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Extensions of Richardson's theorem for infinite digraphs and $(\mathscr{A}, \mathscr{B})$-kernels

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ABSTRACT

Let D be a digraph and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$, where $\mathscr{P}_D = \{P$: P is a non trivial finite path in $D\}$. A subset N of $V(D)$ is said to be an $(\mathscr{A}, \mathscr{B})$-kernel of D if: (1) for every $\{u,v\} \subseteq N$ there exists no uv-path P such that $P \in \mathscr{A}$ (N is $\mathscr{A}$-independent), (2) for every vertex x in $V(D)\backslash N$ there exist y in N and P in $\mathscr{B}$ such that P is an xy-path (N is $\mathscr{B}$-absorbent). As a particular case, the concept of $(\mathscr{A}, \mathscr{B})$-kernel generalizes the concept of kernel when $\mathscr{A} = \mathscr{B} = A(D)$. A classical result in kernel theory is Richardson's theorem which establishes that if D is a finite digraph without odd cycles, then D has a kernel. In this paper, the original results are sufficient conditions for the existence of $(\mathscr{A}, \mathscr{B})$-kernels in possibly infinite digraphs, in particular we will present some generalizations of Richardson's theorem for infinite digraphs. Also we will deduce some conditions for the existence of kernels by monochromatic paths, H-kernels and (k,l)-kernels in possibly infinite digraphs.

KEYWORDS
Kernel; kernel by monochromatic paths; H-kernel; $\mathscr{R}$-kernel; $(\mathscr{A}, \mathscr{B})$-kernel

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1. Introduction

For general concepts we refer the reader to [2, 3]. An arc of the form (x,x) is a *loop*. We will say that two digraphs D_1 and D_2 are equal, denoted by $D_1 = D_2$, if $A(D_1) = A(D_2)$ and $V(D_1) = V(D_2)$. A *directed walk* in a digraph D is a sequence $(v_1, v_2, \ldots, v_n)$ of vertices of D such that $(v_i,v_{i+1}) \in A(D)$ for each $i \in \{1, \ldots, n-1\}$. We will say that the directed walk $(v_1, v_2, \ldots, v_n)$ is *closed* if $v_1 = v_n$. If $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$, it is called a *directed path*. A *directed cycle* is a directed walk $(v_1, v_2, \ldots, v_n, v_1)$ such that $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$. If D is an infinite digraph, an *infinite outward path* is an infinite sequence $(v_1, v_2, \ldots)$ of distinct vertices of D such that $(v_i,v_{i+1}) \in A(D)$ for each $i \in \mathbb{N}$. In this paper we are going to write a walk, path, cycle instead of a directed walk, directed path, directed cycle, respectively. The union of walks will be denoted by $\cup$. Let S_1 and S_2 be subsets of $V(D)$, an arc (u,v) of D will be called an S_1S_2-arc whenever $u \in S_1$ and $v \in S_2$. For $S \subseteq V(D)$ we define the *in-neighborhood* of S as $\Gamma_D^-(S) = \{y \in V(D)$: $(y,v) \in A(D)$ for some $v \in S\}$. We define the *out-neighborhood* of S as $\Gamma_D^+(S) = \{y \in V(D)$: $(v,y) \in A(D)$ for some $v \in S\}$. A digraph D is *strong* if for every pair of vertices x and y there is an xy-path and there is a yx-path in D. A subdigraph G of D is called a *strong component* if it is strong and it is maximal with respect to this property. A strong component G of D is called *initial* (respectively *terminal*) if $\Gamma_D^-(V(G)) \subseteq V(G)$ (respectively $\Gamma_D^+(V(G)) \subseteq V(G)$). For $S \subseteq V(D)$ the subdigraph of D induced by S, denoted by $D[S]$, has $V(D[S]) = S$ and $A(D[S]) = \{(u,v) \in A(D)$: $\{u, v\} \subseteq S\}$. For a nonempty subset B of $A(D)$, the *arc-induced subdigraph* by B, denoted by $D[B]$, is the subdigraph of D such that $V(D[B]) = \{v \in V(D)$: either $(v,w) \in B$ or $(w,v) \in B$ for some $w \in V(D)\}$ and $A(D[B]) = B$. We shall say that a subset $S \subseteq V(D)$ is *independent* if the only arcs in $D[S]$ are loops. A digraph D is *bipartite* if there is a partition (V_1, V_2) of $V(D)$ such that $D[V_i]$ is an independent set for each $i \in \{1, 2\}$. A digraph D is *transitive* whenever $(u,v) \in A(D)$ and $(v,w) \in A(D)$ implies $(u,w) \in A(D)$. A digraph D is *symmetric* whenever $(u,v) \in A(D)$ implies $(v,u) \in A(D)$. A digraph D is said to be *m-colored* if the arcs of D are colored with m colors. Let D be an m-colored digraph. A path is called *monochromatic* if all of its arcs are colored alike. For an arc (u,v) of D we will denote by $c(u,v)$ its color. Let H be a digraph possibly with loops and D a digraph without loops. D is said to be H-colored if its arcs are colored with the vertices of H. A walk (path) W in D is an *H-walk* (*H-path*) if and only if the consecutive colors encountered on W form a walk in H.

Definition 1.1. Let D be a digraph. A subset K of $V(D)$ is said to be a kernel if it is both independent (a vertex in K has no successor in K) and absorbing (a vertex not in K has a successor in K).

In [14] von Neumann and Morgenstern introduced the concept of kernel of a digraph in the context of Game Theory. This concept attracted a lot of attention due to its

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aplications in winning strategies of some combinatorial games, in Mathematical Logic and in Nim-type games, to name a few. Moreover, some problems may be modelled by use of digraph kernels. The main problems when we want to find a kernel are: first, that not every digraph has a kernel, and that the problem of determining if a given digraph has a kernel is NP-complete, which was proved by Chvátal [4]. Kernels have been widely studied, many sufficient conditions for the existence of a kernel are known. An interesting survey on the subject is [7]. The first works on the existence of kernels in digraphs were for finite digraphs. In previous years, some works have been done in the direction of finding sufficient condition for the existence of kernels in infinite digraphs but these are very a few. In addition to the theorical interest that has to consider kernels in infinite digraphs, some applications have arisen throughout the literature. For example, in [6] Duchet and Meyniel considered a two player game on a progressively and locally finite digraph (no vertex is the origin of an infinite path and only a finite number of succesors for every vertex, respectively) and they proved that the first player wins if and only if the digraph has a local-kernel (a non-empty independent subset L of $V(D)$ such that every element of $\Gamma_D^+(L)$ has a successor in L). However, kernel theory in infinite digraphs is still a relatively unexplored subject, so far there are few results that guarantee the existence of kernels in infinite digraphs, see [12, 15, 17].

The first sufficient conditions for a finite digraph to have a kernel were given on the cycles of the digraph. For example, in [14] Neumann and Morgenstern proved that every finite digraph without cycles has a unique kernel. In [16] Richardson generalized this result and he proved that every finite digraph without cycles of odd length has a kernel. Indeed, many extensions of Richardson's theorem have been found. In this paper we will show some generalizations of Richardson's theorem for infinite digraphs.

In the literature we know some generalizations of the concept of kernel in digraphs, manely: *kernel by monochromatic paths* ([18] Sands, Sauer and Woodrow [8], Galeana-Sánchez), *(k,l)-kernel* ([13] Kwaśnik and Borowiecki), *H-kernel* ([5] Galeana-Sánchez and Delgado-Escalante), *H-kernel by walks* ([1] Arpin and Linek), π-kernel ([9] Galeana-Sánchez and Montellano-Ballesteros).

Since the previous concepts have some similarity, in [11] the authors generalized those concepts as follows: Consider a digraph D and define $\mathscr{P}_D = \{P\colon P$ is a non trivial finite path in $D\}$.

Definition 1.2. Let D be a digraph, N a subset of $V(D)$ and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$. N is an $(\mathscr{A},\mathscr{B})$-kernel of D if

1. For every $\{u,v\} \subseteq N$ there exists no uv-path P such that $P \in \mathscr{A}$ (N is $\mathscr{A}$-independent).
2. For every vertex x in $V(D)\backslash N$ there exist y in N and P in $\mathscr{B}$ such that P is an xy-path (N is $\mathscr{B}$-absorbent).

When $\mathscr{A} \subseteq A(D)$ and $\mathscr{B} = A(D)$ then we will say that an $(\mathscr{A},\mathscr{B})$-kernels is called an $\mathscr{A}$- *kernel*. From now on we will write $\mathscr{R}$- *kernel* instead of $\mathscr{A}$- *kernel*

When a digraph has no a kernel we can ask the following, with respect to $\mathscr{A}$: How large can be a set $\mathscr{A}$ such that D has an $\mathscr{A}$-kernel? If N is an $\mathscr{A}$-kernel with $\mathscr{A}$ the maximum number of elements, then N is similar to a kernel, and it is in this sense that such sets are as close as possible to a kernel.

In this work we will show sufficient conditions for the existence of $(\mathscr{A},\mathscr{B})$-kernels in possibly infinite digraphs. This paper is divided into three parts, the first one studies the existence of $\mathscr{R}$-kernels and the main result (Theorem 2.6) established that if D is a digraph without infinite outward path and $\mathscr{R}$ is a nonempty subset of $A(D)$ such that every cycle in D has even $\mathscr{R}$-length, then D has an $\mathscr{R}$-kernel. We can see that this result is a generalization of Richardson's theorem. The second part studies the relation that there exists between an $(\mathscr{A},\mathscr{B})$-kernel and the $\mathscr{R}$-kernel of an asociated digraph to D. And in the third part we show sufficient conditions for the existence of $(\mathscr{A},\mathscr{B})$-kernels, in particular we are going to show more generalizations of Richardson's theorem. And we will see that we can deduce as direct consequences some results which guarantee the existence of kernels by monochromatic paths, (k,l)-kernels and H-kernels.

We need the following.

Theorem 1.3 ([10]). *Let D be a strong digraph (possibly infinite). Then D is bipartite if and only if every directed cycle has even length.*

Theorem 1.4 ([17]). *Let D be a digraph possibly infinite. If D contains no infinite outward path then D contains at least one terminal strong component.*

Theorem 1.5 ([17]). *Let D be a digraph possibly infinite. If D contains no infinite outward path and contains no odd cycle then D is a kernel perfect digraph.*

2. $\mathscr{R}$-kernels and $\mathscr{R}$-semikernels

In this section, the main result is Theorem 2.6. This result is an extension of Richardson's theorem for possibly infinite digraphs and in order to prove it we need some previous definitions and results.

Definition 2.1. Let D be a digraph and $\mathscr{R}$ be a subset of $A(D)$. A subset S of $V(D)$ is an $\mathscr{R}$-semikernel of D if

1. S is an $\mathscr{R}$-independent set.
2. S absorbs $\Gamma^+(S) \cap (V(D)\backslash S)$.

Definition 2.2. Let D be a digraph and $\mathscr{R}$ a subset of $A(D)$. We will say that D is an $\mathscr{R}$-kernel perfect digraph if every induced subdigraph H of D has an $\mathscr{R}_H$-kernel, with $\mathscr{R}_H = A(H) \cap \mathscr{R}$.

We will use the following notation.

Let D be a digraph, $\mathscr{R}$ a subset of $A(D)$, $\{u, v\}$ a subset of $V(D)$ and S a subset of $V(D)$. We will write: $u \to v$ if there exists an arc from u to v in D; $u \to^{\mathscr{R}} v$ if there exists an arc

from u to v in $u \to^{R} v$; $u \not\to v$ is the denial of $u \to v$; $u \not\to^{\mathscr{R}} v$ is the denial of $u \to^{\mathscr{R}} v$; $u \to S$ if there exists a $\{u\}S$-arc in D; $u \to^{\mathscr{R}} S$ if there exists an $\{u\}S$-arc in $\mathscr{R}$; $u \not\to S$ is the denial of $u \to S$; $u \not\to^{\mathscr{R}} S$ is the denial of $u \to^{\mathscr{R}} S$.

Proposition 2.3. *Let D be a digraph (possibly infinite) and $\mathscr{R}$ a subset of $A(D)$. If D has a nonempty $\mathscr{R}$-semi-kernel, then D has a maximal $\mathscr{R}$-semi-kernel.*

Proof. Let $\mathcal{S}$ be the set of all nonempty $\mathscr{R}$-semi-kernels of D. It follows from the hypothesis of Proposition 2.3 that $\mathcal{S} \neq \emptyset$. Clearly the set $\mathcal{S}$ with the inclusion relation is a partially ordered set. We will prove that every chain in $(\mathcal{S}, \subseteq)$ has an upper bound in $\mathcal{S}$. Let $\mathscr{C}$ be a chain in $(\mathcal{S}, \subseteq)$ and $S^{\infty} = \cup_{S \in \mathscr{C}} S$. It is easy to check that S^{∞} is an upper bound for $\mathscr{C}$ and $S^{\infty} \in \mathcal{S}$.

Therefore, it follows from Zorn's lemma that there exists a maximal $\mathscr{R}$-semi-kernel in D. □

The following theorem is inspired in a result of Víctor Neumann-Lara [15] which establishes that if D is a digraph such that every induced subdigraph of D has a nonempty semi-kernel, then D is kernel perfect.

Theorem 2.4. *Let D be a digraph (possibly infinite) and $\mathscr{R}$ a subset of $A(D)$. If every induced subdigraph H of D has a nonempty $\mathscr{R}_H$-semi-kernel, with $\mathscr{R}_H = \mathscr{R} \cap A(H)$, then D has an $\mathscr{R}$-kernel.*

Proof. Let S be a maximal $\mathscr{R}$-semi-kernel of D (S exists by Proposition 2.3) and $\mathscr{D} = \{v \in V(D) \backslash S : v \not\to S\}$.

Consider the following claims.

Claim 1. $\mathscr{D} = \emptyset$.

Proceeding by contradiction, suppose that $\mathscr{D} \neq \emptyset$. Then it follows from the hypothesis that the digraph $H = D[\mathscr{D}]$ has a non-empty $\mathscr{R}_H$-semi-kernel S_1, with $\mathscr{R}_H = \mathscr{R} \cap A(H)$.

Claim 1.1. $S' = S \cup S_1$ is an $\mathscr{R}$-semi-kernel of D.

We are going to prove that S' is an $\mathscr{R}$-independent set in D. Since both S and S_1 are $\mathscr{R}$-independent sets in D, then it remains to prove that $\{(u,v),(v,u)\} \cap \mathscr{R} = \emptyset$ for every $u \in S$ and for every $v \in S_1$.

We are going to prove that $u \not\to^{\mathscr{R}} v$ for each $u \in S$ and for each $v \in S_1$. Proceeding by contradiction, suppose that there exist $u \in S$ and $v \in S_1$ such that $u \to^{\mathscr{R}} v$. Since $v \notin S$ and S is an $\mathscr{R}$-semi-kernel of D, it follows that there exists $w \in S$ such that $v \to w$, contradicting $v \in \mathscr{D}$ (recall that $S_1 \subseteq \mathscr{D}$). Therefore, $u \not\to^{\mathscr{R}} v$ for each $u \in S$ and for each $v \in S_1$.

We will prove that $x \not\to^{\mathscr{R}} y$ for each $x \in S_1$ and for each $y \in S$. Proceeding by contradiction, suppose that there exist $x \in S_1$ and $y \in S$ such that $x \to^{\mathscr{R}} y$. Thus, in particular, we have that $x \to y$, contradicting $x \in \mathscr{D}$. Therefore, $x \not\to^{\mathscr{R}} y$ for each $x \in S_1$ and for each $y \in S$.

So, S' is an $\mathscr{R}$-independent set in D.

We are going to prove that S' absorbs $\Gamma^{+}(S') \cap (V(D) \backslash S')$. Let z be a vertex in $V(D) \backslash S'$ and $u \in S'$ such that $u \to z$. If $u \in S$, then we have that $z \to S$ (because S is an $\mathscr{R}$-semi-kernel of D), which implies that $z \to S'$. Suppose that $u \in S_1$. If $z \in ((V(D) \backslash S) \backslash \mathscr{D})$, then it follows from the definition of $\mathscr{D}$ that $z \to S$, which implies that $z \to S'$. If $z \in \mathscr{D} \setminus S_1$, then $z \to S_1$ (because S_1 is an $\mathscr{R}$-semi-kernel of $D[\mathscr{D}]$), which implies that $z \to S'$.

Therefore, $S \cup S'$ is an $\mathscr{R}$-semi-kernel of D such that $S \subset S'$ (the symbol $\subset$ denotes a proper subset), which contradicts the choice of S.

So, $\mathscr{D} = \emptyset$.

Thus, S is an $\mathscr{R}$-kernel in D. □

We are ready to prove the main result of this section but for this we need consider a definition.

Definition 2.5. Let D be a digraph, $\mathscr{R}$ a subset of $A(D)$ and $W = (v_0, v_1, \ldots, v_n)$ a walk in D. Suppose $e_i = (v_{i-1}, v_i)$ for each i in $\{1, \ldots, n\}$. Let $I = \{k \in \{1, \ldots, n\}: e_k \in \mathscr{R}\}$. We will say that $|I|$ is the $\mathscr{R}$-length of W.

Theorem 2.6. *Let D be a digraph (possibly infinite) without infinite outward path and $\mathscr{R}$ a nonempty subset of $A(D)$. If every cycle in D has even $\mathscr{R}$-length, then D has an $\mathscr{R}$-kernel.*

Proof. We will prove that every induced subdigraph H of D has a nonempty $\mathscr{R}_H$-semi-kernel, with $\mathscr{R}_H = A(H) \cap \mathscr{R}$. Let H be an induced subdigraph of D. If $\mathscr{R}_H = \emptyset$, then $V(H)$ is a nonempty $\mathscr{R}_H$-semi-kernel of H. Suppose that $\mathscr{R}_H \neq \emptyset$.

Let H_R be the subdigraph arc-induced by the set $\{(u,v) \in A(H): (u,v) \in \mathscr{R}\}$ ($H_R = H[\{(u,v) \in A(H): (u,v) \in \mathscr{R}\}]$). Let G be a terminal strong component of H_R (G exists by Theorem 1.4). Suppose that $|V(G)| \geq 2$.

Since every cycle in D has even $\mathscr{R}$-length, it follows that H_R has no cycles of odd length. So, G is a bipartite digraph (by Theorem 1.3). Let (V_1, V_2) be a partition of $V(G)$ where V_1 and V_2 are two independent sets. Consider the set $K = \{v \in V(H_R) \backslash V(G): \Gamma_H^{+}(v) \cap V_2 = \emptyset\}$. If $K = \emptyset$, then $V_2 \cup (V(H) \backslash V(H_R))$ is an $\mathscr{R}_H$-semi-kernel of H. Suppose that $K \neq \emptyset$. Since $H_R[K]$ has no cycles of odd length, it follows that $H_R[K]$ has a kernel N_1 (by Theorem 1.5).

We claim that $N = (V_2 \cup N_1 \cup (V(H) \backslash V(H_R)))$ is a non-empty $\mathscr{R}_H$-semi-kernel of H.

We will prove that N is an $\mathscr{R}_H$-independent set in H. From the definition of H_R we have that both V_2 and N_1 are $\mathscr{R}_H$-independent sets in H. It follows from the definition of K and because of that G is a terminal strong component of H_R that there exists no arc of $\mathscr{R}_H$ between V_2 and N_1. On the other hand, from the definition of H_R we have that $V(H) \backslash V(H_R)$ is an $\mathscr{R}_H$-independent set in H and there exists no arc of $\mathscr{R}_H$ between $V(H) \backslash V(H_R)$ and $V_2 \cup N_1$. Therefore, N is an $\mathscr{R}_H$-independent set in H.

We will prove that N absorbs $\Gamma_H^{+}(N) \cap (V(H) \backslash N)$. Let v be a vertex in $V(H) \backslash N$ such that there exits an $N\{v\}$-arc in H. We consider three cases on v: $v \in V_1$, $v \in K \setminus N_1$ or $v \in V(H_R) \backslash (V(G) \cup K)$. If $v \in V_1$, then there exists $w \in V_2$ such that $(v,w) \in A(H)$ (because G is strong and G is bipartite). If $v \in K \setminus N_1$, then there exists $w \in N_1$ such that $(v,w) \in A(H)$ because N_1 is a kernel of $H_R[K]$. If $v \in V(H_R) \backslash (V(G) \cup K)$, then it follows from the definition of K that $\Gamma_H^{+}(v) \cap V_2 \neq \emptyset$.

Therefore, N is a non-empty $\mathscr{R}_H$-semi-kernel of H.

If $|V(G)| = 1$, then the proof is similar as the case $|V(G)| \geq 2$. When $|V(G)| = 1$ we have that $N = V(G) \cup N_1 \cup (V(H) \backslash V(H_R))$ is a nonempty $\mathscr{R}_H$-semi-kernel of H.

Therefore, every induced subdigraph H of D has a nonempty $\mathscr{R}_H$-semi-kernel, with $\mathscr{R}_H = A(H) \cap \mathscr{R}$.

So, D has $\mathscr{R}$-kernel. □

Corollary 2.7. [17] *Let D be a digraph (possibly infinite) without infinite outward path and without cycles of odd length. Then D has a kernel.*

3 $(\mathscr{A}, \mathscr{B})$-kernels and $\mathscr{B}$-closure

Consider the following definitions.

Definition 3.1. Let D be a digraph and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$. We will say that $\mathscr{A}$ and $\mathscr{B}$ satisfy the property $\mathcal{P}$, $(\mathscr{A}, \mathscr{B})_{\mathcal{P}}$, if for every uv-path, say P, such that $P \in \mathscr{A}$ we have that there exists a uv-path, say P', such that $P' \in \mathscr{B}$.

Definition 3.2. Let D be a digraph and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ such that $\mathscr{A}$ and $\mathscr{B}$ satisfy the property $\mathcal{P}$. The $\mathscr{B}$-closure, $\mathscr{C}_{\mathscr{B}}(D)$, of D is defined as follows:

$V(\mathscr{C}_{\mathscr{B}}(D)) = V(D)$ and

$(u,v) \in A(\mathscr{C}_{\mathscr{B}}(D))$ if there exists a uv-path in D, say P, such that $P \in \mathscr{B}$.

We will use the following notation.

Let $\{u,v\}$ be a subset of $V(\mathscr{C}_{\mathscr{B}}(D))$ and $\mathscr{R}$ a subset of $A(\mathscr{C}_{\mathscr{B}}(D))$. We will write: $u \Rightarrow v$ if there exists an arc from u to v in $\mathscr{C}_{\mathscr{B}}(D)$; $u \Rightarrow^{\mathscr{R}} v$ if there exists an arc from u to v in $\mathscr{R}$.

From now on, D will be a digraph, $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ such that $\mathscr{A}$ and $\mathscr{B}$ satisfy the property $\mathcal{P}$ and $\mathscr{C}_{\mathscr{B}}(D)$ the $\mathscr{B}$-closure of D.

The following theorem shows how we can found an $(\mathscr{A}, \mathscr{B})$-kernel from an $\mathscr{R}$-kernel of $\mathscr{C}_{\mathscr{B}}(D)$.

Theorem 3.3. *$N \subseteq V(D)$ is an $(\mathscr{A}, \mathscr{B})$-kernel of D if and only if N is an $\mathscr{R}$-kernel of $\mathscr{C}_{\mathscr{B}}(D)$, with $\mathscr{R} = \{(x,y) \in A(\mathscr{C}_{\mathscr{B}}(D))$: there exists an xy-path in D, say P, such that $P \in \mathscr{A}\}$.*

Proof. **(Necessity)**. We will prove that N is an $\mathscr{R}$-independent set and N is an absorbent set in $\mathscr{C}_{\mathscr{B}}(D)$.

Proceeding by contradiction, suppose that there exists a subset $\{x,y\}$ of N such that $x \Rightarrow^{\mathscr{R}} y$. Then, from the definition of $\mathscr{R}$ we have that there exists $P \in \mathscr{A}$ such that P is an xy-path in D, contradicting that N is an $\mathscr{A}$-independent set in D. Therefore, N is an $\mathscr{R}$-independent set in $\mathscr{C}_{\mathscr{B}}(D)$.

Let u be a vertex in $V(\mathscr{C}_{\mathscr{B}}(D)) \backslash N$. Since $V(\mathscr{C}_{\mathscr{B}}(D)) = V(D)$ and N is a $\mathscr{B}$-absorbent set in D, it follows that there exist $v \in N$ and $P \in \mathscr{B}$ such that P is a uv-path in D. So, $u \Rightarrow v$. Therefore, N is an absorbent set in $\mathscr{C}_{\mathscr{B}}(D)$.

So, N is an $\mathscr{R}$-kernel in $\mathscr{C}_{\mathscr{B}}(D)$.

(Sufficiency). We will prove that N is an $\mathscr{A}$-independent set and N is a $\mathscr{B}$-absorbent set in D.

Proceeding by contradiction, suppose that there exist a subset $\{x,y\}$ of N and $P \in \mathscr{A}$ such that P is an xy-path in D. Then, from property $\mathcal{P}$, the definition of $(\mathscr{A}, \mathscr{B})$-closure and the definition of $\mathscr{R}$ we conclude that $x \Rightarrow^{\mathscr{R}} y$, contradicting that N is an $\mathscr{R}$-independent set in $\mathscr{C}_{\mathscr{B}}(D)$. Therefore, N is an $\mathscr{A}$-independent set in D.

Let u be a vertex in $V(D) \backslash N$. Since $V(\mathscr{C}_{\mathscr{B}}(D)) = V(D)$ and N is an absorbent set in $\mathscr{C}_{\mathscr{B}}(D)$, it follows that there exists $v \in N$ such that $u \Rightarrow v$, which implies that there exists $P \in \mathscr{B}$ such that P is a uv-path in D (by definition of $\mathscr{C}_{\mathscr{B}}(D)$). Therefore, N is a $\mathscr{B}$-absorbent set in D.

Thus, N is an $(\mathscr{A}, \mathscr{B})$-kernel in D. □

4. $(\mathscr{A}, \mathscr{B})$-kernels and Richardson's theorem

Let D be a digraph and $\mathscr{T}$ a subset of $\mathscr{P}_D$. A sequence $W = (v_0, \ldots, v_n)$ of vertices of D is called a *$\mathscr{T}$-walk* if there exists a $v_i v_{i+1}$-path in D, say P, such that $P \in \mathscr{T}$ for every i in $\{0, \ldots, n-1\}$. We will say that n is the *$\mathscr{T}$-length* of W. If $v_i \neq v_j$ for every $i \neq j$, then W is called a *$\mathscr{T}$-path*; if $v_0 = v_n$, then W is called a *$\mathscr{T}$- closed walk*; if $v_0 = v_n$ and $v_i \neq v_j$ for every $i \neq j$ with $\{i,j\} \neq \{0,n\}$, then W is called a *$\mathscr{T}$-cycle*. An infinite outward *$\mathscr{T}$-path* is a sequence $(v_n)_{n \in \mathbb{N}}$ of distinct vertices of D such that there exists a $v_i v_{i+1}$-path in D, say P_i, with $P_i \in \mathscr{T}$ for every i in $\mathbb{N}$.

Theorem 4.1. *Let D be a digraph (possibly infinite) and $\mathscr{T}$ a subset of $\mathscr{P}_D$. Every $\mathscr{T}$-closed walk in D with odd $\mathscr{T}$-length contains a $\mathscr{T}$-cycle with odd $\mathscr{T}$-length.*

Proof. Let W be a $\mathscr{T}$-closed walk with odd $\mathscr{T}$-length in D. We proceed by induction on $l_{\mathscr{T}}(W)$, the $\mathscr{T}$-length of W.

If $l_{\mathscr{T}}(W) = 3$, then W is already a $\mathscr{T}$-cycle with odd $\mathscr{T}$-length.

Assume that the statement holds for every $\mathscr{T}$-closed walk in D with both odd $\mathscr{T}$-length and $\mathscr{T}$-length less than $2n+1$.

Let $W = (v_0, v_1, \ldots, v_{2n+1} = v_0)$ be a $\mathscr{T}$-closed walk in D with odd $\mathscr{T}$-length. If $v_i \neq v_j$ for every $i \neq j$ with $\{i,j\} \neq \{0,2n+1\}$, then W is the desire $\mathscr{T}$-cycle with odd $\mathscr{T}$-length. Suppose that there exist i and j, with $i \neq j$ and $\{i,j\} \neq \{0,2n+1\}$, such that $v_i = v_j$. Without loss of generality let us suppose that $i < j$ Consider the following two $\mathscr{T}$-closed walks $W_1 = (v_0, W, v_i = v_j) \cup (v_j, W, v_{2n+1} = v_0)$ and $W_2 = (v_i, W, v_j = v_i)$. Since W has odd $\mathscr{T}$-length and $l_{\mathscr{T}}(W) = l_{\mathscr{T}}(W_1) + l_{\mathscr{T}}(W_2)$, then either W_1 or W_2 must have odd $\mathscr{T}$-length. Without loss of generality let us suppose that W_1 has odd $\mathscr{T}$-length. Then by the induction hypothesis it follows that W_1 contains a $\mathscr{T}$-cycle with odd $\mathscr{T}$-length. □

Theorem 4.2. *Let D be a digraph (possibly infinite) and $\mathscr{T}$ a subset of $\mathscr{P}_D$. If D has no infinite outward $\mathscr{T}$-path, then $\mathscr{C}_{\mathscr{T}}(D)$ has a terminal strong component.*

Proof. We will prove that $\mathscr{C}_{\mathscr{T}}(D)$ has no infinite outward path and then we will apply Theorem 1.4 in order to conclude. Proceeding by contradiction, suppose that $\mathscr{C}_{\mathscr{T}}(D)$ has an infinite outward path, say $(v_n)_{n \in \mathbb{N}}$. It follows from the definition of $\mathscr{C}_{\mathscr{T}}(D)$ that there exists a $v_i v_{i+1}$-path in D, say P_i, with $P_i \in \mathscr{T}$ for every i in $\mathbb{N}$, which implies that $(v_n)_{n \in \mathbb{N}}$ is an infinite outward *$\mathscr{T}$-path* in D, a contradiction. Therefore, $\mathscr{C}_{\mathscr{T}}(D)$ has no infinite outward path. □

Let D be a digraph, H a subdigraph of D and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$. We will say that a subset S of $V(H)$ is a *$\mathscr{B}$-semikernel* of H if (1) S is an $\mathscr{A}$-independent set in D, and (2) for every v in $V(H)\backslash S$ if there exists an Sv-path in D, say P, such that P in $\mathscr{B}$, then there exists a vS-path in D, say P', such that $P' \in \mathscr{B}$.

Theorem 4.3. *Let D be a digraph (possibly infinite) and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$. If every induced subdigraph H of D has a nonempty $\mathscr{B}$-semikernel, then D has an $(\mathscr{A},\mathscr{B})$-kernel.*

Proof. Let $\mathscr{R}$ be a subset of $A(\mathscr{C}_{\mathscr{B}}(D))$ with $\mathscr{R} = \{(x,y)\in A(\mathscr{C}_{\mathscr{B}}(D))$: there exists P in $\mathscr{A}$ such that P is an xy-path$\}$. We will prove that $\mathscr{C}_{\mathscr{B}}(D)$ has an $\mathscr{R}$-kernel and for this we will use Theorem 3.3 and Theorem 2.4; that is, we will prove that every induced subdigraph G of $\mathscr{C}_{\mathscr{B}}(D)$ has a nonempty $\mathscr{R}_G$-semikernel, with $\mathscr{R}_G = \mathscr{R} \cap A(G)$.

Let G be an induced subdigraph of $\mathscr{C}_{\mathscr{B}}(D)$ and S a nonempty $\mathscr{B}$-semikernel of $D[V(G)]$. We claim that S is a nonempty $\mathscr{R}_G$-semikernel of G. Notice that S is an $\mathscr{R}_G$-independent set in G; otherwise there exists a subset $\{x,y\}$ of S, with $x \neq y$, such that $(x,y) \in \mathscr{R}_G$ which implies that there exists P in $\mathscr{A}$ such that P is an xy-path in D, in contradiction with S an $\mathscr{A}$-independent set in D. Therefore S is an $\mathscr{R}_G$-independent set in G. On the other hand, let v be a vertex in $V(G)\backslash S$ and s in S such that $(s,v) \in A(G)$, then there exists P in $\mathscr{B}$ such that P is an sv-path in D. Since S is an $\mathscr{B}$-semikernel of $D[V(G)]$, it follows that there exist s' in S and P' in $\mathscr{B}$ such that P' is a vs'-path in D which implies that $(v,s) \in A(G)$. Therefore, S is a nonempty $\mathscr{R}_G$-semikernel of G. Thus, it follows from Theorem 2.4 that $\mathscr{C}_{\mathscr{B}}(D)$ has an $\mathscr{R}$-kernel and so Theorem 3.3 implies that D has an $(\mathscr{A},\mathscr{B})$-kernel. □

Theorem 4.4. *Let D be a digraph (possibly infinite) and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ such that $(\mathscr{A},\mathscr{B})_{\mathcal{P}}$. Suppose that D has no infinite outward $\mathscr{B}$-path and D has no $\mathscr{B}$-cycle with odd $\mathscr{B}$-length. Then D has a nonempty $\mathscr{B}$-semikernel.*

Proof. It follows from Theorem 4.2 that $\mathscr{C}_{\mathscr{B}}(D)$ has a terminal strong component, say H. Let v_0 be a fixed vertex of $V(H)$ and consider the following sets $V_1 = \{w \in V(H)$: there exists a v_0w-$\mathscr{B}$-path with odd $\mathscr{B}$-length$\}$ and $V_2 = \{w \in V(H)$: there exists a v_0w-$\mathscr{B}$-path with even $\mathscr{B}$-length$\}$.

Remark 1. Consider a vertex w in V_1 and let P be a v_0w-$\mathscr{B}$-path with odd $\mathscr{B}$-length; if P' is a wv_0-$\mathscr{B}$-path, then P' has odd $\mathscr{B}$-length, this follows from Theorem 4.1 and the hypothesis that D has no $\mathscr{B}$-cycle with odd $\mathscr{B}$-length. Similarly it holds for w in V_2 that if P is a v_0w-$\mathscr{B}$-path with even $\mathscr{B}$-length and P' is a wv_0-$\mathscr{B}$-path, then P' has even $\mathscr{B}$-length.

Remark 2. For every subset $\{u,v\}$ of $V(H)$ there exists a uv-$\mathscr{B}$-path in D and there exists a vu-$\mathscr{B}$-path in D. This follows from the fact that H is strong and the definition of $\mathscr{C}_{\mathscr{B}}(D)$.

On the other hand, notice that $v_0 \in V_2$. We will prove that V_2 is a nonempty $\mathscr{B}$-semikernel of D. V_2 is an $\mathscr{A}$-independent set in D, otherwise there exist $\{x,y\} \subseteq V_2$, with $x \neq y$, and P in $\mathscr{A}$ such that P is an xy-path in D which implies that there exists P' in $\mathscr{B}$ such that P' is an xy-path in D (because $(\mathscr{A},\mathscr{B})_{\mathcal{P}}$). Since $y \in V_2$, it follows from the definition of V_2, Remarks 2 and 1 that there exists a yv_0-$\mathscr{B}$-path with even $\mathscr{B}$-length in D, say P_1. Also, there exists a v_0x-$\mathscr{B}$-path with even $\mathscr{B}$-length, say P_2. Therefore $P' \cup P_1 \cup P_2$ is a $\mathscr{B}$-closed walk, with odd $\mathscr{B}$-length, which contains a $\mathscr{B}$-cycle with odd $\mathscr{B}$-length (by Theorem 4.1), a contradiction. Thus, V_2 is an $\mathscr{A}$-independent set in D. We will prove that if v is a vertex in $V(D)\backslash V_2$ such that there exist u in V_2 and P in $\mathscr{B}$ such that P is a uv-path, then there exist w in V_2 and P' in $\mathscr{B}$ such that P' is a vw-path. Notice that $v \in V(H)$ because H is a terminal strong component of $\mathscr{C}_{\mathscr{B}}(D)$, which implies that $v \in V_1$ and so by Remark 2 we have that there exists P' in $\mathscr{B}$ such that P' is a vv_0-$\mathscr{B}$-path. □

Theorem 4.5. *Let D be a digraph (possibly infinite) and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ such that $(\mathscr{A},\mathscr{B})_{\mathcal{P}}$. Suppose that D has no infinite outward $\mathscr{B}$-path and D has no $\mathscr{B}$-cycle with odd $\mathscr{B}$-length. Then D has an $(\mathscr{A},\mathscr{B})$-kernel.*

Proof. It follows from Theorems 4.3 and 4.4. □

The following results are consequences of the previous Theorems.

Corollary 4.6 *Let D be a digraph (possibly infinite) and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ such that $(\mathscr{A},\mathscr{B})_{\mathcal{P}}$ and P has odd length for every P in $\mathscr{B}$. Suppose that D has no infinite outward $\mathscr{B}$-path and D has no odd cycles, then D has an $(\mathscr{A},\mathscr{B})$-kernel.*

Proof. We will prove that D has no $\mathscr{B}$-cycle with odd $\mathscr{B}$-length.

Proceeding by contradiction, suppose that D contains a $\mathscr{B}$-cycle with odd $\mathscr{B}$-length, say $\gamma = (v_0, \ldots, v_{2n}, v_0)$. Since there exists a v_iv_{i+1}-path in D, say P_i, such that $P_i \in \mathscr{B}$ for every $i \in \{0, \ldots, 2n\}$ (indices modulo $2n+1$), then $W = P_0 \cup \ldots \cup P_{2n}$ is a closed walk with odd length in D (because P_i has odd length for every $i \in \{0, \ldots, 2n\}$), which implies that W contains an odd cycle, contradicting the hypothesis. So, D has no $\mathscr{B}$-cycle with odd $\mathscr{B}$-length. Therefore, it follows from Theorem 4.4 that D has an $(\mathscr{A},\mathscr{B})$-kernel. □

Theorem 4.7 is another extension of Richardson's theorem.

Let D be a digraph and $\mathscr{U}$ and $\mathscr{T}$ two subsets of $\mathscr{P}_D$. If $\gamma = (v_0, v_1, \ldots, v_n = v_0)$ is a $\mathscr{U}$-cycle and $I = \{i \in \{0, \ldots, n-1\}$: there exists a v_iv_{i+1}-path, say P, such that $P \in \mathscr{T}\}$, then we will say that $|I|$ is the $\mathscr{U}_{\mathscr{T}}$-length of γ.

Theorem 4.7. *Let D be a digraph (possibly infinite) and $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ such that $(\mathscr{A},\mathscr{B})_{\mathcal{P}}$. Suppose that D has no infinite outward $\mathscr{B}$-path and every $\mathscr{B}$-cycle in D has even $\mathscr{B}_{\mathscr{A}}$-length. Then D has an $(\mathscr{A},\mathscr{B})$-kernel.*

Proof. We will prove that $\mathscr{C}_{\mathscr{B}}(D)$ has an $\mathscr{R}$-kernel, with $\mathscr{R} = \{(x,y) \in A(\mathscr{C}_{\mathscr{B}}(D))$: there exists an xy-path, say P, such that $P \in \mathscr{A}\}$. We will see that every cycle in $\mathscr{C}_{\mathscr{B}}(D)$ has

even $\mathscr{R}$-length. Let $\gamma = (v_0, \ldots, v_n = v_0)$ be a cycle in $\mathscr{C}_{\mathscr{B}}(D)$ and suppose that $a_i = (v_{i-1}, v_i)$ for every $i \in \{1, \ldots, n\}$ and $\{a_{i_1}, \ldots, a_{i_m}\} \subseteq \mathscr{R}$. From the definition of $\mathscr{C}_{\mathscr{B}}(D)$ it follows that there exists a $v_{h-1}v_h$-path, say P_h, such that $P_h \in \mathscr{B}$ for each $h \in \{1, \ldots, n\}$. On the other hand, since $\{a_{i_1}, \ldots, a_{i_m}\} \subseteq \mathscr{R}$, we deduce from the definition of $\mathscr{R}$ that there exists a $v_{i_j-1}v_{i_j}$-path, say P'_{i_j}, such that $P'_{i_j} \in \mathscr{A}$ for each $j \in \{1, \ldots, m\}$. So, γ is a $\mathscr{B}$-cycle in D and the $\mathscr{B}_{\mathscr{A}}$-length of γ is m, which implies that m is even by the hypothesis. Therefore, every cycle in $\mathscr{C}_{\mathscr{B}}(D)$ has even $\mathscr{R}$-length. So, from Theorem 2.6 we have that $\mathscr{C}_{\mathscr{B}}(D)$ has an $\mathscr{R}$-kernel (notice that $\mathscr{C}_{\mathscr{B}}(D)$ has no infinite outward path because D has no infinite outward $\mathscr{B}$-path), which implies that D has an $(\mathscr{A}, \mathscr{B})$-kernel (by Theorem 3.3). □

We are going to show some consequences of Theorem 4.7 and for that we need the following definitions.

Definition 4.8. Let D be a digraph, l a positive integer and $(v_0, v_1, \ldots, v_n = v_0)$ a sequence of vertices of D such that $v_j \neq v_s$ for each $j \neq s$. $(v_0, v_1, \ldots, v_n = v_0)$ is called a γ_l-cycle if for every $i \in \{0, \ldots, n\}$ there exists a v_iv_{i+1}-path of length less than or equal to l (indices modulo $n+1$).

Definition 4.9. Let D be a digraph, $\mathscr{A}$ and $\mathscr{B}$ two subsets of $\mathscr{P}_D$ and $\{k,l\} \subseteq \mathbb{N}\backslash\{0\}$. An $(\mathscr{A}, \mathscr{B})$-kernel is said to be a (k,l)-kernel when $\mathscr{A} = \{P \in \mathscr{P}_D$; the length of P is less than or equal to $k - 1\}$ and $B = \{P \in \mathscr{P}_D$; the length of P is less than or equal to $l\}$.

Corollary 4.10. *Let D be a digraph (possibly infinite) without infinite outward path and k and l two positive integers such that $1 \leq k - 1 \leq l$. Suppose that every γ_l-cycle $(v_0, v_1, \ldots, v_n = v_0)$ holds that $|\{i \in \{0, \ldots, n\}; d((v_i, v_{i+1}) \leq k - 1\}|$ is even, then D has a (k,l)-kernel.*

Proof. Apply Theorem 4.7 considering the sets $\mathscr{A} = \{P \in \mathscr{P}_D$: the length of P is less than or equal to $k - 1\}$ and $\mathscr{B} = \{P \in \mathscr{P}_D$: the length of P is less than or equal to $l\}$. □

Corollary 4.11. *Let D be a digraph (possibly infinite) without infinite outward path and k and l two positive integers such that $l = k - 1$. Suppose that every γ_l-cycle has an even number of vertices, then D has a (k,l)-kernel.*

Proof. Apply Theorem 4.7 considering the sets $\mathscr{A} = \{P \in \mathscr{P}_D$: the length of P is less than or equal to $k - 1\}$ and $\mathscr{B} = \{P \in \mathscr{P}_D$: the length of P is less than or equal to $l\}$. □

Consider the following definitions.

Definition 4.12. *Let H be a digraph (possibly with loops) and D an H-colored digraph. A subset N of $V(D)$ is said to be an H-kernel if for every pair of different vertices in N there is no H-path between them, and for every vertex u in $V(D)\backslash N$ there exists an H-path in D from u to N.*

Definition 4.13. Let H be a digraph, D an H-colored digraph and $(v_0, v_1, \ldots, v_n)$ a walk in D. We will say that there is an obstruction on v_i (or v_i is an obstruction) if $(c(v_{i-1}, v_i), c(v_i, v_{i+1}))$ is not an arc of $A(H)$ (*if* $v_0 = v_n$ we will take indices modulo n).

Corollary 4.14. *Let H be a digraph (possibly with loops) and D an H-colored digraph (possibly infinite) without infinite outward H-path and without odd cycles. Suppose that every vertex in every closed walk is an obstruction, then D has an H-kernel.*

Proof. Notice that the sets $\mathscr{A} = \mathscr{B} = \{P \in \mathscr{P}_D : P$ is an H-path$\}$ satisfy the property $\mathcal{P}$. We are going to prove that every $\mathscr{B}$-cycle in D has even $\mathscr{B}_{\mathscr{A}}$-length.

Claim. Every $\mathscr{B}$-cycle in D is a cycle.

Let $W = (v_0, \ldots, v_n = v_0)$ be a $\mathscr{B}$-cycle in D. It follows that there exists a v_iv_{i+1}-path P_i for each $i \in \{0, \ldots, n - 1\}$ such that $P_i \in \mathscr{B}$. Since $\cup_{i=1}^{n-1} P_i$ is a closed walk and every vertex of $\cup_{i=1}^{n-1} P_i$ is an obstruction, it follows that the length of P_i is 1 for every $i \in \{0, \ldots, n - 1\}$ (because P_i is an H-walk). Therefore, W is a cycle.

Since $\mathscr{A} = \mathscr{B}$ and every $\mathscr{B}$-cycle in D is a cycle, it follows that the $\mathscr{B}_{\mathscr{A}}$-length of every $\mathscr{B}$-cycle $W = (v_0, \ldots, v_n = v_0)$ is n. Since D has no odd cycles, we have that every $\mathscr{B}$-cycle has even $\mathscr{B}_{\mathscr{A}}$-length. Therefore, it follows from Theorem 4.7 that D has an $(\mathscr{A}, \mathscr{B})$-kernel N, which is an H-kernel.

Notice that if $A(H) = \{(x,x): x \in V(H)\}$, then N is a kernel by monochromatic paths. □

Corollary 4.15. *Let D be an m-colored digraph (possibly infinite) without monochromatic infinite outward path and without odd cycles. If every closed walk $(v_0, \ldots, v_n, v_0)$ is such that $c(v_{i-1}, v_i) \neq c(v_i, v_{i+1})$ for every $i \in \{1, \ldots, n\}$ (indices modulo $n+1$), then D has a kernel by monochromatic paths.*

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