# On the maximum spectral radius of multipartite graphs 

Jian Wu \& Haixia Zhao

To cite this article: Jian Wu \& Haixia Zhao (2020): On the maximum spectral radius of multipartite graphs, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/ j.akcej.2020.01.006

To link to this article: https://doi.org/10.1016/j.akcej.2020.01.006

© 2020 The Author(s). Published with license by Taylor \& Francis Group, LLC.

Published online: 16 Jun 2020.

Submit your article to this journal

Article views: 70

View related articles

View Crossmark data

# On the maximum spectral radius of multipartite graphs 

Jian Wu ${ }^{\text {a,b }}$ and Haixia Zhao ${ }^{\text {c }}$<br>${ }^{\text {a }}$ School of Information and Computer Science, Taiyuan University of Technology Taiyuan, Shanxi, P. R. China; ${ }^{\text {b }}$ School of Applied Mathematics, Shanxi University of Finances and Economics, Taiyuan, Shanxi, P. R. China; 'School of Statistics, Shanxi University of Finance and Economics, Taiyuan, Shanxi, P. R. China


#### Abstract

Let $r \geq 2$ be an integer. A graph $G=(V, E)$ is called $r$-partite if $V$ admits a partition into $r$ parts such that every edge has its ends in different parts. All of the $r$ - partite graphs with given integer $r$ consist of the class of multipartite graphs. Let $\mathcal{G}(r, n, D)$ be the set of multipartite graphs with $r$ vertex parts, $n$ nodes and diameter $D$. In this paper, we characterize the graphs with the maximum spectral radius in $\mathcal{G}(r, n, D)$. Furthermore, we show that the maximum spectral radius is not only a decreasing function on $D$, but also an increasing function on $r$.


## KEYWORDS

Multipartite graph; diameter; spectral radius; chromatic number

## 2000 MSC

05C50

## 1. Introduction

All graphs considered here are connected, simple and undirected. Let $r \geq 2$ be an integer. A graph $G=(V, E)$ is called $r$ - partite if $V$ admits a partition into $r$ parts such that every edge has its ends in different parts:vertices in the same part must not be adjacent. Instead of 2-partite, we usually say bipartite. All of the $r$ - partite graphs with given integer $r$ consist of the class of multipartite graphs. An $r$ - partite graph is called complete, if every two vertices from different parts are adjacent. Let $\mathcal{G}(r, n, D)$ be the set of multipartite graphs with $r$ vertex parts, order $n$ and diameter $D$. The diameter, denoted by $D$, is the maximum distance between any two vertices.

For $S \subseteq V(G)$, let $G[S]$ be the subgraph induced by $S$. We will use $V \backslash V^{\prime}$ to denote the set that arises from $V$ by removing the subset $V^{\prime} \subset V$. We use $G+u v$ to denote the graph arising from $G$ by adding an edge $u v \notin E(G)$, where $u, v \in V(G)$.

Let $A(G)$ be the adjacency matrix of a graph $G$. It follows immediately that $A(G)$ is a real symmetric $(0,1)$ matrix in which every diagonal entry is zero, and that all of its eigenvalues are real. The spectral radius, $\rho(G)$, of $G$ is the largest eigenvalue of $A(G)$. By the Perron-Frobenius Theorem, the spectral radius is simple and has a unique positive eigenvector. We will refer to such an eigenvector as the Perron vector of $G$.

The problem [3] concerning maximum spectral radius of a given class of graphs has been studied extensively. The chromatic number $\chi(G)$ of a graph $G$ is the minimum number of colors such that $G$ can be colored in a way such that no two adjacent vertices have the same color. According to the result of Brooks [2], we have $\chi(G) \leq \Delta(G)+1$ with
equality if and only if $G$ is an odd cycle or a complete graph. By using the spectral radius of a graph, Cao [4] and Wilf [14] improved this result. Tian et al. [12] considered the spectral radius of graphs with given chromatic number. So, it is important to estimating the bounds of the spectral radius of graphs. For given order $n$ and diameter $D$, Zhai et al. [15] determined bipartite graphs with the maximum spectral radius. Van Dam [13] determined the extremal graphs with maximum spectral radius among the ones on $n$ nodes with diameter $D$. It is well known that every graph belongs to a class of multipartite graphs. So, it is important to consider the maximum spectral radius of multipartite graphs. This paper focuses on the maximum spectral radius of multipartite graphs with given $r(r>2)$ vertex parts, order $n$ and diameter $D$. That is, we will consider the following problem:

Of all the $r$-partite graphs in $\mathcal{G}(r, n, D)$, which achieves the maximum spectral radius?

Let $T\left(\left\lceil\frac{D-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right) \in \mathcal{G}(r, n, D)$ be the graph obtained from a complete $r$-partite graph on $n-D+1$ vertices in which each vertex part has either $\left\lfloor\frac{n-D+1}{r}\right\rfloor$ or $\left\lceil\frac{n-D+1}{r}\right\rceil$ vertices: joining one pendent vertex of the path $P_{\left[\frac{D-1}{2}\right]}$ with all vertices in one vertex part which has $\left\lfloor\frac{n-D+1}{r}\right\rfloor$ vertices and joining one pendent vertex of the path $P_{\left\lceil\frac{D-1}{2}\right\rceil}$ with all vertices in the other $r-1$ vertex parts. In this paper, we will show that the $r$ - partite graph $T\left(\left\lceil\frac{D-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right)$ is just the extremal graph with the maximum spectral radius in $\mathcal{G}(r, n, D)$. Note that the case $D=n-1$ is contained in [15] and for $D=1$, the extremal graph is just the complete graphs $K_{n}$ (see [6]). It is easy to check that $3 \leq r \leq n-D+1$.

The rest of this work is organized as follows. In the rest of section 1, we give some results used in this paper. In
section 2 , the main results are given. In section 3,4 and 5 , the main results are proved. In section 6 , conclusion of our paper is drawn. Next, there are some useful results used in this paper.

The characteristic polynomial of $G$ is just $\operatorname{det}(x I-A(G))$, which is denoted by $\Phi(G, x)$ or simply by $\Phi(G)$.
Lemma 1 [8]. Let $v$ be a pendent vertex in $G, v w \in E(G)$. Then

$$
\Phi(G, x)=x \Phi(G-v ; x)-\Phi(G-v-w ; x)
$$

By the Perron-Frobenius Theorem, we can obtain the following results which can also be found in [5] and [7], respectively.

Lemma 2. Let $G_{1}$ and $G_{2}$ be two graphs. Then
(i) If $\Phi\left(G_{2}, x\right)>\Phi\left(G_{1}, x\right)$ for $x \geq \rho\left(G_{2}\right)$ then $\rho\left(G_{1}\right)>\rho\left(G_{2}\right)$;
(ii) If $G_{2}$ is a proper subgraph of a connected graph $G_{1}$, then $\rho\left(G_{2}\right)<\rho\left(G_{1}\right)$.

The complete $r$-partite graph on $n$ vertices in which each part has either $\left\lfloor\frac{n}{r}\right\rfloor$ or $\left\lceil\frac{n}{r}\right\rceil$ vertices is denoted by $T_{r, n}$. Let $e\left(T_{r, n}\right)$ denotes the number of edges of graph $T_{r, n}$. The following result can be found in [1].

Lemma 3. Let $G$ is a complete $r$-partite graph on $n$ vertices. Then $e(G) \leq e\left(T_{r, n}\right)$, and the equality holds if and only if $G \cong T_{r, n}$.

## 2. Main results

Theorem 1. Let $n, D$ and $r$ be integers with $3 \leq r \leq$ $n-D+1$ and $2 \leq D \leq n-2$. Then the graph

$$
T\left(\left\lceil\frac{D-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right)
$$

is the unique graph with the maximum spectral radius in $\mathcal{G}(r, n, D)$.

Theorem 2. Let $r \in[3, n-D+1]$ and order $n$ be fixed integers. Then for $D \in[2, n-2]$,

$$
\rho\left(T\left(\left\lceil\frac{D-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right)\right)
$$

the spectral radius of the extremal graphs in $\mathcal{G}(r, n, D)$ is a decreasing function on $D$.

Theorem 3. Let diameter $D \in[2, n-2]$ and order $n$ be fixed integers. Then for $r \in[3, n-D+1]$,

$$
\rho\left(T\left(\left\lceil\frac{D-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right)\right)
$$

the spectral radius of the extremal graphs in $\mathcal{G}(r, n, D)$ is an increasing function on $r$.

## 3. Proof of Theorem 1

The proof of this result will be established in a number of steps. Note that for $G \in \mathcal{G}(r, n, D)$, there exists a distance partition $\Pi$ :

$$
\begin{gathered}
U_{0}=\{u\} \\
U_{i}=\left\{v \mid d_{G}(u, v)=i, i=1,2, \cdots, D, u \in U_{0}, v \in V(G) \backslash U_{0}\right\} \\
V(G)=\cup_{i=0}^{D} U_{i}
\end{gathered}
$$

Next, we will characterize the extremal graphs with the distance partition $\Pi$ when the spectral radius of graph $G \in$ $\mathcal{G}(r, n, D)$ achieves the maximum.

Proposition 3.1. Let $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximal spectral radius. Then for each $i \in\{0,1,2, \cdots, D-1\}$,
(i) $G\left[U_{i} \cup U_{i+1}\right]$ induce a complete $k$-partite subgraph with $2 \leq k \leq r$;
(ii) There exist two partition sets $U_{i_{0}}, U_{i_{0}+1}$ such that $G\left[U_{i_{0}} \cup U_{i_{0}+1}\right]$ is a complete $r$-partite subgraph.

Proof of Proposition 3.1. (i) Suppose that for each $i \in$ $\{0,1,2, \cdots, D-1\}, G\left[U_{i} \cup U_{i+1}\right]$ induce a $k$-partite subgraph with $k>r$ or $k=1$. The inequality $k>r$ contradicts the fact that $G \in \mathcal{G}(r, n, D)$. According to the partition $U_{0}$, $U_{1}, \cdots, U_{D}$ of $V(G), U_{i}$ and $U_{i+1}$ are two different partition sets. Thus, the integer $k$ is not equal to 1 . So, the inequality $2 \leq k \leq r$ holds. On the other hand, Suppose that for each $i \in\{0,1,2, \cdots, D-1\}, G\left[U_{i} \cup U_{i+1}\right]$ induce a $k$-partite subgraph which is not complete. Let $U_{j}(j=1,2, \cdots, k)$ be the partition classes of the subgraph $G\left[U_{i} \cup U_{i+1}\right]$. Moreover, for any different integers $j_{1}, j_{2} \in\{1,2, \cdots, k\}$, we can assume, without loss of generality, that there exist two vertices $u \in U_{j_{1}}, v \in U_{j_{2}}$ which are not adjacent. Now, we add the edge $u v$ to the induced subgraph. Let $G^{\prime}$ be a graph obtained by adding an edge between $u$ and $v$. Then $G^{\prime} \in \mathcal{G}(r, n, D)$ and $e\left(G^{\prime}\right)>e(G)$. Thus, we have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction.
(ii) This case is clear because $G$ is a graph in $\mathcal{G}(r, n, D)$.

Proposition 3.2. Let $G \in \mathcal{G}(r, n, D)$ with distance partition $\Pi$ and maximum spectral radius. If $V^{\prime}$ is a vertex part of induced subgraph $H_{i}=G\left[U_{i} \cup U_{i+1}\right], i \in\{0,1,2, \cdots, D-1\}$, then $V^{\prime} \subseteq U_{i}$ or $V^{\prime} \subseteq U_{i+1}$.

Proof of Proposition 3.2. Let X be the Perron vector of $G$, where $x_{\omega}$ corresponds to the vertex $\omega \in V(G)$. For each $i \in$ $\{0,1,2, \cdots, D\}$, we assume $N_{G}^{i}(\omega)$ be the sets of $\omega$ 's neighbors in vertex sets $U_{i}$.

Suppose that there exist two vertices $u \in U_{i} \cap V^{\prime}$ and $v \in$ $U_{i+1} \cap V^{\prime}$. Hence, these two vertices are not adjacent in the subgraph $G\left[U_{i} \cup U_{i+1}\right]$. Now we consider the following two cases to finish the proof.

Case 1: For the fixed integer $i \in\{0,1,2, \cdots, D-1\}$, $G\left[U_{i} \cup U_{i+1}\right]$ induce a $k$-partite subgraph with $2 \leq k<r$. Let $G^{\prime}$ be a graph obtained by adding an edge between $u$ and $v$. Clearly, $G^{\prime} \in \mathcal{G}(r, n, D)$ and $e\left(G^{\prime}\right)>e(G)$. Thus, we have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction.

Case 2: For the fixed integer $i \in\{0,1,2, \cdots, D-1\}$, $G\left[U_{i} \cup U_{i+1}\right]$ induce a $r$-partite subgraph. Now we consider the following three subcases to finish the proofs of Case 2.

Subcase 2.1: For the fixed integer $i \in\{1, \cdots, D-2\}$, $G\left[U_{i} \cup U_{i+1}\right]$ induce a $r$-partite subgraph. Without loss of generality, we assume that $x_{u} \leq x_{v}$. Let $G^{\prime}$ be the graph obtained by deleting all edges between the vertex $u$ and all its neighbours in $U_{i-1}$, and joining $u$ with all neighbours of the vertex $v$ in $U_{i+2}$. Clearly, $G^{\prime} \in \mathcal{G}(r, n, D)$ and

$$
\begin{aligned}
\rho\left(G^{\prime}\right)-\rho(G) & \geq X^{T} A\left(G^{\prime}\right) X-X^{T} A(G) X \\
& =2 x_{u} \cdot\left[\sum_{\omega \in N_{G}^{i+2}(v)} x_{\omega}-\sum_{\gamma \in N_{G}^{i-1}(u)} x_{\gamma}\right] .
\end{aligned}
$$

Since $A(G) X=\rho(G) X$,

$$
\begin{aligned}
\rho(G) x_{v} & =\sum_{\omega \in N_{G}^{i+2}(v)} x_{\omega}+\sum_{\omega \in N_{G}^{i+1}(v)} x_{\omega}+\sum_{\omega \in N_{G}^{i}(v)} x_{\omega}, \\
\rho(G) x_{u} & =\sum_{\gamma \in N_{G}^{i-1}(u)} x_{\gamma}+\sum_{\gamma \in N_{G}^{i}(u)} x_{\gamma}+\sum_{\gamma \in N_{G}^{i+1}(u)} x_{\gamma}
\end{aligned}
$$

with

$$
\sum_{\omega \in N_{G}^{i+1}(v)} x_{\omega}=\sum_{\gamma \in N_{G}^{i+1}(u)} x_{\gamma}, \sum_{\omega \in N_{G}^{i}(v)} x_{\omega}=\sum_{\gamma \in N_{G}^{i}(u)} x_{\gamma} .
$$

Thus

$$
\rho\left(G^{\prime}\right)-\rho(G) \geq 2 x_{u} \rho(G)\left(x_{v}-x_{u}\right) \geq 0
$$

If $\rho\left(G^{\prime}\right)=\rho(G)$ then $x_{v}=x_{u}$; for the vertex $\omega \in N_{G}^{i+2}(v)$, we have $\quad x_{u}=\rho\left(G^{\prime}\right) x_{\omega}-\rho(G) x_{\omega}=0, \quad$ a contradiction. Consequently, we have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction.

Subcase 2.2: For $i=0, \quad G\left[U_{i} \cup U_{i+1}\right]$ induce a $r$-partite subgraph.

In this subcase, we can obtain the distance between $u$ and $v$ will be greater than 1 , a contradiction.

Subcase 2.3: For $i=D-1, G\left[U_{i} \cup U_{i+1}\right]$ induce a $r$-partite subgraph.

By Case 1, Subcase 2.1 and Subcase 2.2, for each integer $i \in\{0,1,2, \cdots, D-2\}$, if $u \in U_{i}$ and $v \in U_{i+1}$ then $u v \in$ $E(G)$ and $G\left[U_{i} \cup U_{i+1}\right]$ induce a complete multipartite subgraph with $k(2 \leq k \leq r)$ vertex parts. Therefore, if $G\left[U_{i+1}\right] \cong G\left[U_{D}\right] \cong\left|U_{D}\right| \cdot K_{1}$ then $G\left[U_{D-2} \cup U_{D-1}\right]$ induce a $k$-partite subgraph with $k>r$, implying a contradiction. Thus we assume that $G\left[U_{i+1}\right] \cong G\left[U_{D}\right]$ induce a $k$-partite subgraph with $2 \leq k \leq r$. Then let $G^{\prime}$ be the graph obtained by joining $v$ with all neighbours of the vertex $u$ in $U_{i-1}=$ $U_{D-2}$. Clearly, $G^{\prime} \in \mathcal{G}(r, n, D)$ and $e\left(G^{\prime}\right)>e(G)$. Thus, we have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction. Therefore, Case 1 and Case 2 complete the proof.

The following result can be deduced from Proposition 3.1 and Proposition 3.2.

Proposition 3.3. Let $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximal spectral radius. Then, for each integer $i \in\{0,1,2, \cdots, D-1\}$, if $u \in U_{i}$ and $v \in U_{i+1}$ then $u v \in E(G)$.

Remark 3.4. Let $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ of $V(G)$ have the maximal spectral radius. By the Proposition 3.1, Proposition 3.2 and Proposition 3.3, $G\left[U_{i}\right]$ induce a complete $k$-partite subgraph with $2 \leq k \leq r-1$ or
$G\left[U_{i}\right] \cong\left|U_{i}\right| \cdot K_{1}$ (here, we call this subgraph an empty graph or 1 -partite graph). Thus, we can give some useful symbols. Let $U_{i}=\cup_{m=1}^{k_{i}} U_{i, m}$, where $i=0,1, \cdots, D, 1 \leq k_{i} \leq r-1$ and $G\left[U_{i, m}\right] \cong\left|U_{i, m}\right| \cdot K_{1}$. In particular, if $k_{i}=1$ then $U_{i}=$ $V_{i, 1}$ and $G\left[U_{i}\right] \cong\left|V_{i}\right| \cdot K_{1} \cong\left|U_{i, 1}\right| \cdot K_{1}$. Clearly, $U_{0}=U_{0,1}$. By Proposition 3.1(ii), there exist two vertex parts $U_{i_{0}}, U_{i_{0}+1}$ such that $G\left[U_{i_{0}} \cup U_{i_{0}+1}\right]$ induce a complete $r$-partite subgraph. In fact, the size of the other vertex parts can also be determined.

Proposition 3.5. Let $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximal spectral radius. Then, there exist at most two partition sets $U_{i}$ and $U_{j}$ containing more than one vertex. Furthermore, $|i-j|=1$.

Proof of Proposition 3.5. The cases $D=2,3$ are trivial. Now let $D \geq 4$ and $G$ be an extremal graph in $\mathcal{G}(r, n, D)$. By Remark 3.4, we have $U_{i}=\cup_{m=1}^{k_{i}} U_{i, m}$. Let X be the Perron vector of $G$. We observe that $x_{u}=x_{v}$ for any two vertices $u, v \in U_{i, m}$, since they have the same neighbors. Let $x_{i, m}$ correspond to the vertices in $U_{i, m}$. By Proposition 4.1(ii), there exist two vertex parts $U_{i_{0}}, U_{i_{0}+1}$ such that $G\left[U_{i_{0}} \cup\right.$ $U_{i_{0}+1}$ ] induce a complete $r$-partite subgraph. Now we have to show that the vertex parts contain no more than one vertex other than $U_{i_{0}}, U_{i_{0}+1}$.

Suppose that $\left|U_{i}\right|,\left|U_{j}\right| \geq 2$ with $|i-j| \geq 2$ and $i, j \notin$ $\left\{i_{0}, i_{0}+1\right\}$. Without loss of generality, we assume that

$$
x_{i, l} \geq x_{j, h}
$$

and by Remark 3.4,

$$
\begin{aligned}
& U_{i}=\left(\bigcup_{m=1}^{l-1} U_{i, m}\right) \cup\left(\bigcup_{m=l+1}^{k_{i}} U_{i, m}\right) \cup U_{i, l} \\
& U_{j}=\left(\bigcup_{m=1}^{h-1} U_{j, m}\right) \cup\left(\bigcup_{m=h+1}^{k_{j}} U_{j, m}\right) \cup U_{j, h}
\end{aligned}
$$

Select a vertex $u \in U_{j, h}$, and let $G^{\prime}$ be a graph obtained by deleting all edges incident to $u$ and joining $u$ with all vertices in $\left(U_{i-1} \cup U_{i+1}\right) \cup\left(U_{i} \backslash U_{i, l}\right)$. Clearly, $G^{\prime} \in$ $\mathcal{G}(r, n, D)$ and

$$
\begin{aligned}
\rho\left(G^{\prime}\right)-\rho(G) \geq & X^{T} A\left(G^{\prime}\right) X-X^{T} A(G) X \\
= & 2 x_{j, h} \cdot\left[\sum_{m=1}^{k_{i-1}}\left|U_{i-1, m}\right| \cdot x_{i-1, m}+\sum_{m=1}^{k_{i+1}}\left|U_{i+1, m}\right| \cdot x_{i+1, m}\right. \\
& \left.+\sum_{m=1}^{l-1}\left|U_{i, m}\right| \cdot x_{i, m}+\sum_{m=l+1}^{k_{i}}\left|U_{i, m}\right| \cdot x_{i, m}\right] \\
& -2 x_{j, h} \cdot\left[\sum_{m=1}^{k_{j-1}}\left|U_{j-1, m}\right| \cdot x_{j-1, m}+\sum_{m=1}^{k_{j+1}}\left|U_{j+1, m}\right| \cdot x_{j+1, m}\right. \\
& +\sum_{m=1}^{h-1}\left|U_{j, m}\right| \cdot x_{j, m} \\
& \left.+\sum_{m=h+1}^{k_{j}}\left|U_{j, m}\right| \cdot x_{j, m}\right] .
\end{aligned}
$$

Since $A(G) X=\rho(G) X$,

$$
\begin{aligned}
\rho(G) x_{i, l}= & \sum_{m=1}^{k_{i-1}}\left|U_{i-1, m}\right| \cdot x_{i-1, m}+\sum_{m=1}^{k_{i+1}}\left|U_{i+1, m}\right| \cdot x_{i+1, m} \\
& +\sum_{m=1}^{l-1}\left|U_{i, m}\right| \cdot x_{i, m}+\sum_{m=l+1}^{k_{i}}\left|U_{i, m}\right| \cdot x_{i, m} \\
\rho(G) x_{j, h} & =\sum_{m=1}^{k_{j-1}}\left|U_{j-1, m}\right| \cdot x_{j-1, m}+\sum_{m=1}^{k_{j+1}}\left|U_{j+1, m}\right| \cdot x_{j+1, m} \\
& +\sum_{m=1}^{h-1}\left|U_{j, m}\right| \cdot x_{j, m}+\sum_{m=h+1}^{k_{j}}\left|U_{j, m}\right| \cdot x_{j, m} .
\end{aligned}
$$

Thus

$$
\rho\left(G^{\prime}\right)-\rho(G) \geq 2 x_{j, h} \rho(G)\left(x_{i, l}-x_{j, h}\right) \geq 0
$$

If $\rho\left(G^{\prime}\right)=\rho(G)$ then $x_{i, l}=x_{j, h}, \rho\left(G^{\prime}\right) X=A\left(G^{\prime}\right) X$. Thus, we have

$$
\begin{aligned}
\rho(G) x_{i-1,1}= & \sum_{m=1}^{k_{i-2}}\left|U_{i-2, m}\right| \cdot x_{i-2, m}+\sum_{m=1}^{k_{i}}\left|U_{i, m}\right| \cdot x_{i, m} \\
& +\sum_{m=2}^{k_{i-1}}\left|U_{i-1, m}\right| \cdot x_{i-1, m} \\
\rho\left(G^{\prime}\right) x_{i-1,1}= & \sum_{m=1}^{k_{i-2}}\left|U_{i-2, m}\right| \cdot x_{i-2, m}+\sum_{m=1}^{k_{i}}\left|U_{i, m}\right| \cdot x_{i, m} \\
& +\sum_{m=2}^{k_{i-1}}\left|U_{i-1, m}\right| \cdot x_{i-1, m}+x_{j, h}
\end{aligned}
$$

a contradiction. Consequently, we have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction.

Proposition 3.6. Let $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximal spectral radius. Then $\left|U_{D}\right|=1$ for $D \geq 3$.

Proof of Proposition 3.6. Suppose that $\left|U_{D}\right| \geq 2$. Then by Proposition $3.5,\left|U_{i}\right|=1(i=0,1,2 \cdots, D-1)$ or $\left|U_{i}\right|=$ $1(i=0,1,2 \cdots, D-2)$. Let $G^{\prime}$ be a graph obtained by deleting the edge incident to $u_{0} \in U_{0}$ and joining $u_{0}$ with all vertices in $U_{D}$. Clearly, $G^{\prime} \in \mathcal{G}(r, n, D)$ and $e\left(G^{\prime}\right)>e(G)$. Thus, we have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction.

By Proposition 3.5 and 3.6 , if $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximal spectral radius then there exist at most two consecutive partition classes $U_{i_{0}}$ and $U_{i_{0}+1}$ of $\Pi$ containing more than one vertex, and $G\left[U_{i_{0}} \cup U_{i_{0}+1}\right]$ is a induced complete $r$-partite subgraph. That is, each of the rest of the partition classes of $\Pi$ contains only one vertex.

Let $G \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximum spectral radius. Without loss of generality, we assume that the partition classes $U_{a}$ and $U_{a+1}$ satisfy the conditions of Proposition 3.1-3.6. That is, for integer $D \geq$ 3, $U_{a}, U_{a+1}$ induce a complete $(\alpha+\beta)$-partite graph with 2 $\leq \alpha+\beta \leq \chi(G)$. Each of the partition classes other than $U_{a}$ and $U_{a+1}$ has only one vertex. Hence, the distance partition $\Pi$ of extremal graph $G \in \mathcal{G}(r, n, D)$ with maximum spectral radius can be displayed as

$$
\Pi: U_{0}, U_{1}, \cdots, U_{a-1}, U_{a}, U_{a+1}, U_{a+2}, \cdots, U_{D}
$$

The extremal graph $G=G_{a, b}$ with the distance partition $\Pi$ can be obtained by

- constructing the path $P_{a}: u_{0} u_{1} \cdots u_{a-1}\left(u_{i} \in U_{i}, i=0\right.$, $1, \cdots, a-1)$, and joining the end vertex $u_{a-1} \in U_{a-1}$ to every vertices in $U_{a}$;
- constructing the path $P_{b}: u_{a+2} u_{a+3} \cdots u_{D}\left(u_{j} \in U_{j}, j=\right.$ $a+2, a+3, \cdots, D)$, and joining the end vertex $u_{a+2} \in$ $U_{a+2}$ to every vertices in $U_{a+1}$.

Moreover, the vertex set, partition classes and diameter of extremal graph $G_{a, b}$ satisfy

$$
\begin{aligned}
V(G)=V\left(P_{a}\right) & \cup U_{a} \cup U_{a+1} \cup V\left(P_{b}\right), \\
U_{a} & =\cup_{i=1}^{\alpha} U_{a, i} \\
U_{a+1} & =\cup_{j=1}^{\beta} U_{a+1, j} \\
G\left[U_{a, i}\right] & \cong\left|U_{a, i}\right| \cdot K_{1} \\
G\left[U_{a+1, j}\right] & \cong\left|U_{a+1, j}\right| \cdot K_{1} \\
a+b & =D-1
\end{aligned}
$$

Proposition 3.7. Let $G=G_{a, b} \in \mathcal{G}(r, n, D)$ with the distance partition $\Pi$ and maximal spectral radius. Suppose that $X$ be a Perron vector of $G$. For any vertex $\omega \in V(G)$, its corresponding component in $X$ is denoted by $x_{\omega}$. Then
(1) $G\left[U_{a} \cup U_{a+1}\right]$ is an induced $r$-partite subgraph with $n-D+1$ vertices in $G_{a, b}$; note that each vertex part have $\left\lfloor\frac{n-D+1}{r}\right\rfloor$ vertices or $\left\lceil\frac{n-D+1}{r}\right\rceil$ vertices;
(2) Either $\alpha=1$ and $G\left[U_{a}\right] \cong\left\lfloor\frac{n-D+1}{r}\right\rfloor \cdot K_{1}$ or $\beta=1$ and $G\left[U_{a+1}\right] \cong\left\lfloor\frac{n-D+1}{r}\right\rfloor \cdot K_{1}$;
(3) $a-b \in\{0,1\}$, if $x_{a-1} \geq x_{a+2}$.

Proof of Proposition 3.7. (1) By Proposition 3.1-3.6, it is true that $\left|U_{a} \cup U_{a+1}\right|=n-D+1$, and that $G\left[U_{a} \cup U_{a+1}\right]$ is a complete induced $(\alpha+\beta)$-partite subgraph with $2 \leq \alpha+$ $\beta \leq r$. If $\alpha+\beta<r$, then $G\left[U_{a-1} \cup U_{a}\right]$ a complete induced $(\alpha+1)$-partite subgraph with $\alpha+1<r ; G\left[U_{a+1} \cup U_{a+2}\right]$ is a complete induced $(\beta+1)$-partite subgraph with $\beta+1$ $<r$. Thus, we have $G \notin \mathcal{G}(r, n, D)$, a contradiction. So, we must have $\alpha+\beta=r$. Furthermore, each vertex part in $G\left[U_{a} \cup U_{a+1}\right]$ have $\left\lfloor\frac{n-D+1}{r}\right\rfloor$ vertices or $\left\lceil\frac{n-D+1}{r}\right\rceil$ vertices by Lemma 3, since it is well known that adding edges may increase the value of the spectral radius of any graph $G$.
(2) Suppose that $\alpha>1$ and $\beta>1$, and

$$
x_{u}=x_{v}, u, v \in U_{a, i} \quad\left(\text { or } u, v \in U_{a+1, j}\right)
$$

Without loss of generality, we assume that

$$
\begin{gathered}
x_{a-1} \leq x_{a+2} \\
\left|U_{a, 1}\right| \geq\left|U_{a, 2}\right| \geq, \cdots,\left|U_{a, \alpha}\right|
\end{gathered}
$$

The corresponding components of vertices $u_{a-1} \in U_{a-1}, u_{a, i}$ $\in U_{a, i}, u_{a+1, j} \in U_{a+1, j}, u_{a+2} \in U_{a+2}$ in $X$ are denoted by $x_{a-1}, x_{a, i}, x_{a+1, j}, x_{a+2}$, respectively. Let graph $G^{\prime}$ is obtained by

- deleting the edges between the vertices in $U_{a-1}$ and the vertices in $U_{a} \backslash U_{a, \alpha}$;
- joining the vertex $u_{a+2} \in U_{a+2}$ to every vertices in $U_{a} \backslash U_{a, \alpha}$.

Then, we have that

$$
\begin{aligned}
\rho\left(G^{\prime}\right)-\rho(G) & \geq X^{T} A\left(G^{\prime}\right) X-X^{T} A(G) X \\
& =2\left(\sum_{i=1}^{\alpha-1}\left|U_{a, i}\right| \cdot x_{a, i}\right) \cdot\left(x_{a+2}-x_{a-1}\right) \\
& \geq 0
\end{aligned}
$$

If $\rho\left(G^{\prime}\right)=\rho(G)$ then $x_{a-1}=x_{a+2}, \rho\left(G^{\prime}\right)=X^{T} A\left(G^{\prime}\right) X$. Thus, we have

$$
\begin{aligned}
& \rho\left(G^{\prime}\right) x_{a+2}=x_{a+3}+\sum_{j=1}^{\beta}\left|U_{a+1, j}\right| \cdot x_{a+1, j}+\sum_{i=1}^{\alpha-1}\left|U_{a, i}\right| \cdot x_{a, i}, \\
& \rho(G) x_{a+2}=x_{a+3}+\sum_{j=1}^{\beta}\left|U_{a+1, j}\right| \cdot x_{a+1, j},
\end{aligned}
$$

a contradiction. So, we must have $\rho\left(G^{\prime}\right)>\rho(G)$, a contradiction. Hence, it is true that $\alpha=1$, and that $G\left[U_{a}\right] \cong$ $\left\lfloor\frac{n-D+1}{r}\right\rfloor \cdot K_{1}$. Carrying out analogical proofs, for $x_{a-1} \geq$ $x_{a+2}$, we have $\beta=1$ and $G\left[U_{a+1}\right] \cong\left\lfloor\frac{n-D+1}{r}\right\rfloor \cdot K_{1}$.
(3) Suppose that $a-b \geq 2$. Then, the graph $G_{a, b}$ satisfy (1) and (2) of this present proposition. By Lemma 1, we have

$$
\begin{align*}
\Phi\left(G_{a, b} ; x\right) & =x \Phi\left(G_{a-1, b} ; x\right)-\Phi\left(G_{a-2, b} ; x\right)  \tag{3.1}\\
\Phi\left(G_{a-1, b+1} ; x\right) & =x \Phi\left(G_{a-1, b} ; x\right)-\Phi\left(G_{a-1, b-1} ; x\right) \tag{3.2}
\end{align*}
$$

From the formulas (3.1) and (3.2), we have

$$
\begin{aligned}
\Phi\left(G_{a, b} ; x\right)-\Phi\left(G_{a-1, b+1} ; x\right) & =\Phi\left(G_{a-1, b-1} ; x\right)-\Phi\left(G_{a-2, b} ; x\right) \\
& =\cdots \cdots \\
& =\Phi\left(G_{a-b, 0} ; x\right)-\Phi\left(G_{a-b-1,1} ; x\right)
\end{aligned}
$$

By Lemma 2(ii), we have

$$
\rho\left(G_{a-1, b+1}\right)>\rho\left(G_{a-b-1,1}\right) \geq \rho\left(G_{a-b, 0}\right)
$$

Now, suppose that

$$
\Phi\left(G_{a-b, 0} ; x\right)-\Phi\left(G_{a-b-1,1} ; x\right)<0
$$

Then, by Lemma 2(i), for $x \geq \rho\left(G_{a-1, b+1}\right)>\rho\left(G_{a-b-1,1}\right) \geq$ $\rho\left(G_{a-b, 0}\right)$, we have

$$
\rho\left(G_{a-b, 0}\right)>\rho\left(G_{a-b-1,1}\right)
$$

a contradiction. In fact, $\rho\left(G_{a-b, 0}\right) \leq \rho\left(G_{a-b-1,1}\right)$, since $e\left(G_{a-b, 0}\right)<e\left(G_{a-b-1,1}\right)$.

So, we must have

$$
\Phi\left(G_{a-b, 0} ; x\right)-\Phi\left(G_{a-b-1,1} ; x\right)>0
$$

That is

$$
\Phi\left(G_{a, b} ; x\right)-\Phi\left(G_{a-1, b+1} ; x\right)>0
$$

But, by the Lemma 2(ii), for $x \geq \rho\left(G_{a-1, b+1}\right)>\rho\left(G_{a-b-1,1}\right)$ $\geq \rho\left(G_{a-b, 0}\right)$, we also have $\rho\left(G_{a, b}\right)<\rho\left(G_{a-1, b+1}\right)$, which
contradicts the fact that $G_{a, b}$ achieves the maximum spectral radius. Therefore, $a-b \in\{0,1\}$.

By Proposition 3.1-Proposition 3.6, the extremal graph $G$ $\in \mathcal{G}(r, n, D)$ with maximum spectral radius has a special distance partition $\Pi$. According to the Proposition 3.7, the the extremal graph $G=G_{a, b}$. Therefore, the presented Theorem 1 is true.

## 4. Proof of Theorem 2

Clearly, the spectral radius of extremal graphs in $\mathcal{G}(r, n, D)$ is a function on $D$. We will show that it is a decreasing function on $D$.

For any integers $D_{1}, D_{2} \in[2, n-2]$, suppose that $D_{1}>$ $D_{2}$. Let $T\left(\left\lceil\frac{D_{1}-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D_{1}-1}{2}\right\rfloor\right)$ denoted by $T_{1}$ and $T\left(\left\lceil\frac{D_{2}-1}{2}\right\rceil, r-1,1,\left\lfloor\frac{D_{2}-1}{2}\right\rfloor\right)$ denoted by $T_{2}$ be the extremal graphs in $\mathcal{G}\left(r, n, D_{1}\right)$ and $\mathcal{G}\left(r, n, D_{2}\right)$, respectively. Next we will show that $\rho\left(T_{1}\right)<\rho\left(T_{2}\right)$.

Let $V\left(T_{1}\right)=V\left(P_{a}\right) \cup V_{a} \cup V_{a+1} \cup V\left(P_{b}\right)$, where $\quad V_{a}=$ $\cup_{i=1}^{r-1} V_{a, i}, \quad a=\left\lceil\frac{D_{1}-1}{2}\right\rceil, V\left(P_{a}\right)=\left\{v_{0}, v_{1}, \cdots, \quad v_{a-1}\right\}, \quad V\left(P_{b}\right)=$ $\left\{v_{a+2}, v_{a+3}, \cdots, v_{D_{1}}\right\}$. Now let $T_{3}$ be the graph obtained from $T_{1}$ by
(1) selecting the $p$ vertices $v_{0}, v_{1}, \cdots, v_{p-1} \in V\left(P_{a}\right)$ and the $q$ vertices $v_{D_{1}-q+1}, v_{D_{1}-q+2}, \cdots, v_{D_{1}} \in V\left(P_{b}\right)$, where $p+$ $q=D_{1}-D_{2}$;
(2) deleting edges incident to these $p+q$ vertices; joining these $p+q$ vertices with all vertices in $V_{a}$; joining these $p+q$ vertices with the vertex $v_{a+2} \in$ $V_{a+2}$.

Clearly, $T_{3} \in \mathcal{G}\left(r, n, D_{2}\right)$ and $\rho\left(T_{1}\right)<\rho\left(T_{3}\right)$, since $e\left(T_{1}\right)<$ $e\left(T_{3}\right)$. By Theorem 1, we have $\rho\left(T_{3}\right) \leq \rho\left(T_{2}\right)$. Thus $\rho\left(T_{1}\right)<$ $\rho\left(T_{2}\right)$. According to the arbitrariness of $D_{1}$ and $D_{2}$, we can get the result.

## 5. Proof of Theorem 3

Next we will show that the spectral radius of the extremal graphs in $\mathcal{G}(r, n, D)$ is an increasing function on $r$.

For any integers $r_{1}, r_{2} \in[3, n-D+1]$, suppose that $r_{1}<$ $r_{2}$. Let $T\left(\left\lceil\frac{D-1}{2}\right\rceil, \quad r_{1}-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right)$ denoted by $T_{1}$ and $T\left(\left\lceil\frac{D-1}{2}\right\rceil, r_{2}-1,1,\left\lfloor\frac{D-1}{2}\right\rfloor\right)$ denoted by $T_{2}$ be the extremal graphs in $\mathcal{G}\left(r_{1}, n, D\right)$ and $\mathcal{G}\left(r_{2}, n, D\right)$, respectively. Now we will show that $\rho\left(T_{1}\right)<\rho\left(T_{2}\right)$.

Let $V\left(T_{1}\right)=V\left(P_{a}\right) \cup V_{a} \cup V_{a+1} \cup V\left(P_{b}\right)$, where $\quad V_{a}=$ $\cup_{i=1}^{r_{1}-1} V_{a, i}, a=\left\lceil\frac{D-1}{2}\right\rceil$. To obtain the graph $T_{3}$ from $T_{1}$, a total of $r_{2}-r_{1}$ times the following operations will be carried out:
(1) joining one vertex $u \in V_{a, i}\left(\left|V_{a, i}\right| \geq 2, i=1,2, \cdots, r_{1}-\right.$ 1) to all vertices in $V_{a, i} \backslash\{u\}$;
(2) joining one vertex $v \in V_{a+1}\left(\left|V_{a+1}\right| \geq 2\right)$ to all vertices in $V_{a+1} \backslash\{v\}$.

Clearly, $T_{3} \in G\left(r_{2}, n, D\right)$ and $\rho\left(T_{1}\right)<\rho\left(T_{3}\right)$, since $e\left(T_{1}\right)<$ $e\left(T_{3}\right)$. By Theorem 1, we have $\rho\left(T_{3}\right) \leq \rho\left(T_{2}\right)$. Thus $\rho\left(T_{1}\right)<$ $\rho\left(T_{2}\right)$. According to the arbitrariness of $r_{1}$ and $r_{2}$, the proof is finished.

## 6. Conclusion

It is well known that every graph belongs to a class of multipartite graphs, and that estimating the spectral radius is helpful for estimating the bounds for some parameters of graphs. So, it is important to study the spectral radius of multipartite graphs (see [9-11]). In this paper, we consider the spectral radius of the class of multipartite graphs, and get the extremal graphs with the maximum spectral radius. To some extent, we extend the results in [15] to the class of multipartite graphs (Theorems 1, 2 and 3). Furthermore, the result (Theorem 1) is more specific than the result in [13]. That is, for estimating the maximum spectral radius of specific graph $G$, it is more accurate to using the result (Theorem 1) in this work than to using the result in [13].

## Acknowledgments

Thanks to Professor Weihua Yang of the School of Mathematics of Taiyuan University of Technology for his useful suggestions during the discussion of this problem.

## Disclosure statement

No potential conflict of interest was reported by the authors.

## Funding

This article was supported by the Teaching project of Shanxi University of Finance and Economics of P. R. China: The exploration
of College Students' entrepreneurship training model based on social network index random graph model, Grant 2017336.

## References

[1] Bondy, J. A, Murty, U. S. R. (1976). Graph Theory with Applications. New York: Macmillan.
[2] Brooks, R. L. (1941). On coloring the node of a network. Math. Proc. Cambridge Philos. Soc. 37(2): 194-197.
[3] Brualdi, R. A, Solheid, E. S. (1986). On the spectral radius of complementary acyclic matrices of zeros and ones. SIAM J. Algebr. Discrete Methods 7(2): 265-272.
[4] Cao, D. S. (1998). Bounds on eigenvalues and chromatic number. Linear Algebra Appl. 270(1-3): 1-13.
[5] Cvetković, D, Rowlinson, P. (1987). Spectra of unicyclic graphs. Graphs Combin. 3(1): 7-23.
[6] Hansen, P, Stevanović, D. (2008). On bags and bugs. Discrete Appl. Math. 156(7): 986-997.
[7] Li, Q, Feng, K. (1979). On the largest eigenvalue of graphs. Acta Math. Appl. Sinica 2: 167-175 (in Chinese).
[8] Liu, H., Lu, M, Tian, F. (2004). On the spectral radius of graphs with cut edges. Linear Algebra Appl. 389(1): 139-145.
[9] Saha, S., Adiga, A., Prakash, B. A, Vullikanti, A. K. S. (2015). Approximation algorithms for reducing the spectral radius to control epidemic spread. Comput. Sci. 568-576.
[10] Santos, A, Moura, J. M. F, Xavier J. (2013a). Epidemics in Multipartite Networks: Emergent Dynamics. Compute. Sci. Available at: https://arxiv.org/abs/1306.6812
[11] Santos, A, Moura, J. M. F, Xavier J. (2013b). Emergent Behavior in Multipartite Large Networks: Multi-Virus Epidemics. Compute. Sci. Available at: https://arxiv.org/abs/1306.6198
[12] Tian, F., Li, h., Li, Q, Zhang, X.-D. (2007). Spectral radii of graphs with given chromatic number. Appl. Math. Lett. 20(2): 158-162.
[13] Van Dam, E. R. (2007). Graphs with given diameter maximizing the spectral radius. Linear Algebra Appl. 426(2-3): 454-457.
[14] Wilf, H. S. (1967). The eigenvalues of a graph and its chromatic number. J. Lond. Math. Soc. s1-42(1): 330-332.
[15] Zhai, M. Q., Liu, R. F, Shu, J. L. (2009). On the spectral radius of bipartite graphs with given diameter. Linear Algebra Appl. 430(4): 1165-1170.

