{\chi}_T\|_{H^{1/2}(\Gamma)} + \|(v - v_T)\widetilde{\chi}_T\|_{H^1(\Gamma)} \lesssim \|\nabla_\Gamma v\|_{L^2(\pi_\bullet(T))}. \tag{B.5}$$

We fix (independently of $\mathcal{T}_\bullet$) a bounded domain $U \subset \mathbb{R}^d$ with $U_T \subset U$ for all $T \in \mathcal{T}_\bullet$. Let $\widetilde{\mathfrak{K}} : H^{1/2}(\Gamma)^D \to H^1(U \setminus \Gamma)^D$ denote the *double-layer potential* from [19, Theorem 6.11]. With these preparations and if $\mathfrak{P}$ has no lower-order terms, i.e. $b_i = 0$ for all $i \in \{1, \dots, d\}$ as well as $c = 0$, the proof of (A4) follows as in [5, Section 5 and Section 6]. In particular, one exploits the fact that

$$\widetilde{\mathfrak{K}} v_T = -v_T \text{ on } \Omega \quad \text{and} \quad \widetilde{\mathfrak{K}} v_T = 0 \text{ on } \Omega^{\mathrm{ext}} := \mathbb{R}^d \setminus \overline{\Omega}, \tag{B.6}$$

which follows from the representation formula [19, Theorem 6.10] together with the assumption that $\mathfrak{P}$ has no lower-order terms. The proof of (B.4) then employs the Poincaré inequality (B.5), the property (B.6), the stability of $\mathfrak{K} : H^1(\Gamma)^D \to H^1(\Gamma)^D$ and of $\mathfrak{W} : H^1(\Gamma)^D \to L^2(\Gamma)^D$ from [22, Corollary 3.38] or [19, page 209], and the Caccioppoli inequality (71) in combination with the transmission property [19, Theorem 6.11].

To prove (A12) for general $\mathfrak{P}$ with lower-order terms, where (B.6) is in general false, we recall the principal part $\mathfrak{P}_0$ from (91). In the following five steps, we show for the corresponding double-layer operator $\mathfrak{K}_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and the hyper-singular operator $\mathfrak{W}_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ that

$$\mathfrak{K} - \mathfrak{K}_0 : H^{1/2}(\Gamma) \to H^1(\Gamma) \quad \text{and} \quad \mathfrak{W} - \mathfrak{W}_0 : H^{1/2}(\Gamma) \to L^2(\Gamma) \tag{B.7}$$

are continuous. Since (B.4) is satisfied for the operators corresponding to $\mathfrak{P}_0$, this and the trivial estimate $w_\bullet \lesssim \|w_\bullet / h_\bullet^{1/2}\|_{L^\infty(\Gamma)}$ will conclude the proof.

Step 1: Let $\widetilde{\mathfrak{N}}, \widetilde{\mathfrak{N}}_0$ be the *Newton potentials* from [19, Theorem 6.1] corresponding to $\mathfrak{P}, \mathfrak{P}_0$. According to [19, Theorem 6.1], they satisfy the mapping property $\widetilde{\mathfrak{N}}, \widetilde{\mathfrak{N}}_0 : H^\sigma(\mathbb{R}^d)^D \to H^{\sigma+2}(\mathbb{R}^d)^D$ for all $\sigma \in \mathbb{R}$. In the proof of the latter stability, the fundamental solution is defined in terms of the Fourier transformation. The definition involves a multivariate polynomial $P : \mathbb{R}^d \to \mathbb{C}^{D\times D}$ resp. $P_0 : \mathbb{R}^d \to \mathbb{C}^{D\times D}$ associated to $\mathfrak{P}$ resp. $\mathfrak{P}_0$ (which is obtained from the differential operator by replacing the derivatives with variables) such that $|P(\xi)| \lesssim |P_0(\xi)^{-1}| = \mathcal{O}(|\xi|^{-2})$ for $\xi \in \mathbb{R}^d$ and $|\xi| \to \infty$; see [19, Equation (6.7)]. Indeed, the latter inequality is the key of the proof. As elementary analysis even shows that $|P(\xi)^{-1} - P_0(\xi)^{-1}| = |P(\xi)^{-1}[I - P(\xi)P_0(\xi)^{-1}]| = \mathcal{O}(|\xi|^{-3})$ with the identity matrix $I \in \mathbb{C}^{D\times D}$, one sees along the lines of [19, Theorem 6.1] the additional regularity

$$\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}_0 : H^\sigma(\mathbb{R}^d)^D \to H^{\sigma+3}(\mathbb{R}^d)^D. \tag{B.8}$$

Since multiplication with a fixed compactly supported smooth function is stable (see, e.g. [19, Theorem 3.20]), we also see for arbitrary $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^d)$ the continuity of

$$\mathfrak{A} : H^\sigma(\mathbb{R}^d)^D \to H^{\sigma+3}(\mathbb{R}^d)^D, \quad g \mapsto \chi_1(\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}_0)(g\chi_2). \tag{B.9}$$

Step 2: For sufficiently large $\lambda > 0$, the sesquilinear forms $(\cdot\,;\,\cdot)_{\mathfrak{P}+\lambda}$ and $(\cdot\,;\,\cdot)_{\mathfrak{P}_0+\lambda}$ from (18) are both even elliptic on $H_0^1(\Omega)^D$. Let $u_\lambda \in H^1(\Omega)$ be the unique weak solution of $(\mathfrak{P} + \lambda)u_\lambda = 0$ and $u_\lambda|_\Gamma = v$. Similarly, let $u_{0,\lambda} \in H^1(\Omega)$ be the solution of $(\mathfrak{P}_0 + \lambda)u_{0,\lambda} = 0$ and $u_{0,\lambda}|_\Gamma = v$. We extend both functions by zero outside of Ω. Since $\mathfrak{P}u_\lambda = -u_\lambda$ and $\mathfrak{P}_0 u_{0,\lambda} = -u_{0,\lambda}$, the representation formula [19, Theorem 6.10] yields that

$$(\widetilde{\mathfrak{K}} - \widetilde{\mathfrak{K}}_0)v = (u_\lambda - u_{0,\lambda}) - \lambda(\widetilde{\mathfrak{N}} u_\lambda - \widetilde{\mathfrak{N}}_0 u_{0,\lambda}) + (\widetilde{\mathfrak{V}}\mathfrak{D}_\nu u_\lambda - \widetilde{\mathfrak{V}}_0 \mathfrak{D}_\nu u_{0,\lambda}). \tag{B.10}$$

We note that $\mathfrak{K} - \mathfrak{K}_0 = (\widetilde{\mathfrak{K}} - \widetilde{\mathfrak{K}}_0)(\cdot)|_\Gamma$ and $\mathfrak{W} - \mathfrak{W}_0 = -\mathfrak{D}_\nu(\widetilde{\mathfrak{K}} - \widetilde{\mathfrak{K}}_0)$ with the conormal derivative $\mathfrak{D}_\nu(\cdot)$. To see (B.7) and hence to conclude the proof, we thus only have to bound the trace as well as the conormal derivative of each summand in (B.10) separately, which will be done in the following three steps.

Step 3: By definition, the trace of the first summand in (B.10) vanishes, i.e. $(u_\lambda - u_{0,\lambda})|_\Gamma = 0$. According to (A9) (where the differential operator can also be chosen as $\mathfrak{P} + \lambda$ instead of $\mathfrak{P}$), the normal derivative satisfies that

$$\|\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda})\|_{L^2(\Gamma)} \overset{(A9)}{\lesssim} \|(u_\lambda - u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} + \|u_\lambda - u_{0,\lambda}\|_{H^1(\Omega)} + \|(\mathfrak{P} + \lambda)(u_\lambda - u_{0,\lambda})\|_{L^2(\Omega)}.$$

Since the first summand vanishes and $\|(\mathfrak{P} + \lambda)(u_\lambda - u_{0,\lambda})\|_{L^2(\Omega)} \lesssim \|u_{0,\lambda}\|_{H^1(\Omega)}$ due to $(\mathfrak{P} + \lambda)(u_\lambda - u_{0,\lambda}) = -(\sum_{i=1}^d b_i \partial_i u_{0,\lambda}) - c u_{0,\lambda}$, the stability of the solution mapping $v \mapsto u_\lambda$ and $v \mapsto u_{0,\lambda}$ gives that

$$\|\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda})\|_{L^2(\Gamma)} \lesssim \|v\|_{H^{1/2}(\Gamma)}. \tag{B.11}$$

Step 4: Due to our assumption that the trace operator $(\cdot)|_\Gamma : H^{3/2}(\Omega)^D \to H^1(\Gamma)^D$ is well-defined and continuous, the stability of the Newton potentials, the stability of the solution mapping $v \mapsto u_\lambda$ and $v \mapsto u_{0,\lambda}$, the trace of the second summand in (B.10) satisfies that

$$\begin{aligned}
\|(\widetilde{\mathfrak{N}} u_\lambda - \widetilde{\mathfrak{N}}_0 u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} &\le \|(\widetilde{\mathfrak{N}} u_\lambda)|_\Gamma\|_{H^1(\Gamma)} + \|(\widetilde{\mathfrak{N}}_0 u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} \\
&\lesssim \|\widetilde{\mathfrak{N}} u_\lambda\|_{H^{3/2}(\Omega)} + \|\widetilde{\mathfrak{N}}_0 u_{0,\lambda}\|_{H^{3/2}(\Omega)} \lesssim \|u_\lambda\|_{H^{-1/2}(\mathbb{R}^d)} + \|u_{0,\lambda}\|_{H^{-1/2}(\mathbb{R}^d)} \\
&\lesssim \|u_\lambda\|_{L^2(\mathbb{R}^d)} + \|u_{0,\lambda}\|_{L^2(\mathbb{R}^d)} = \|u_\lambda\|_{L^2(\Omega)} + \|u_{0,\lambda}\|_{L^2(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

Note that $\widetilde{\mathfrak{N}}$ and $\widetilde{\mathfrak{N}}_0$ are indeed potentials, i.e. $\mathfrak{P}\widetilde{\mathfrak{N}} = 0$ weakly and $\mathfrak{P}_0\widetilde{\mathfrak{N}}_0 = 0$ weakly. With (B.1) (applied for $\mathfrak{P}$ and $\mathfrak{P}_0$), the stability of the Newton potentials, and the estimates for the trace of $\widetilde{\mathfrak{N}}u_\lambda$ and $\widetilde{\mathfrak{N}}_0 u_{0,\lambda}$, we thus see that

$$\|\mathfrak{D}_\nu(\widetilde{\mathfrak{N}}u_\lambda - \widetilde{\mathfrak{N}}_0 u_{0,\lambda})\|_{L^2(\Gamma)} \overset{(A9)}{\lesssim} \|(\widetilde{\mathfrak{N}}u_\lambda)|_\Gamma\|_{H^1(\Gamma)} + \|\widetilde{\mathfrak{N}}u_\lambda\|_{H^1(\Omega)} + \|(\widetilde{\mathfrak{N}}_0 u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} + \|\widetilde{\mathfrak{N}}_0 u_{0,\lambda}\|_{H^1(\Omega)}$$
$$\lesssim \|v\|_{H^{1/2}(\Gamma)} + \|u_\lambda\|_{H^{-1}(\mathbb{R}^d)} + \|u_{0,\lambda}\|_{H^{-1}(\mathbb{R}^d)} \lesssim \|v\|_{H^{1/2}(\Gamma)} + \|u_\lambda\|_{L^2(\mathbb{R}^d)} + \|u_{0,\lambda}\|_{L^2(\mathbb{R}^d)}.$$

Again, the stability of the solution mappings allows to bound the last terms by $\|v\|_{H^{1/2}(\Gamma)}$.

Step 5: To deal with the third summand in (B.10), we first rewrite it as follows:

$$\widetilde{\mathfrak{V}}\mathfrak{D}_\nu u_\lambda - \widetilde{\mathfrak{V}}_0\mathfrak{D}_\nu u_{0,\lambda} = \widetilde{\mathfrak{V}}\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda}) + (\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda}. \tag{B.12}$$

Due to the stability of $\widetilde{\mathfrak{V}}(\cdot)|_\Gamma = \mathfrak{V} : L^2(\Gamma) \to H^1(\Gamma)$ (see (21) and (76)) and (B.11), we have for the first summand in (B.12) that

$$\|(\widetilde{\mathfrak{V}}\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda}))|_\Gamma\|_{H^1(\Gamma)} \lesssim \|v\|_{H^{1/2}(\Gamma)}. \tag{B.13}$$

To deal with the conormal derivative, we apply again (B.1) together with the fact that $\widetilde{\mathfrak{V}}$ is a potential, i.e. $\mathfrak{P}\widetilde{\mathfrak{V}} = 0$ weakly. This leads to

$$\|\mathfrak{D}_\nu\widetilde{\mathfrak{V}}\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda})\|_{L^2(\Gamma)} \overset{(A9)}{\lesssim} \|(\widetilde{\mathfrak{V}}\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda}))|_\Gamma\|_{H^1(\Gamma)} + \|\widetilde{\mathfrak{V}}\mathfrak{D}_\nu(u_\lambda - u_{0,\lambda})\|_{H^1(\Omega)}.$$

The first summand can be bounded as in (B.13). For the second summand, we use the stability (75) in combination with $\|\cdot\|_{H^{-1/2}(\Gamma)} \lesssim \|\cdot\|_{L^2(\Gamma)}$ and (B.11).

Finally, it only remains to bound the trace as well as the conormal derivative of the second summand in (B.12). Choosing $\sigma = -1$ in the additional regularity (B.9), one sees as in the proof of [19, page 203] (which proves stability of $\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(\Omega)^D$) that

$$\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0 : H^{-1/2}(\Gamma)^D \to H^2(\Omega)^D \tag{B.14}$$

With the assumption $(\cdot)|_\Gamma : H^{3/2}(\Omega)^D \to H^1(\Gamma)^D$, this implies that

$$\|((\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} \lesssim \|(\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda}\|_{H^{3/2}(\Omega)} \lesssim \|\mathfrak{D}_\nu u_{0,\lambda}\|_{H^{-1/2}(\Gamma)}. \tag{B.15}$$

Recall that $(\mathfrak{P}_0 + \lambda)u_{0,\lambda} = 0$ with $u_{0,\lambda}|_\Gamma = v$. Thus [19, Theorem 4.25] (which states boundedness of the Dirichlet to Neumann mapping) gives that

$$\|((\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} \overset{(A23)}{\lesssim} \|\mathfrak{D}_\nu u_{0,\lambda}\|_{H^{-1/2}(\Gamma)} \lesssim \|v\|_{H^{1/2}(\Gamma)}. \tag{B.16}$$

Moreover, (A9), the fact that $\mathfrak{P}(\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0) = -(\sum_{i=1}^d b_i\partial_i\widetilde{\mathfrak{V}}_0) - c\widetilde{\mathfrak{V}}_0$, the stability (75), and (B.16) show for the conormal derivative that

$$\|\mathfrak{D}_\nu(\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda}\|_{L^2(\Gamma)} \lesssim \|((\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda})|_\Gamma\|_{H^1(\Gamma)} + \|(\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda}\|_{H^1(\Omega)}$$
$$+ \|\mathfrak{P}(\widetilde{\mathfrak{V}} - \widetilde{\mathfrak{V}}_0)\mathfrak{D}_\nu u_{0,\lambda}\|_{L^2(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)}.$$

Overall, we have estimated the trace as well as the conormal derivative of all terms in (B.10). Since $\mathfrak{K} - \mathfrak{K}_0 = (\widetilde{\mathfrak{K}} - \widetilde{\mathfrak{K}}_0)(\cdot)|_\Gamma$ and $\mathfrak{W} - \mathfrak{W}_0 = -\mathfrak{D}_\nu(\widetilde{\mathfrak{K}} - \widetilde{\mathfrak{K}}_0)$, this verifies the stability (B.7) and thus concludes the proof. ■