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The game chromatic number of corona of two graphs

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ABSTRACT

In this paper, we compute an upper bound for the game chromatic number of corona of any two simple graphs G and H, denoted by $G \circ H$. Also, we determine the game chromatic number of $P_n \circ P_m, P_n \circ C_m, P_n \circ K_{a,b}, P_n \circ W_m, K_{1,m} \circ P_n, K_n \circ K_n$ and $K_n \circ K_{n,n}$.

KEYWORDS

Graphs; game chromatic number; corona of two graphs

1. Introduction

Let $G = (V, E)$ be a finite simple graph and X be a set of colors. The game chromatic number of G is defined through a two person game. Two players, say Alice and Bob, alternately color a vertex of G with a color from the color set X so that no two adjacent vertices receive the same color. Alice wins the game if all the vertices of G are colored. Bob wins the game if at any stage of the game, there is an uncolored vertex which is adjacent to vertices of all colors from X. The *game chromatic number*, $\chi_g(G)$, of G is the least number of colors in the color set X for which Alice has a winning strategy in the coloring game on G. This parameter is well defined since Alice always wins if $|X| = |V|$.

A variation of the game chromatic number called the game coloring number was introduced by Zhu [4] as a tool in the study of the game chromatic number. It is also defined through a two person game, say Alice and Bob, with Alice starting first. The players fix a positive integer k and instead of coloring vertices, they mark an unmarked vertex each turn. Bob wins if at some time some unmarked vertex has k marked neighbours, otherwise Alice wins the game. The *game coloring number* of G, denoted by $col_g(G)$, is defined as the least number k for which Alice has a winning strategy in the marking game on the graph G. Clearly, if Alice can win the marking game for some integer k, then she can also win the coloring game with k colors. Thus $\chi(G) \leq \chi_g(G) \leq col_g(G) \leq \triangle(G) + 1$, where $\chi(G)$ is the usual chromatic number of G and $\triangle(G)$ is the maximum degree of G.

In 2007, T. Bartnicki, B. Bresar, J. Grytczuk, M. Kovsse, Z. Miechowicz and I. Peterin [1] studied the game chromatic number of cartesian product of two graphs G and H. In 2009, Charmaine Sia [3] determined the exact value of $\chi_g(S_m \Box P_n), \chi_g(S_m \Box C_n), \chi_g(P_2 \Box W_n)$ and $\chi_g(P_2 \Box K_{m,n})$. In 2009, Andre Raspaud, Jiaojiao Wu [2] found the game chromatic number of toroidal grids. In this paper we study the game chromatic number of the corona of any two simple graphs G and H.

We say a color i is an available color for an uncolored vertex x if no neighbours of x has been colored by color i. As given by Raspaud and Wu [2], during the game, at any instant, an uncolored vertex x is called color-i critical if the following hold at that instant.

- i is the only available color for x.
- x has a neighbour y such that i is an available color for y.

Note that, at some point of the game, if a vertex x is color-i critical and if it is

⋆ Bob's turn to color, then Bob wins the game. Because Bob colors a neighbor y of x (for which i is an available color) with the color i.

⋆ Alice's turn to color, then Alice must defend the vertex x so that after her move, vertex x is no more color-i critical. To achieve this, Alice can do one of the following.

- she can color vertex x with color i
- she can choose a vertex z such that color i is available for z and it is adjacent to all those neighbours of x for which i is an available color and color the vertex z with color i.
- If x has only one neighbour y for which color i is one of the available colors, then she can color y with some color other than color i.

2. Corona of two graphs

Definition 2.1. The corona of two simple graphs G and H is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.

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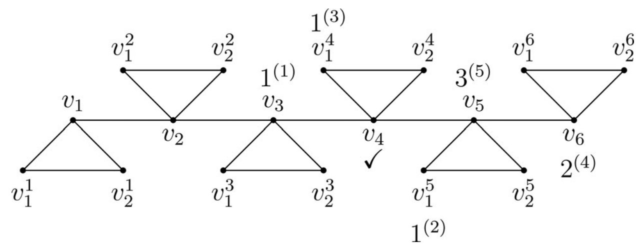

Figure 1. Illustration for Case 1 with $n = 6$.

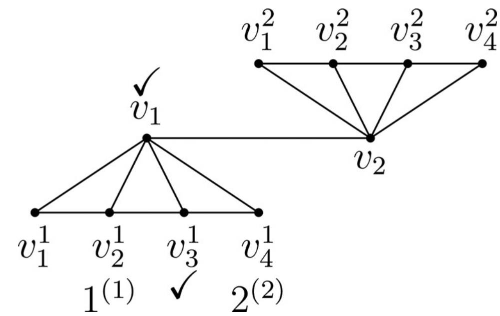

Figure 2. Illustration for Case 2 with $n = 2$ and $m = 4$.

Observation: It can be easily observed that the graphs $G \circ H$ and $H \circ G$ are non isomorphic.

Notations:

P_n is the path on n vertices. C_n is the cycle on n vertices. K_n is the complete graph on n vertices. $K_{a,b}$ is the complete bipartite graph with partite sets of sizes a and b. W_n is the wheel on $n+1$ vertices.

Let G be a graph with n vertices, say $v_1, v_2, \ldots, v_n$ and H be a graph with m vertices. The n copies of H in $G \circ H$ are denoted by $F_1, F_2, F_3, \ldots, F_n$. The vertices of F_i are denoted by $v_1^i, v_2^i, v_3^i, \ldots, v_m^i, 1 \leq i \leq n$. Let the colors in X be $\{1, 2, 3, \ldots\}$. In the figures, a vertex v is labelled $i^{(j)}$ if the vertex v is given color i in the j^{th} move of the game and at some point of the game, a vertex v is labelled ✓ if vertex v is color-i critical, for some i.

3. An upper bound for $\chi_g(G \circ H)$

Theorem 3.1. [3] *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs. Suppose that $G = (V, E)$ is a graph with $E = E_1 \cup E_2$, then $\chi_g(G) \leq col_g(G) \leq col_g(G_1) + \triangle(G_2)$.*

Theorem 3.2. *For any two simple graphs G and H,*

$$\chi_g(G \circ H) \leq col_g(G \circ H) \leq max\{\triangle(G), \triangle(H)\} + 2.$$

Proof. Let $G \circ H = (V, E)$. Observe that the edges $v_i v_1^i, v_i v_2^i, v_i v_3^i, \ldots, v_i v_m^i$, where $1 \leq i \leq n$, form a copy of star in $G \circ H$. Let it be S_i, $1 \leq i \leq n$. Let $G_1 = (V, E_1)$ where V is the set of vertices of $G \circ H$ and E_1 is the set of edges of $S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n$. Let $G_2 = (V, E_2)$ where V is the set of vertices of $G \circ H$ and E_2 is given by $E_2 = E - E_1$. It is clear that $col_g(G_1) = 2$ and the maximum degree of G_2, $\triangle(G_2) = max\{\triangle(G), \triangle(H)\}$. Therefore by using Theorem 3.1 we get, $\chi_g(G \circ H) \leq col_g(G \circ H) \leq 2 + max\{\triangle(G), \triangle(H)\}$. □

4. The game chromatic number of corona of two graphs

Theorem 4.1. *For any two integers n and m with $n \geq 6$ and $m = 2$ and with $n \geq 2$ and $m \geq 3$, $\chi_g(P_n \circ P_m) = 4$.*

Proof. Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $V(F_i) = \{v_1^i, v_2^i, v_3^i, \ldots, v_m^i\}$. Note that a vertex of P_n in $P_n \circ P_m$ has degree either $m+1$ or $m+2$ and a vertex of F_i has degree either 2 or 3. Now we find the exact value of the game chromatic number of $P_n \circ P_m$ for different values of n and m. First we show that $\chi_g(P_n \circ P_m) > 3$ for $n \geq 6$, $m = 2$ and $n \geq 2$, $m \geq 3$. To prove this, we give a strategy for Bob to win the game using three colors. The strategy of Bob for different values of n and m is as follows.

Case 1: $n \geq 6$ and $m = 2$.

In the first move, Alice can play either in a vertex of P_n or in a vertex of F_i, $1 \leq i \leq n$, as discussed in Sub-case 1(a) and Sub-case 1(b).

Sub-case 1(a): Alice plays in P_n.

Suppose she plays in a vertex of P_n, say v_i with color 1. Bob replies with a vertex of F_{i+2} if $(i+3) \leq n$ or F_{i-2} if $(i-3) \geq 1$ with color 1. If $(i+3) \leq n$ and $(i-3) \geq 1$, then Bob can choose either F_{i+2} or F_{i-2} arbitrarily and color a vertex with color 1. Let F_j be the copy of P_2 on which Bob plays in this move. Let F_k be the copy of P_2 which lies between F_i and F_j. Now Alice plays anywhere. Now it is Bob's turn and he checks for the following situations and plays accordingly (Figure 1).

- **Both the vertices of F_k are uncolored.**
  In this situation, Bob colors a vertex of F_k with color 2. This makes the vertex v_k color-3 critical. Now the game is in Bob's hand.
- **Exactly one vertex of F_k is uncolored.** [This means that Alice should have colored the other vertex of F_k in the previous move and she should have used color 1. Had she used color 2 or 3, the game will be in Bob's hand.]
  In this situation, Bob colors a vertex v_l, $(l \neq k)$, with color 2 such that v_l is adjacent to v_j. This makes v_j color-3 critical and the game is in Bob's hand.

Sub-case 1(b): Suppose in the first move, Alice colors a vertex of F_i, $1 \leq i \leq n$, say, with color 1. Then Bob colors the other vertex of F_i. This forces Alice to color v_i. Now Bob will be using a similar strategy as given in Sub-case 1(a) for the course of the game and wins.

Case 2: $n \geq 2, m \geq 3$.

In the first move, Alice can color either a vertex of P_n or a vertex of F_i $(1 \leq i \leq n)$ and the course of the game is as discussed below (Figures 2 and 3).

The vertex in which Alice colors in the first move	The vertex with which Bob responds
◇ A vertex of P_n, say v_i with color 1.	v_1^j where $j = i+1$ if $(i+1) \leq n$ or $j = i-1$ if $(i-1) \geq 1$ with color 2. If both $(i+1) \leq n$ and $(i-1) \geq 1$, then Bob can choose j as either $i+1$ or $i-1$ arbitrarily. This forces Alice to color v_j with color 3 and hence v_2^j becomes color-1 critical. Hence Bob wins.
◇ A vertex of degree two in F_i, say v_1^i with color 1.	v_3^i color 2 and hence v_i and v_2^i becomes color-3 critical. Hence Bob wins.
◇ A vertex of degree three in F_i, say v_2^i with color 1, $1 \leq i \leq n$.	(*i*) When $m > 3$, Bob colors v_4^i with color 2. This makes the vertices v_3^i and v_i color-3 critical. Hence Bob wins. (*ii*) When $m = 3$, Bob colors v_1^i with color 2. This forces Alice to color v_i with color 3. Now Bob colors v_1^j where $j = i+1$ if $(i+1) \leq n$ or $j = i-1$ if $(i-1) \geq 1$ with color 2. If both $(i+1) \leq n$ and $(i-1) \geq 1$, then Bob chooses j as either $i+1$ or $i-1$ arbitrarily. This forces Alice to color v_j with color 1 and hence v_2^j becomes color-3 critical. Hence Bob wins.

Thus $\chi_g(P_n \circ P_m) \geq 4$ for $n \geq 6$, $m = 2$ and $n \geq 2, m \geq 3$. Now to show that $\chi_g(P_n \circ P_m) = 4$, we give a strategy for Alice to win the game using four colors. The strategy of Alice is as follows.

Alice colors any vertex of P_n in the first move. In the following moves, whenever

- Bob colors a vertex of F_i, Alice colors v_i, if it is uncolored. Otherwise she colors a vertex of F_i, if there is one uncolored. If not, she replies with a vertex of P_n, if there is one uncolored. Otherwise she colors any uncolored vertex of F_j $(j \neq i)$.
- Bob colors a vertex of P_n, Alice colors an uncolored vertex of P_n, if there is one uncolored. Otherwise she replies in any uncolored vertex of the graph.

Note: Recall that in $P_n \circ P_m$ the maximum degree of a vertex of P_n is $m+2$ and that of F_i, $1 \leq i \leq n$, is 3. At any point of the game, using this strategy, any vertex of P_n is adjacent to at most three distinctly colored vertices. Hence one color is available for all the vertices.
Thus $\chi_g(P_n \circ P_m) = 4$. □

Theorem 4.2. *For any two integers n and m with $n \geq 2$ and $m \geq 3$, $\chi_g(P_n \circ C_m) = 4$.*

Proof. Observe that in $P_n \circ C_m$, a vertex of P_n has degree either $m+1$ or $m+2$ and a vertex of F_i, $1 \leq i \leq n$, has degree 3. Each F_i in $P_n \circ C_m$ has just one more edge than the corresponding F_i in $P_n \circ P_m$. Hence, by using a similar argument given for $P_n \circ P_m$, it is easy to check that $\chi_g(P_n \circ C_m) = 4$. □

Theorem 4.3. *For any three integers n, a and b with $n \geq 2, a \geq 2$ and $b \geq 2, \chi_g(P_n \circ K_{a,b}) = 4$.*

Proof. Let $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $V(F_i) = \{v_1^i, v_2^i, v_3^i, ..., v_m^i\}$ where $m = a+b$. Let us denote the two partite sets of F_i, $1 \leq i \leq n$, by A_1^i and A_2^i. First we show that $\chi_g(P_n \circ K_{a,b}) > 3$. To prove this, we give a strategy for Bob to win the game using three colors. The strategy of Bob is as follows (Figure 4).

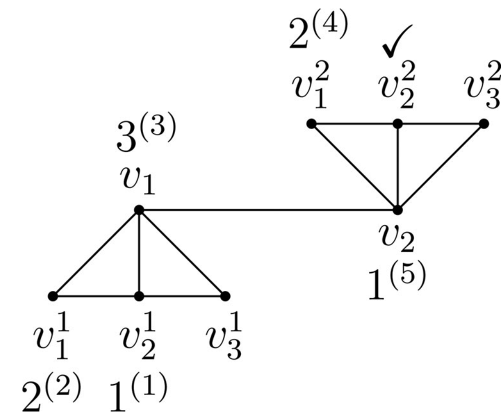

Figure 3. Illustration for Case 2 with $n = 2$ and $m = 3$.

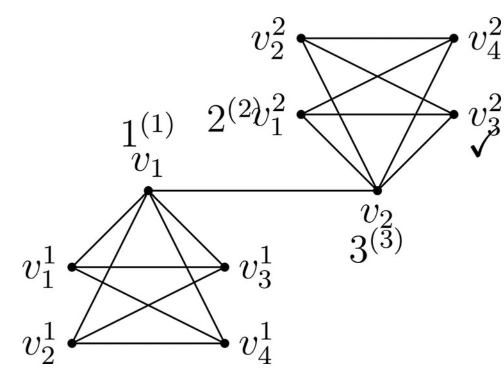

Figure 4. Illustration for Bob's strategy with $n = a = b = 2$.

The vertex in which Alice colors in the first move	The vertex with which Bob responds
◇ A vertex of P_n, say v_i with color 1.	A vertex of A_1^j with color 2 such that $j = i+1$ if $(i+1) \leq n$ or $j = i-1$ if $(i-1) \geq 1$. If both $(i+1) \leq n$ and $(i-1) \geq 1$, then Bob can choose j as either $i+1$ or $i-1$ arbitrarily. This makes v_j color-3 critical. By defending this, Alice herself makes all the vertices of A_2^j color-1 critical. Hence Bob wins.
◇ A vertex of F_i, say v_k^i with color 1	A vertex of the same partite set of F_i where Alice colored in the previous move with color 2. This makes the vertices of the other partite set of F_i and v_i color-j critical, for some j, $1 \leq j \leq 3$. Hence Bob wins.

Thus $\chi_g(P_n \circ K_{a,b}) \geq 4$. Now to show that $\chi_g(P_n \circ K_{a,b}) = 4$, we give a strategy for Alice to win the game using four colors. The strategy of Alice is as follows.

In the first move, Alice colors a vertex of P_n, say v_i. In the following moves, whenever

- **Bob colors a vertex of F_j, say a vertex of** A_1^j, Alice will look at the following situations and plays accordingly.
  - If exactly one vertex of A_1^j is colored, then Alice responds in v_j, if it is uncolored. Otherwise she colors a vertex of A_2^j.

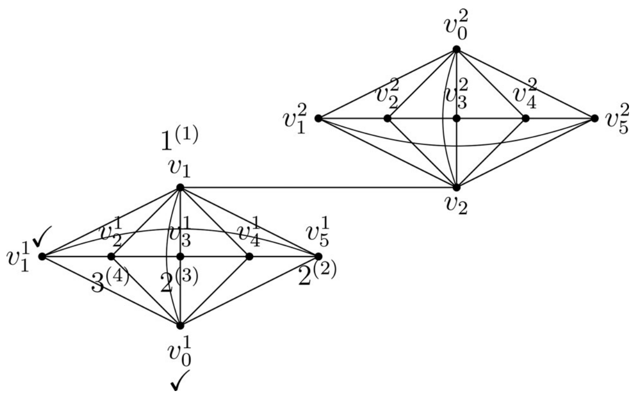

Figure 5. Illustration for Bob's strategy with $n = 2$ and $m = 5$.

- If two vertices of A_1^j are colored with distinct colors, then Alice replies in a vertex of A_2^j.
- Otherwise she colors any uncolored vertex of F_j, if there is one uncolored. If not, she colors any uncolored vertex of P_n, if there is one uncolored. Otherwise she colors in a vertex of a partition of F_i, $(i \neq j)$, if no vertex of that partition is already colored. If such a partition does not exist, then she colors any uncolored vertex of the graph.

- **Bob colors a vertex of** P_n, Alice replies in a vertex of P_n, if there is one uncolored. Otherwise she colors any uncolored vertex.

Note: Using this strategy, at any point of the game, any uncolored vertex is adjacent to at most three distinctly colored vertices. Hence at least one color is available for all the vertices.

Thus $\chi_g(P_n \circ K_{a,b}) = 4$. □

Theorem 4.4. *For any two integers n and m with $n \geq 2$ and $m \geq 5$, $\chi_g(P_n \circ W_m) = 5$.*

Proof. Let $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $V(F_i) = \{v_0^i, v_1^i, v_2^i, ..., v_m^i\}$ where v_0^i is the center vertex of F_i, the i^{th} copy of W_m. First we show that $\chi_g(P_n \circ W_m) > 4$. To prove this, we give a strategy for Bob to win the game using four colors. The strategy of Bob is as follows (Figure 5).

The vertex in which Alice colors in the first move	The vertex with which Bob responds
◇ A vertex of P_n, say v_i.	v_m^i. Now Alice can color any vertex. Note that v_0^i is uncolored because if v_0^i is colored by Alice, then the game is in Bob's hand. Now Bob colors one of v_2^i or v_{m-2}^i with the color different from the color of v_m^i. This makes (v_1^i and v_0^i) or (v_{m-1}^i and v_0^i) color-4 critical. Hence Bob wins.
◇ The vertex $v_0^i, 1 \leq i \leq n$.	v_m^i. Now Alice can color any vertex. Note that v_i is uncolored because if v_i is colored by Alice, then the game is in Bob's hand. Now Bob colors one of v_2^i or v_{m-2}^i with the color different from the color of v_m^i. This makes (v_1^i and v_i) or (v_{m-1}^i and v_i) color-4 critical. Hence Bob wins.
◇ A vertex of F_i, say v_j^i with color 1, $1 \leq j \leq m, 1 \leq i \leq n$.	A vertex v_k^i which is at a distance two from v_j^i with color 2. Alice can play anywhere. Note that both v_i and v_0^i are uncolored because if any one of v_i or v_0^i is colored by Alice, then the game is in Bob's hand. Now Bob colors a vertex of F_i with color 3. This makes v_i and v_0^i color-4 critical. Hence Bob wins.

Thus $\chi_g(P_n \circ W_m) \geq 5$. Now to show that $\chi_g(P_n \circ W_m) = 5$, we give a strategy for Alice to win the game using five colors. The strategy of Alice is as follows.

In the first move, Alice colors a vertex of P_n. In the following moves, whenever

- **Bob colors a vertex of P_n,** Alice colors a vertex of P_n, if there is one uncolored. Otherwise she colors any uncolored vertex of the graph.
- **Bob colors a vertex of F_i, say v_k^i,** Alice colors the vertex v_i, if it is uncolored. Otherwise she colors any uncolored vertex of the graph. Observe that while choosing any uncolored vertex of the graph, Alice's first preference is the vertex v_0^i, her second preference is a vertex of F_i, her third preference is a vertex of P_n and her last preference is a vertex of any other F_i.

Note: Using this strategy, at any point of the game, any uncolored vertex is adjacent with at most four distinctly colored vertices and hence at least one color is available for all the vertices.

Thus $\chi_g(P_n \circ W_m) = 5$. □

Theorem 4.5. *For any two integers n and m with $n \geq 3$ and $m \geq 3$, $\chi_g(K_{1,n} \circ P_m) = 4$.*

Proof. Let $V(K_{1,n}) = \{v_0, v_1, v_2, ..., v_n\}$ where v_0 is the center vertex of $K_{1,n}$ and $V(F_i) = \{v_1^i, v_2^i, ..., v_m^i\}, 1 \leq i \leq n$. First we show that $\chi_g(K_{1,n} \circ P_m) > 3$. To prove this, we give a strategy for Bob to win the game using three colors. The strategy of Bob is as follows (Figure 6).

The vertex in which Alice colors in her first move	The vertex with which Bob responds
◇ v_0 with color 1.	v_1^1 with color 2. This forces Alice to color v_1. This makes v_2^1 color-1 critical. Hence Bob wins.
◇ v_i with color 1, $1 \leq i \leq n$.	v_1^0 with color 2. This forces Alice to color v_0. This makes v_2^0 color-1 critical. Hence Bob wins.
◇ v_k^i with color 1.	(*i*) When $m > 3$ Bob colors a vertex of v_l^i which is at a distance two from v_k^i with color 2. This makes the two common neighbors of v_k^i and v_l^i color-3 critical. Hence Bob wins. (*ii*) When $m = 3$ Bob colors another vertex of F_i with color 2. This forces Alice to color v_i with color 3. Now Bob colors v_1^0 with color 2. This forces Alice to color v_0. This makes v_2^0 color-3 critical. Hence Bob wins.

Thus $\chi_g(K_{1,n} \circ P_m) \geq 4$. Now to show that $\chi_g(K_{1,n} \circ P_m) = 4$, we give a strategy for Alice to win the game using four colors. The strategy of Alice is as follows.

In the first move, Alice colors a vertex v_0. In the following moves, whenever

- **Bob colors a vertex of** $K_{1,n}$, Alice colors a vertex of $K_{1,n}$, if there is one uncolored. Otherwise she colors any uncolored vertex of the graph.

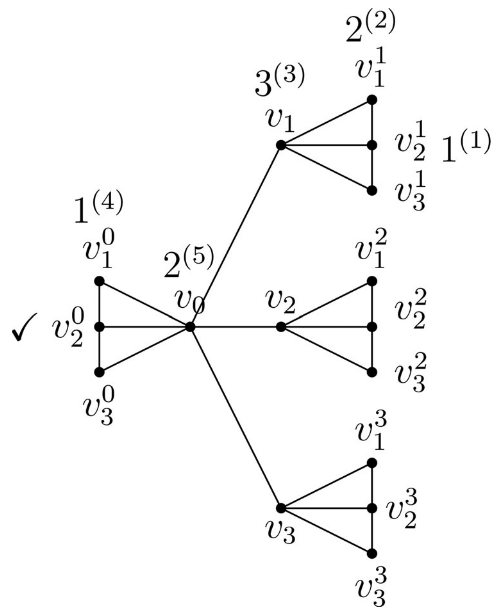

Figure 6. Illustration for Bob's strategy with $n = m = 3$.

- **Bob colors a vertex of F_i,** Alice colors a vertex v_i, if it is uncolored. If not, she colors a vertex of $K_{1,n}$, if there is one uncolored. Otherwise she colors any uncolored vertex of the graph.

Note: Using this strategy, at any point of the game, a vertex of $K_{1,n}$ is adjacent to at most three distinctly colored vertices and the maximum degree of a vertex of F_i, $0 \leq i \leq n$ is three and hence at least one color is available.
Thus $\chi_g(K_{1,n} \circ P_m) = 4$. □

Theorem 4.6. *For any integer $n \geq 3$,*

(i) $\chi_g(K_n \circ K_n) = n+1$;
(ii) $\chi_g(K_n \circ K_{n,n}) = n+1$.

Proof. (*i*) As K_{n+1} is an induced subgraph of $K_n \circ K_n$ and $\chi(K_{n+1}) = n+1$, we have $\chi_g(K_n \circ K_n) \geq n+1$. Now we show that $\chi_g(K_n \circ K_n) = n+1$. To prove this, we give a strategy for Alice to win the game using $n+1$ colors. The strategy of Alice is as follows.

In the first move, Alice colors a vertex of K_n. In the following moves, whenever

- **Bob colors a vertex of F_i,** $1 \leq i \leq n$, Alice colors v_i, if it is uncolored. Otherwise she colors a vertex of F_i, if there is one uncolored. If not she colors a vertex of K_n, if there is one uncolored. Otherwise she colors any uncolored vertex of the graph.
- **Bob colors a vertex of K_n,** Alice colors a vertex of K_n, if there is one uncolored. If not she colors any uncolored vertex of the graph.

Note: Using this strategy, at any point of the game, any uncolored vertex is adjacent to at most n distinctly colored vertices and hence at least one color is available for all the vertices.
Thus $\chi_g(K_n \circ K_n) = n+1$. □
(*ii*) Let $V(K_n) = \{v_1, v_2, v_3, ..., v_n\}$ and $V(F_i) = \{v_1^i, v_2^i, v_3^i, ..., v_m^i\}$ where $m = 2n$. Let A_1^i and A_2^i are the two partite sets of F_i, $1 \leq i \leq n$. First we show that $\chi_g(K_n \circ K_{n,n}) > n$. To prove this, we give a strategy for Bob to win the game using n colors. The strategy of Bob is as follows.

Alice starts the game by coloring any vertex. Bob colors a vertex of K_n. In the following moves, irrespective of Alice's moves, Bob keeps coloring the vertices of K_n until all but two vertices of K_n, say v_i and v_j, are colored. Each time when Bob colors a vertex of K_n, his first preference would be a vertex v_k, if a vertex of F_k is colored by Alice in the previous move. Note that, at this stage $n - 2$ colors are used. Let the two remaining unused colors be color $n-1$ and color n. Now it may be Bob's turn or Alice's turn.

(i) *If it is Bob's turn, then he colors a vertex of F_i with color $n - 1$. This forces Alice to color v_i which in turn makes v_j color-(n-1) critical. Hence Bob wins.*
(ii) *If it is Alice's turn, then we have the following two cases.*
  - If Alice colors v_i or v_j, then the game is in Bob's hand.
  - If Alice colors any vertex other than v_i and v_j, then Bob plays in a way similar to his response given above in (*i*).

Thus $\chi_g(K_n \circ K_{n,n}) \geq n+1$. Now to show that $\chi_g(K_n \circ K_{n,n}) = n+1$, we give a strategy for Alice to win the game using $n+1$ colors. The strategy of Alice is as follows.

Alice colors a vertex of K_n in the first move. In the following moves, whenever

- **Bob colors a vertex of F_i, say a vertex in A_1^i,** Alice considers the partite set where Bob has colored in the last move. The course of the game is discussed below.
  - If exactly one vertex of A_1^i is colored, then Alice responds in v_i, if it is uncolored. Otherwise she colors a vertex of A_2^i.
  - If two vertices of A_1^i are colored with distinct colors, then Alice replies in a vertex of A_2^i.
  - Otherwise she colors any uncolored vertex of F_i, if there is one uncolored. If not, she colors any uncolored vertex of K_n, if there is one uncolored. Otherwise she colors in a vertex of a partition of F_j, $j / = i$, if no vertex of that partition is already colored. If such a partition does not exist, then she colors any uncolored vertex of the graph.
- **Bob colors a vertex of K_n,** Alice colors a vertex of K_n, if there is one uncolored. Otherwise she colors in a vertex of a partition of F_j, $j \neq i$, if no vertex of that partition is already colored. If such a partition does not exists, then she colors any uncolored vertex of the graph.

Note: Using this strategy, at any point of the game, any uncolored vertex is adjacent to at most n distinctly colored vertices. Hence at least one color is available for all the vertices.

Thus $\chi_g(K_n \circ K_{n,n}) = n + 1$. □

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