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AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Chakradhar Padamutham & Venkata Subba Reddy Palagiri (2020): Complexity of Roman {2}-domination and the double Roman domination in graphs, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2020.01.005

To link to this article: https://doi.org/10.1016/j.akcej.2020.01.005

Published online: 04 May 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
https://doi.org/10.1016/j.akcej.2020.01.005

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Complexity of Roman {2}-domination and the double Roman domination in graphs

Chakradhar Padamutham and Venkata Subba Reddy Palagiri

Department of Computer Science and Engineering, NIT Warangal, Telangana, India

ABSTRACT

For a simple, undirected graph $G = (V, E)$, a Roman {2}-dominating function (R2DF) $f : V \rightarrow \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N(v)$, with $f(u) = 2$, or at least two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The weight of an R2DF is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an R2DF is called the Roman {2}-domination number and is denoted by $\gamma_{\{R2\}}(G)$. A double Roman dominating function (DRDF) on G is a function $f : V \rightarrow \{0, 1, 2, 3\}$ such that for every vertex $v \in V$ if $f(v) = 0$, then v has at least two neighbors $x, y \in N(v)$ with $f(x) = f(y) = 2$ or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then v must have at least one neighbor w with $f(w) \geq 2$. The weight of a DRDF is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a DRDF is called the double Roman domination number and is denoted by $\gamma_{dR}(G)$. Given an graph G and a positive integer k, the R2DP (DRDP) problem is to check whether G has an R2DF (DRDF) of weight at most k. In this article, we first show that the R2DP problem is NP-complete for star convex bipartite graphs, comb convex bipartite graphs and bisplit graphs. We also show that the DRDP problem is NP-complete for star convex bipartite graphs and comb convex bipartite graphs. Next, we show that $\gamma_{\{R2\}}(G)$, and $\gamma_{dR}(G)$ are obtained in linear time for bounded tree-width graphs, chain graphs and threshold graphs, a subclass of split graphs. Finally, we propose a $2(1 + \ln(\Delta + 1))$-approximation algorithm for the minimum Roman {2}-domination problem and $3(1 + \ln(\Delta + 1))$-approximation algorithm for the minimum double Roman domination problem, where Δ is the maximum degree of G.

KEYWORDS
Roman {2}-domination; double Roman domination; tree convex bipartite graphs; NP-complete; approximation algorithm

2010 MSC
05C69; 68Q25

1. Introduction

Consider $G = (V, E)$ be a simple, undirected and connected graph. For a vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u | u \in V, (u, v) \in E\}$ and the *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. We shall follow [11] for graph theoretic terminology. A bipartite graph $G = (X, Y, E)$ is called *tree convex* if there exists a tree $T = (X, F)$ such that, for each y in Y, the neighbors of y induce a subtree in T. When T is a star (comb), G is called *star* (*comb*) *convex bipartite* graph [10]. A *bisplit graph* is a graph in which vertex set can be partitioned into an independent set and a complete bipartite graph.

A vertex v in G dominates the vertices of its closed neighborhood. A set of vertices $S \subseteq V$ is a *dominating set* (DS) in G if for every vertex $u \in V \setminus S$, there exists at least one vertex $v \in S$ such that $(u, v) \in E$, i.e., $N_G[S] = V(G)$. The *domination number* is the minimum cardinality of a dominating set in G and is denoted by $\gamma(G)$ [8].

The concept of Roman domination was introduced in 2004 by Cockayne et al. [5]. A function $f : V \rightarrow \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V) = \sum_{u \in V} f(u)$. The *Roman domination number* is the minimum weight of an RDF on G and is denoted by $\gamma_R(G)$.

Roman {2}-domination was introduced in 2016 by Chellali et al. [4]. A Roman {2}-dominating function (R2DF) $f : V \rightarrow \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N(v)$, with $f(u) = 2$, or at least two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The weight of an R2DF is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an R2DF is called the *Roman {2}-domination number* and is denoted by $\gamma_{\{R2\}}(G)$.

Double Roman domination was initiated in 2016 by Robert et al. [3]. A *double Roman dominating function* (DRDF) on G is a function $f : V \rightarrow \{0, 1, 2, 3\}$ such that for every vertex $v \in V$ if $f(v) = 0$, then v has at least two neighbors $x, y \in N(v)$ with $f(x) = f(y) = 2$ or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then v must have at least one neighbor w with $f(w) \geq 2$. The weight of a DRDF is the value $f(V) = \sum_{v \in V} f(v)$. The *double Roman domination number* equals the minimum weight of a DRDF on G, denoted by $\gamma_{dR}(G)$.

Given a graph G and a positive integer k, the R2DP (DRDP) problem is to check whether G has an R2DF

CONTACT Chakradhar Padamutham corneliusp7@gmail.com Department of Computer Science and Engineering, NIT Warangal, Telangana 506 004, India.

(DRDF) of weight at most k. Chellali et al. [4] have proved that the R2DP problem is NP-complete for bipartite graphs. Ahangar et al. [1] have proved that the DRDP problem is NP-complete for bipartite and chordal graphs. Motivated by their work [1, 4], we investigate the complexity of R2DP and DRDP problems in subclasses of bipartite graphs and chordal graphs.

2. Complexity results

In this section, we show that the decision versions of the R2DF and DRDF problems are NP-complete for some subclasses of bipartite graphs by giving a polynomial time reduction from a well-known NP-complete problem, Exact-3-Cover (*X3C*) [9], which is defined as follows.
EXACT-3-COVER (X3C)
INSTANCE: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
QUESTION: Is there a subcollection C' of C such that every element of X appears in exactly one member of C'?
The decision versions of Roman {2}-domination and double Roman domination problems are defined below.
ROMAN {2}-DOMINATION PROBLEM (R2DP)
INSTANCE: Graph $G = (V,E)$ and a positive integer $k \leq |V|$.
QUESTION: Does G have an R2DF of weight at most k?

DOUBLE ROMAN DOMINATION PROBLEM (DRDP)
INSTANCE: Graph $G = (V,E)$ and a positive integer $k \leq 2|V|$.
QUESTION: Does G have a DRDF of weight at most k?

Theorem 1. *R2DP is NP-complete for star convex bipartite graphs.*

Proof. Given a graph G and a function f, whether f is an R2DF of size at most k can be checked in polynomial time. Hence R2DP is a member of NP. Now we show that R2DP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3C, where $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{c_1, c_2, ..., c_t\}$, to an instance $\langle G, k\rangle$ of R2DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. Let $A = \{a\} \cup \{x_i : 1 \leq i \leq 3q\}$ and $B = \{y_i : 1 \leq i \leq 3q\} \cup \{c_i : 1 \leq i \leq t\} \cup \{a_1, a_2, a_3\}$. The subgraph induced by A is a star with vertex a as central vertex and the neighbors of each element of B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time.

Next we show that, *X3C* has a solution if and only if G has an R2DF with weight at most $4q + 2$. Let $k = 4q + 2$. Suppose C' is a solution for *X3C* with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } v \in \{y_i : 1 \leq i \leq 3q\} \cup \{c_i : c_i \in C'\} \\ 2, & \text{if } v = a \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

It can be easily verified that f is an R2DF of G and $f(V) = 4q + 2 = k$.

Conversely, suppose that G has an R2DF g with weight k. Let $M = \{a, a_1, a_2, a_3\}$. Clearly, $\sum_{u \in M} g(u) \geq 2$, and so we may assume, without loss of generality, $g(a) = 2$ and $g(a_1) = g(a_2) = g(a_3) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0. Clearly, $g(x_i) = 0$ and $g(y_i) = 0$ case does not occur.

Claim 1. *If $g(V) = k$ then for each pair of vertices (x_i, y_i), $g(x_i) = 0$ and $g(y_i) = 1$.*

Proof. (Proof by contradiction) Assume $g(V) = k$ and there exist some pairs (x_i, y_i) such that $g(x_i) + g(y_i) > 1$. Let m be the number of pairs of (x_i, y_i) with $g(x_i) + g(y_i) = 2$. The number of pairs of (x_i, y_i) with $g(x_i) = 0$ and $g(y_i) = 1$ is $3q - m$. Since g is an R2DF, each x_i with $g(x_i) = 0$, where $g(y_i) = 1$, should have neighbor c_j with $g(c_j) = 1$. Then the minimum number of c_j's required with $g(c_j) = 1$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 3q + 2 + m + \lceil \frac{3q-m}{3} \rceil$, which is greater than k. Our assumption leads to a contradiction. Therefore for each pair (x_i, y_i), $g(x_i) = 0$ and $g(y_i) = 1$. Hence the claim. □

Since each c_i has exactly three neighbors in X, clearly, there exist exactly q number of c_i's with weight at least 1 such that $(\cup_{g(c_i) \geq 1} N(c_i)) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. □

Theorem 2. *R2DP is NP-complete for comb convex bipartite graphs.*

Proof. Clearly, R2DP for comb convex bipartite graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of *X3C*, where $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{c_1, c_2, ..., c_t\}$, to an instance $\langle G, k\rangle$ of R2DP as follows.

Create vertices x_i, x'_i and y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a', a_1, a_2 and a_3. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i and (c_j, x_i) if $x_i \in c_j$. Next add edges (c_j, a) and (c_j, a') for each c_j. Also add edges by joining each c_j to every x'_i. Let $A = \{a, a'\} \cup \{x_i, x'_i : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. The subgraph induced by A is a comb with the elements $\{x'_i : 1 \leq i \leq 3q\} \cup \{a'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\} \cup \{a\}$ as teeth and the neighbors of each element of B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time. Next, we show that, *X3C* has a solution if and only if G has an R2DF with weight at most $4q + 2$.

Suppose C' is a solution for *X3C* with $|C'| = q$. We construct an R2DF f, on G, same as in Theorem 1. Clearly, $f(V) = 4q + 2 = k$.

The proof of the converse is similar to the proof given in Theorem 1. □

Theorem 3. *R2DP is NP-complete for bisplit graphs.*

Proof. It is clear that R2DP for bisplit graphs is in NP. We transform an instance of *X3C*, where $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{c_1, c_2, ..., c_t\}$, to an instance $\langle G, k\rangle$ of R2DP as follows.

Create vertices x_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. Let $P = \{x_i : 1 \le i \le 3q\}$, $Q = \{c_i : 1 \le i \le t\} \cup \{a_1, a_2, a_3\}$ and $R = \{a\}$. In the constructed graph G, P forms an independent set and $Q \cup R$ is a complete bipartite graph. Hence, making G a bisplit graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time. Next we show that, $X3C$ has a solution if and only if G has an R2DF with weight at most $2q + 2$. Let $k = 2q + 2$.

Suppose C' is a solution for $X3C$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a\} \cup \{c_i : c_i \in C'\} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

It can be easily verified that f is an R2DF of G and $f(V) = 2q + 2 = k$.

Conversely, suppose that G has an R2DF g with weight k. Clearly, as in Theorem 1, $g(a) = 2$ and $\forall a_i,\ g(a_i) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0.

Claim 2. *If $g(V) = k$ then for each $x_i \in X$, $g(x_i) = 0$.*

Proof. (Proof by contradiction) Assume $g(V) = k$ and there exist some x_i's such that $g(x_i) \neq 0$. Let $m = |\{x_i : g(x_i) \neq 0\}|$. The number of x_i's with $g(x_i) = 0$ is $3q - m$. Since g is an R2DF, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 2 + m + 2\lceil \frac{3q-m}{3} \rceil$, which is greater than k. Our assumption leads to a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. Hence the claim. □

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $(\cup_{g(c_i)=2} N(c_i)) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C. □

Theorem 4. *DRDP is NP-complete for star convex bipartite graphs.*

Proof. The proof is obtained with similar arguments as in Theorem 1, in which replace the assigned values 1 with 2 and 2 with 3. □

Theorem 5. *DRDP is NP-complete for comb convex bipartite graphs.*

Proof. The proof is obtained with similar arguments as in Theorem 2, in which replace the assigned values 1 with 2 and 2 with 3. □

3. Threshold graphs

In this section, we determine the Roman {2}-domination number and double Roman domination number of threshold graphs. A *threshold graph* is a graph that can be constructed from one vertex graph by repeated applications of the following two operations: (i) Addition of a single isolated vertex to the graph. (ii) Addition of a single dominating vertex to the graph. For the graph to be connected the last vertex added must be a dominating vertex. Since every threshold graph is a split graph, $V = C \cup I$, where C is a clique constituting all dominating vertices and I is an independent set constituting all isolated vertices. Let $C = \{c_1, c_2, ..., c_n\}$ and $I = \{i_1, i_2, ..., i_m\}$. If the clique vertices are added in the order $c_1, c_2, ..., c_n$ and the independent vertices are added in the order $i_1, i_2, ..., i_m$ then by the definition it follows that $N_G[c_1] \subseteq N_G[c_2] \subseteq N_G[c_3] \subseteq \cdots \subseteq N_G[c_n]$ and $N_G(i_1) \supseteq N_G(i_2) \supseteq N_G(i_3) \supseteq \cdots \supseteq N_G(i_m)$. For example, consider a threshold graph in Figure 1, where the vertices are added in the order i_1, i_2, c_1, i_3, c_2, c_3, …, i_{m-1}, i_m and c_n. The vertices constituting the clique are enclosed in a dotted rectangle in Figure 1. If $|V| = 1$ then, clearly, $\gamma_{\{R2\}}(G) = 1$ and $\gamma_{dR}(G) = 2$.

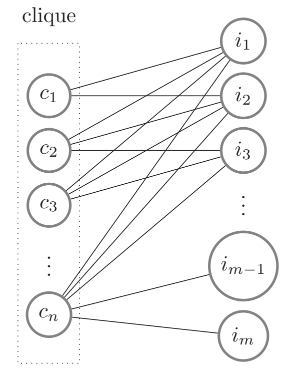

Figure 1. A threshold graph.

Theorem 6. *Let G be a threshold graph. Then $\gamma_{\{R2\}}(G) = k + 1$ and $\gamma_{dR}(G) = 2k + 1$, where k is the number of connected components in G.*

Proof. Let G be a threshold graph with n clique vertices such that $N_G[c_1] \subseteq N_G[c_2] \subseteq N_G[c_3] \subseteq \cdots \subseteq N_G[c_n]$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } \deg(v) = 0 \\ 2, & \text{if } v = c_n \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Clearly, f is an R2DF and $\gamma_{\{R2\}}(G) \le k + 1$. From the definition of R2DF, it follows that $\gamma_{\{R2\}}(G) \ge k + 1$. Therefore, $\gamma_{\{R2\}}(G) = k + 1$.

Similarly, let $g : V \to \{0, 1, 2, 3\}$ be a function on G defined as follows.

$$g(v) = \begin{cases} 2, & \text{if } \deg(v) = 0 \\ 3, & \text{if } v = c_n \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Clearly, g is a DRDF and $\gamma_{dR}(G) \le 2k + 1$. From the definition of DRDF, it follows that $\gamma_{dR}(G) \ge 2k + 1$. Therefore, $\gamma_{dR}(G) = 2k + 1$. □

4. Chain graphs

In this section, we propose a method to compute Roman {2}-domination number and double Roman domination number of a chain graph in linear time. A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $x_1, x_2, ..., x_p$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq ... \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\alpha = (x_1, x_2, ..., x_p, y_1, y_2, ..., y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq ... \supseteq N_G(y_q)$. Every chain graph admits a chain ordering [12]. The following is a proposition without proof.

Proposition 1. *Let $G = K_{r,s}$ be a complete bipartite graph with $r \leq s$.*

(a) *If $r = 1$ then $\gamma_{\{R2\}}(G) = 2$.*
(b) *If $r = 2$ then $\gamma_{\{R2\}}(G) = 3$.*
(c) *If $r \geq 3$ then $\gamma_{\{R2\}}(G) = 4$.*

If G is a complete bipartite graph then $\gamma_{\{R2\}}(G)$ is obtained directly from Proposition 1. Otherwise, the following theorem holds.

Theorem 7. *Let G $(\neq K_{r,s})$ be a connected chain graph. Then,*

$$\gamma_{\{R2\}}(G) = \begin{cases} 3, & \text{if } |X| = 2 \text{ or } |Y| = 2 \\ 4, & \text{otherwise} \end{cases} \quad (5)$$

Proof. Let $G(X, Y, E)$ be a connected chain graph with $|X| = n$ and $|Y| = m$, where $n, m \geq 2$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

Case (1): $|X| = 2$ and $|Y| = 2$ then $f(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$

Case (2): $|X| = 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v = x_2 \\ 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

Case (3): $|X| \neq 2$ and $|Y| = 2$ then same condition holds as in case (1).

Clearly, f is an R2DF and $\gamma_{\{R2\}}(G) \leq 3$. From the definition of R2DF, it follows that $\gamma_{\{R2\}}(G) \geq 3$. Therefore $\gamma_{\{R2\}}(G) = 3$.

Case (4): $|X| \neq 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v \in \{x_n, y_1\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is an R2DF and $\gamma_{\{R2\}}(G) \leq 4$. By contradiction, it can be easily verified that $\gamma_{\{R2\}}(G) \geq 4$. Therefore $\gamma_{\{R2\}}(G) = 4$. □

If the chain graph G is disconnected then weight of the R2DF is increased by k, where k is the number of isolated vertices in G.

The following propositions are proved in [2, 3].

Proposition 2 ([2]). *For any complete bipartite graph $K_{p,q}$, with $p, q \geq 3$, $\gamma_{dR}(K_{p,q}) = 6$.*

Proposition 3 ([3]). *For any complete bipartite graph $K_{p,q}$ with $p \leq q$, $\gamma_{dR}(K_{1,q}) = 3$ and $\gamma_{dR}(K_{2,q}) = 4$.*

If G is a complete bipartite graph then $\gamma_{dR}(G)$ is obtained directly from Propositions 2 and 3. Otherwise, the following theorem holds.

Theorem 8. *Let G $(\neq K_{r,s})$ be a connected chain graph. Then,*

$$\gamma_{dR}(G) = \begin{cases} 5, & \text{if } |X| = 2 \text{ or } |Y| = 2 \\ 6, & \text{otherwise} \end{cases} \quad (6)$$

Proof. The proof is same as in Theorem 7 in which replace the assigned values 1 with 2 and 2 with 3. □

If the chain graph G is disconnected then weight of the DRDF is increased by $2k$, where k is the number of isolated vertices in G.

5. Bounded tree-width graphs

Let G be a graph, T be a tree and v be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair $(T, v\ )$ is called a *tree-decomposition* of G if it satisfies the following three conditions: (i) $V(G) = \cup_{t \in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t, and (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_1, t_2, $t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The *width* of $(T, v\ )$ is the number $max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the minimum width of any tree-decomposition of G. By Courcelle's thoerem, it is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [7]. We show that R2DP and DRDP problems can be expressed in CMSOL.

Theorem 9 (Courcelle's theorem [7]). *Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

Theorem 10. *Given a graph G and a positive integer k, R2DP can be expressed in CMSOL.*

Proof. Let $f : V \to \{0, 1, 2\}$ be a function on a graph G, where $V_i = \{v | f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the R2DP is expressed as follows.

$$Rom\_\{2\}\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p((p \in V_0 \wedge ((\exists q \in V_2 \wedge adj(p, q)) \vee (\exists r, s \in V_1 \wedge \ adj(p, r) \wedge adj(p, s)))) \vee (p \in V_1)\ \ \vee (p \in V_2))$$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G. □

Now, the following result is immediate from Theorems 9 and 10.

Theorem 11. *R2DP problem can be solvable in linear time for bounded tree-width graphs.*

Theorem 12. *Given a graph G and a positive integer k, DRDP can be expressed in CMSOL.*

Proof. Let $g : V \longrightarrow \{0,1,2,3\}$ be a function on a graph G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0,1,2,3\}$. The CMSOL formula for the DRDP is expressed as follows.

$$Double\_Rom\_Dom(V) = (g(V) \le k) \wedge \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r \in V_2 \wedge adj(p,q) \wedge adj(p,r)) \vee (\exists s \in V_3 \wedge adj(p,s))) \vee (p \in V_1 \wedge (\exists t \in V_2 \wedge adj(p,t) \vee (\exists u \in V_3 \wedge adj(p,u))))) \vee (p \in V_2) \vee (p \in V_3))$$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G. □

Now, the following result is immediate from Theorems 9 and 12.

Theorem 13. *DRDP problem can be solvable in linear time for bounded tree-width graphs.*

6. Approximation results

In this section, we design approximation algorithms for optimization versions of Roman {2}-domination and double Roman domination problems based on the approximation result known for MINIMUM DOMINATION problem, which is given below.

MINIMUM DOMINATION
Instance: A simple, undirected graph $G = (V, E)$.
Solution: Minimum cardinality dominating set D of G.
Measure: Cardinality of D.

The minimum Roman {2}-domination and the minimum double Roman domination problems are defined as follows.

MINIMUM ROMAN {2}-DOMINATION PROBLEM (MR2DP)
Instance: A simple, undirected graph $G = (V, E)$.
Solution: An R2DF of G.
Measure: Weight of R2DF.

MINIMUM DOUBLE ROMAN DOMINATION PROBLEM (MDRDP)
Instance: A simple, undirected graph $G = (V, E)$.
Solution: A DRDF of G.
Measure: Weight of DRDF.

Now, we propose a $2(1 + \ln(\Delta + 1))$-approximation algorithm for MR2DP.
The following approximation result has been obtained in [6] for MINIMUM DOMINATION problem.

Theorem 14 ([6]). *The MINIMUM DOMINATION problem in a graph with maximum degree Δ can be approximated with an approximation ratio of $1 + \ln(\Delta + 1)$.*

By Theorem 14, let APPROX-DOM-SET be an approximation algorithm that gives a dominating set D of a graph G such that $|D| \le (1 + \ln(\Delta + 1))\gamma(G)$, where Δ is the maximum degree of the graph G.

Next, we propose an algorithm APPROX-R2D to compute an approximate solution of MR2DP problem. In our algorithm, first we compute a dominating set D of the input graph G using the approximation algorithm APPROX-DOM-SET. Next, we construct a triple T_r in which every vertex in D will be assigned with weight 2 and the remaining vertices will be assigned with weight 0.

Now, let $T_r = (D', \emptyset, D)$ be the triple obtained by using the APPROX-R2D algorithm. It can be easily seen that every vertex $v \in V$ is assigned with weight either 0 or 2. Since D is a dominating set of G, every vertex $v \in D'$ with weight 0 is adjacent to a vertex $u \in D$ with weight 2. Thus, T_r gives an R2DF of G.

Algorithm 1: APPROX-R2D(G)

Input: A simple, undirected graph G.
Output: A Roman 2-dominating triple T_r of G.
1: $D \longleftarrow$ APPROX-DOM-SET(G)
2: $T_r \longleftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

We note that the algorithm APPROX-R2D computes a Roman 2-dominating triple T_r of the given graph G in polynomial time. Hence, we have the following result.

Theorem 15. *The MR2DP in a graph with maximum degree Δ can be approximated with an approximation ratio of $2(1 + \ln(\Delta + 1))$.*

Proof. Let D be the dominating set produced by the algorithm APPROX-DOM-SET, T_r be the Roman {2}-dominating triple produced by the algorithm APPROX-R2D and W_r be the weight of T_r.

It can be observed that $W_r = 2|D|$. It is known that $|D| \le (1 + \ln(\Delta + 1))\gamma(G)$. Therefore, $W_r \le 2(1 + \ln(\Delta + 1))\gamma(G)$. Since $\gamma(G) \le \gamma_{\{R2\}}(G)$ [4], it follows that $W_r \le 2(1 + \ln(\Delta + 1))\gamma_{\{R2\}}(G)$. □

Similar to Algorithm 1, we propose an approximation algorithm namely, APPROX-DRD, which produces a double Roman dominating quadruple as follows.

Algorithm 2: APPROX-DRD(G)

Input: A simple, undirected graph G.
Output: A double Roman dominating quadruple Q_r of G.
1: $D \longleftarrow$ APPROX-DOM-SET(G)
2: $Q_r \longleftarrow (V \setminus D, \emptyset, \emptyset, D)$
3: return Q_r.

We also note that the algorithm APPROX-DRD computes a double Roman dominating quadruple Q_r of a given graph G in polynomial time. Hence, the following theorem holds.

Theorem 16. *The MDRDP in a graph with maximum degree Δ can be approximated with an approximation ratio of $3(1 + \ln(\Delta + 1))$.*

Proof. The proof is obtained with similar arguments as in Theorem 15. □

We have the following corollaries of Theorems 15 and 16, respectively.

Corollary 1. *MR2DP problem for bounded degree graphs is in APX.*

Corollary 2. *MDRDP problem for bounded degree graphs is in APX.*

7. Conclusion

In this article, we have shown that the decision versions of Roman {2}-domination and the double Roman domination problems are NP-complete for some subclasses of bipartite graphs. Next, we have shown that R2DF and DRDF problems are linear time solvable for bounded tree-width graphs, threshold graphs and chain graphs. We have also given polynomial time approximation algorithms for MR2DP and MDRDP. In future work, one can investigate the algorithmic complexity of R2DP and DRDP for other subclasses of bipartite graphs and chordal graphs. Since, the MINIMUM DOMINATION problem is APX-hard for bounded degree graphs, the intuition suggests that MR2DP and the MDRDP could be APX-hard. Hence, determining whether or not MR2DP and MDRDP are APX-hard for bounded degree graphs remains open.

Acknowledgments

The authors are grateful to the referees for their constructive comments and suggestions that lead to the improvements in the article.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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