# Trigonometric and cylindrical polynomials and their applications in electromagnetics 

Y. Shestopalov

To cite this article: Y. Shestopalov (2019): Trigonometric and cylindrical polynomials and their applications in electromagnetics, Applicable Analysis, DOI: 10.1080/00036811.2019.1584290

To link to this article: https://doi.org/10.1080/00036811.2019.1584290


Article views: 598

View related articles


View Crossmark data〔

# Trigonometric and cylindrical polynomials and their applications in electromagnetics 

Y. Shestopalov<br>Faculty of Engineering and Sustainable Development, University of Gävle, Gävle, Sweden


#### Abstract

A development of the theory of trigonometric polynomials (TPs) is considered that involves generalization of the notion of TP and extension of the methods and results to cylindrical polynomials (CPs). On the basis of the proposed augmentation of TPs and CPs, a general approach is presented to the analysis of guided waves and resonances in electromagnetics and beyond. The method employs the known explicit forms of dispersion equations (DEs) describing eigenoscillations and normal waves in layered structures and is based on the development of the theory of generalized TPs and CPs performed in the study. The approach enables one to complete rigorous proofs of existence and determine domains of localization of the TP and CP zeros and the DE roots and validate iterative numerical solution techniques.


## ARTICLE HISTORY

Received 4 January 2019
Accepted 14 February 2019

## COMMUNICATED BY

Michael Klibanov

## KEYWORDS

Trigonometric polynomials; cylindrical polynomials; dispersion equation; zeros; electromagnetic waves

## 2010 MATHEMATICS SUBJECT CLASSIFICATIONS

26C10; 33B10; 33C10; 34B24; 78M22

## 1. Introduction

Trigonometric polynomials (TPs) arise in many areas of pure and applied mathematics: algebra of polynomials and complex analysis [1,2], Fourier analysis, approximation theory, numerical methods and interpolation [3] to name a few, as well as in various applications including signal processing [4]. TPs occur (see e.g. [5] and references therein) in the search for eigenvalues of non-selfadjoint Sturm-Liouville problems on the semi-axis, whole axis, or an interval with piecewise constant coefficients in the equations and local or non-local boundary or transmission conditions or conditions at infinity containing spectral parameter.

Analysis of TPs including the existence and distribution of their (real or complex) zeros constitutes a specific direction of real and complex analysis developed in many classical works. Information about the existence and location of zeros is essential property of TPs (as it is the case of any polynomial). The proofs of the existence and investigations of distribution of the TP zeros constitute complicated mathematical problems addressed during the last two centuries by many famous mathematicians: Littlewood [2,6,7], Erdelyi [8,9], Lanczos [10], and others [11].

Let us give an example (taken from an extended collection of problems [12]) and a typical sophisticated task of the TP theory. A 'regular' TP

$$
\begin{equation*}
L_{N}(\theta)=\sum_{j=0}^{N}\left(C_{j} \cos j \theta+S_{j} \sin j \theta\right) \tag{1}
\end{equation*}
$$

[^0]

Figure 1. $\operatorname{TP} L_{100}(x)=\sum_{m=1}^{100} \cos n_{m} x, x \in(0, \pi)$, with randomly distributed digits $1 \leq n_{m} \leq 106$.
where coefficients $C_{j}$ and $S_{j}$ do not vanish simultabeously has exactly $2 N$ zeros. However, generally, the occurrence of zeros of a TP may constitute a severe problem. Indeed, a cosine TP

$$
\begin{equation*}
L_{M}(\theta)=\sum_{j=1}^{M} \cos n_{j} \theta \tag{2}
\end{equation*}
$$

was considered by Littlewood in [2] who posed the following problem: if the $n_{j}$ are integral and all different, what is the lower bound on the number of real zeros of TP (2)? Possibly M-1, or not much less. No progress has been made on this problem in the last half century until 2008 when it has been shown [13] that there exists a cosine polynomial (2) with the $n_{j}$ integral and all different so that the number of its real zeros in the period $[-\pi, \pi)$ is $O\left(M^{5 / 6} \log M\right)$. Below we make another contribution to the analysis of the Littlewood TPs and prove a statement (Corollary 5.3) estimating the number of zeros for a certain subfamily of TPs (2). Figure 1 gives an example of a TP (2) of order 100 with the estimated number $2 \cdot 100^{5 / 6}$ of zeros located in $[-\pi, \pi)$ to be around 93 .

Many approaches in the TP theory deal with introduction of various TP families and obtaining different sufficient conditions providing the existence of the TP zeros valid for each particular family or class of TPs. In many cases, the developed approaches enable one to describe distribution of zeros and determine their number (over a period). Let us give two examples illustrating such methods. In [9] it is examined the size of a real TP of degree at most $n$ having at least $k$ zeros in $K:=\mathrm{R}(\bmod 2 \pi)$ (counting multiplicities). In [14] the following sufficient conditions are obtained for the existence of zeros of a family of cosine TPs

$$
\begin{equation*}
\mathcal{C}_{M}(x)=\frac{1}{2}+\sum_{m=1}^{M} a_{m} \cos m x, \quad a_{M}=0 \tag{3}
\end{equation*}
$$

where $\left\{a_{m}\right\}, a_{0}=1$, is a monotonically decreasing $\left(a_{m+1}-a_{m}<0, m=0, \ldots, M-1\right)$ and convex ( $a_{m+2}-2 a_{m+1}+a_{m}<0, m=0, \ldots, M-2$ ) number sequence: TP (3) has one simple zero on every interval $(m \pi / M,(m+1) \pi / M), m=1, \ldots, M-1$.

An example of the TP generalization according to [11] has the form

$$
\begin{equation*}
T_{M}(x)=\frac{1}{2} a_{0}+\sum_{m=1}^{M} a_{m} \cos m \theta, \quad a_{M} \neq 0 \tag{4}
\end{equation*}
$$

where $\theta=\Psi(x)$ is a continuous function with positive (or negative, but not both) first derivative for $x$ real. In particular, $T_{k}(x)=\cos (k \arccos x), x \in[-1,1]$, are the Chebyshev polynomial of degree $k$.

Cylindrical polynomials (CPs) have much in common with TPs; however, their theory is much less developed. CPs occur when particular solutions to Maxwell equations are sought in polar or cylindrical coordinates in domains where boundary and interface contours or surfaces possess circular symmetry. Standard reference books on cylindrical functions (see e.g. [15-17]) consider two-term sums of products of two cylindrical functions occurring, e.g. in the form [18,19] $J_{n}\left(\kappa_{1} x\right) Y_{n}\left(\kappa_{2} x\right)-J_{n}\left(\kappa_{2} x\right) Y_{n}\left(\kappa_{1} x\right)(n=0,1,2, \ldots)$ of cross-products of Bessel functions. Extension to the greater number of factors in the products or greater number of product terms has not been performed, to the best of our knowledge.

This study follows and contribute to the three main directions of the TP and CP theories: creating a universal form of generalized TPs; extension of the methods of the TP analysis to the class of CPs involving cylindrical functions of different kinds and order; and obtaining new sufficient conditions providing the existence and description of distribution of zeros of certain families of TPs and CPs. The latter employs non-periodic TPs and analysis of their parameter dependence.

Another objective is application of the obtained results in the electromagnetic field theory, namely, to the determination of real and complex oscillations and waves. The latter are reduced to multiparameter eigenvalue problems [20,21] and then in many cases (when, e.g. the Helmholtz equation admits closed-form solutions using separation of variables) to dispersion equations (DEs). Their complete study is a sophisticated task which requires deep analytical-numerical investigations. In this work, a major attention is paid to creating an introduction to a general method that enables obtaining (sufficient) conditions of the existence and description of localization of the DE roots providing thus justification for the methods of determination of oscillations and waves in multi-layered structures which can be easily implemented in calculations. The method is based on the studies of the so-called generalized TPs (GTPs) and generalized CPs (GCPs) aimed in particular to finding zeros of GTPs and GCPs using different approaches.

The approach proposed in this study makes use of the method [22,23] employing GCPs applied in [23] to the rigorous analysis and determination of real waves in a dielectric waveguide (DW) and a Goubau line (GL) of circular cross section. The technique involves recursive procedures of the determination of zeros of TPs, GTPs, and GCPs.

## 2. Trigonometric polynomials

TPs are weighted linear combinations

$$
\begin{equation*}
\mathcal{T}_{M}(x)=\sum_{q=1}^{2} \sum_{m=0}^{M_{q}} P_{q}^{(m)}(x) \mathcal{U}_{q}\left(n_{q, m} x\right) \quad \text { or } \quad \mathcal{E}_{M}(t)=\sum_{m=0}^{M} P^{(m)}(t) \mathrm{e}^{i n_{m} t} \tag{5}
\end{equation*}
$$

of complex exponents or trigonometric functions $\mathcal{U}_{1}(y)=\sin y(q=1)$ or $\mathcal{U}_{2}(y)=\cos y(q=2)$, where $M=M_{1}+M_{2}$ (for $\mathcal{T}_{M}$ ), $n_{q, m}$ and $n_{m}$ are integers forming integer 'frequency' parameter vectors $\bar{\kappa}_{q, M}=\left(n_{q, 1}, \ldots, n_{q, M}\right)$, and $P_{q}^{(m)}(x)$ and $P^{(m)}(x)\left(m=1,2, \ldots, M_{q}, q=1,2\right)$ are constants or bounded continuous functions considered on the whole line, half-line $x>0$, or an interval. A cosine TP (2) considered by Littlewood has all $P^{(m)}(x)=1$ and is therefore $2 \pi$-periodic. One also separates the cases $M_{1}=M_{2}=M_{0}$ and $n_{1, m}=n_{2, m}=m$ or $n_{m}=m$ and all $P_{q}^{(m)}(x)=$ const $(m=$ $1,2, \ldots, M_{0}$ ) giving classical ( $2 \pi$-periodic) TPs, even (a cosine TP with all $P_{1}^{(m)}(x)=0$ ) or odd (a
sine TP with all $\left.P_{2}^{(m)}(x)=0\right)$. A 'regular' TP (1) constitutes, as well as TPs (2)-(4), examples of such polynomials; several more are presented in the appendix.

Together with TPs (5) we will study more general cases of non-periodic TPs considering wieghted trigonometric sums (TSs)

$$
\begin{equation*}
\mathcal{G}_{M}(x)=\sum_{q=1}^{2} \sum_{m=0}^{M_{q}} P_{q}^{(m)}(x) \mathcal{U}_{q}\left(\kappa_{q, m} x\right) \tag{6}
\end{equation*}
$$

with generally non-integer 'frequency' parameter vectors $\bar{\kappa}_{q, M}=\left(\kappa_{q, 1}, \ldots, \kappa_{q, M}\right), q=1,2$. TS (6) is a TP having the form (5) of the order $M=M_{1}+M_{2}$.

Many standard examples of TPs and TSs calculated in the closed form for which it is possible to determine all zeros explicitly are considered in the appendix.

## 3. Cylindrical polynomials

In this study, we call CPs the weighted linear combinations

$$
\begin{equation*}
\mathcal{C}_{M}(x)=\sum_{m=0}^{M} Q^{(m)}(x) \mathcal{I}_{m}\left(\kappa_{m} x\right) \tag{7}
\end{equation*}
$$

of cylindrical functions $\mathcal{I}_{m}$, where each $\mathcal{I}_{m}$ is either Bessel, $J_{p}$, or Neumann, $Y_{p}$, function ( $p=$ $0,1,2, \ldots), \kappa_{m}$ are real numbers forming parameter vectors $\bar{\kappa}_{M}=\left(\kappa_{1}, \ldots, \kappa_{M}\right)$, and $Q^{(m)}(x)(m=$ $0,1, \ldots, M)$ are constants or bounded continuous functions considered on the whole line, half-line $x>0$, or an interval. Unlike classical $2 \pi$-periodic TPs and TSs (A1)-(A6) with integer parameter vectors, CPs (7) are non-periodic (and in this respect they are similar to TPs (6)) although every term in (7) is an almost-periodic function. Namely, the following statements hold: (i) every function $\mathcal{I}_{m}$ has a countable sequence of positive simple zeros $\left\{z_{k}^{(m)}\right\}$ increasing w.r.t. index $k$; (ii) the distance $d_{k}^{(m)}=z_{k+1}^{(m)}-z_{k}^{(m)}(k=0,1,2, \ldots)$ between two neighboring zeros of $\mathcal{I}_{m}$ tends to a constant (in particular, to $\pi$ for the Bessel functions) as $x \rightarrow \infty$; (iii) there is a number $r>0$ such that $r=\min _{k=0,1, \ldots} d_{k}^{(m)}(m=1,2, \ldots)$; and (iv) zeros of every two different cylindrical functions $J_{m}(x)$, $J_{n}(x), Y_{m}(x), Y_{n}(x)(m \neq n)$, or $J_{m}(x), Y_{n}(x)(m, n=0,1, \ldots)$ alternate.

Analysis of CPs including the existence and distribution of their (real or complex) zeros constitutes a specific set of complicated mathematical problems that has been rarely addressed in the literature as compared with TPs (where one may refer, e.g. to [1,2,10,12]).

## 4. Generalized TPs and CPs

Consider weighted linear combinations

$$
\begin{equation*}
\mathcal{I}_{N, M}(x)=\sum_{m=1}^{M} Q^{(m)}(x) \mathcal{U}^{\left(m, N_{m}\right)}(x) \tag{8}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathcal{G}_{N, M}(x)=\sum_{m=1}^{M} P^{(m)}(x) \mathcal{W}^{\left(m, N_{m}\right)}(x) \tag{9}
\end{equation*}
$$

of the products $\mathcal{U}^{\left(m, N_{m}\right)}=\Pi_{n=1}^{N_{m}} \mathcal{U}_{1,2}\left(\kappa_{n, T}^{(m)} x\right)$ or $\mathcal{W}^{\left(m, N_{m}\right)}(x)=\Pi_{n=0}^{N_{m}} \mathcal{I}_{n}^{(m)}\left(\kappa_{n, G}^{(m)} x\right)$ of trigonometric, $\mathcal{U}_{1,2}$, or cylindrical, $\mathcal{I}_{n}^{(m)}$, functions, where $N=\sum_{m=1}^{M} N_{m}, P^{(m)}(x)$ and $Q^{(m)}(x)$ are constants or
bounded continuous functions for $x>0$ and $\kappa_{n, T}^{(m)}$ and $\kappa_{n, G}^{(m)}$ are real parameters forming the 'frequency' parameter vectors $\bar{\kappa}_{T}^{(m)}=\left(\kappa_{1, T}^{(m)}, \ldots, \kappa_{N_{m}, T}^{(m)}\right)$ and $\bar{\kappa}_{G}^{(m)}=\left(\kappa_{n, G}^{(1)}, \ldots, \kappa_{n, G}^{\left(N_{m}\right)}\right), m=1,2, \ldots, M$.

In this study, functions (8) or (9) are referred to, respectively, as generalized TPs (GTPs) or GCPs of order ( $N, M$ ). In particular, for TPs (5) and (6)

$$
\begin{equation*}
\mathcal{G}_{M}(x)=\sum_{m=1}^{M_{1}} P_{1}^{(m)}(x) \sin \left(\kappa_{1, m} x\right)+\sum_{m=0}^{M_{2}} P_{2}^{(m)}(x) \cos \left(\kappa_{2, m} x\right)=\mathcal{T}_{M, M}(x) \tag{10}
\end{equation*}
$$

all $N_{m}=1$ so that they are GTPs of order $(M, M)$ with $M=M_{1}+M_{2}$.
In the studies of cylindrical functions (e.g. in [15-17]) zeros are considered of the twoterm sums of products of two cylindrical functions; that is, of $\mathcal{G}_{4,2}(x)=\sum_{m=1}^{2} \mathcal{W}^{(m, 2)}(x)$, where $\mathcal{W}^{(m, 2)}(x)=\mathcal{I}_{n_{1, m}}^{(m)}\left(\kappa_{n_{1, m}}^{(m)} x\right) \mathcal{I}_{n_{2, m}}^{(m)}\left(\kappa_{n_{2, m}}^{(m)} x\right)$ with $n_{1,2, m}$ denoting the order of two cylindrical functions $\mathcal{I}_{n_{1, m}}^{(m)}$ and $\mathcal{I}_{n_{2, m}}^{(m)}$ that have either different kind, or the same kind and different order or different parameters $\kappa_{n_{1, m}}^{(m)}$ and $\kappa_{n_{2, m}}^{(m)}(m=1,2)$. Such two-term sums of products, e.g. $J_{n}\left(\kappa_{1} x\right) Y_{n}\left(\kappa_{2} x\right)-J_{n}\left(\kappa_{2} x\right) Y_{n}\left(\kappa_{1} x\right)(n=0,1,2, \ldots)$ as in [19] are called cross-product Bessel functions.

Lemma 4.1: A product $\mathcal{U}^{(m, N)}(x)=\Pi_{n=1}^{N} \mathcal{U}_{1,2}\left(\kappa_{n, T}^{(m)} x\right)$ of trigonometric functions $\mathcal{U}_{1,2}$ can be represented as a GTP of order $\left(M^{\prime}, M^{\prime}\right)$ with a certain $M^{\prime}=M^{\prime}(N)$, that is, as a TP (6) of order $M^{\prime}$.

Proof: The first step of the induction proof is to check basic trigonometric identities $\sin x \sin y=$ $0.5(\cos (x-y)-\cos (x+y)), \sin x \cos y=0.5(\sin (x-y)+\sin (x+y))$, and $\cos x \cos y=0.5(\cos (x$ $-y)+\cos (x+y))$ and to apply them to the triple products, e.g. $\sin x \sin y \sin z=0.5(\sin z \cos (x-$ $y)-\sin z \cos (x+y))=0.25(\sin (z-x+y)+\sin (x+y-z)+\sin (z+x-y)+\sin (-x-y-z))$ so that $\prod_{n=1}^{3} \mathcal{U}_{1}\left(\kappa_{n} t\right)=0.25 \sum_{m=1}^{4} \mathcal{U}_{1}\left(\kappa_{n}^{\prime} t\right)=\mathcal{T}_{4,4}(x)=\mathcal{G}_{4}(x)$ with $x=\kappa_{1} t, y=\kappa_{2} t, z=\kappa_{3} t, \kappa_{1}^{\prime}=$ $-\kappa_{1}+\kappa_{2}+\kappa_{3}, \kappa_{2}^{\prime}=\kappa_{1}+\kappa_{2}-\kappa_{3}, \kappa_{3}^{\prime}=\kappa_{1}-\kappa_{2}+\kappa_{3}$, and $\kappa_{4}^{\prime}=-\kappa_{1}-\kappa_{2}-\kappa_{3}$.

Assuming now that

$$
\begin{equation*}
\Pi_{n=1}^{N-1} \mathcal{U}_{1,2}\left(\kappa_{n} x\right)=\sum_{m=0}^{M^{\prime}(N-1)} P_{q} \mathcal{U}_{1,2}\left(\kappa_{n}^{\prime} x\right) \tag{11}
\end{equation*}
$$

one can easily verify, after converting each product of two trigonometric functions to a sum, that

$$
\Pi_{n=1}^{N} \mathcal{U}_{1,2}\left(\kappa_{n} x\right)=\mathcal{U}_{1,2}\left(\kappa_{N} x\right) \Pi_{n=1}^{N-1} \mathcal{U}_{1,2}\left(\kappa_{n} x\right)=\sum_{m=0}^{M^{\prime}(N)} P_{m}^{\prime} \mathcal{U}_{1,2}\left(\kappa_{n}^{\prime \prime} x\right)
$$

Corollary 4.2: A GTP (8) of order ( $N, M$ ) can be represented as a TP (6) of a certain order $M_{1}=$ $M_{1}(M, N)$.

Proof: Every product $\mathcal{U}^{\left(m, N_{m}\right)}$ in (8) can be written according to Lemma 4.1 as a GTP of order ( $M_{m}^{\prime}, M_{m}^{\prime}$ ) with a certain $M_{m}^{\prime}=M_{m}^{\prime}\left(N_{m}\right)$. Consequently, GTP (8) of order ( $M, N$ ) can be represented as a sum of TPs (6) and finally as a TP (6) of the order $M=\sum_{m=1}^{M} M_{m}^{\prime}$.

Lemma 4.3: Denote by $\Delta_{m}^{(1)}(\alpha)=\left(x_{s}^{(1, m)}(\alpha), x_{s}^{(1, m+1)}(\alpha)\right)$ and $\Delta_{l}^{(2)}(\beta)=\left(x_{c}^{(1, l)}(\beta), x_{c}^{(1, l+1)}(\beta)\right)$, $m, l=1,2, \ldots$, intervals formed by two pairs of neighboring positive zeros of $\sin \alpha x$ and $\cos \beta x$ given by
(A9) and (A10) with $\alpha, \beta \neq 0$. Then for any three distinct positive numbers $x_{1}>0, x^{*}$, and $x_{2}, x_{1}<$ $x^{*}<x_{2}$, there are $\alpha>0, \beta>0$, and integers $m^{\prime}$ and $l^{\prime}$ such that $x^{*} \in \Delta_{m^{\prime}}^{(1)}(\alpha)$ and $\Delta_{m^{\prime}}^{(1)}(\alpha) \subset\left(x_{1}, x_{2}\right)$ or $x^{*} \in \Delta_{l^{\prime}}^{(2)}(\beta)$ and $\Delta_{l^{\prime}}^{(2)}(\beta) \subset\left(x_{1}, x_{2}\right)$.

Proof: Endpoints of $\Delta_{m}^{(1)}(\alpha)$ and $\Delta_{l}^{(2)}(\beta)$ form equidistant (uniform) grids of points on the halfline $x>0$ separated by $d_{1}=\pi \alpha^{-1}$ and $d_{2}=\pi \beta^{-1}$. Therefore, there is an interval between certain neighboring grid points of each grid that contains one, two, or three of the numbers $x_{1}, x^{*}$, and $x_{2}$ (we assume that $x^{*}$ does not coincide with any of the grid points). Choosing $\alpha$ or $\beta$ such that $d_{1,2}<\min \left\{x^{*}-x_{1}, x_{2}-x^{*}\right\}<x_{2}-x_{1}$ we see that there are integers $m^{\prime}$ and $l^{\prime}$ such that $x^{*} \in \Delta_{m^{\prime}}^{(1)}(\alpha)$ or $x^{*} \in \Delta_{l^{\prime}}^{(2)}(\beta)$ and $\Delta_{, m^{\prime}, l^{\prime}}^{(1,2)} \subset\left(x_{1}, x_{2}\right)$.

## 5. Zeros of GTPs and GCPs

In order to formulate sufficient conditions that guarantee the existence of zeros of GTPs and GCPs and describe their localization, we will use the following

Lemma 5.1: Let $f_{j}(x) \in C[a, b], j=1,2,3$ (continuous in a closed interval $\left.[a, b]\right), f_{2}(a)=f_{2}(b)=0$, and $f_{1}(a) f_{1}(b)<0$ or, equivalently, there is an $s \in(a, b)$ such that $f_{1}(s)=0$; then the equation

$$
f(x) \equiv f_{2}(x) f_{3}(x)+f_{1}(x)=0
$$

has a root $x^{*} \in(a, b)$.
Proof: Indeed, $f(a)=f_{1}(a)$ and $f(b)=f_{1}(b)$ so that $f(a) f(b)<0$ and therefore $f(x)=0$ has a root $x=x^{*} \in(a, b)$.

### 5.1. GTPs

We have already noted that TPs (A1)-(A6) have each infinitely many zeros for any number of their terms $N=2,3, \ldots$ and that infinitely many zeros of 'neighboring' TPs $S_{N}, S_{N-1}, C_{N}, C_{N-1}$, and $T_{N}, T_{N-1}(N=2,3, \ldots)$ alternate. This statement can be in a certain sense generalized for arbitrary GTP using Lemma 5.1, validating simultaneously a recursive procedure of proving the existence and determining the location of zeros of TPs, GTPs, and GCPs.

Theorem 5.2: There are (in general, non-integer) frequency vectors $\bar{\kappa}_{q, M}=\left(\kappa_{q, 1}, \ldots, \kappa_{q, M}\right), q=1,2$, such that a TP (6) of order $2 M$ where $P^{(m)}(x)$ are arbitrary constants or bounded continuous functions that do not vanish on the half-line $x>0$ has a positive zero.

Proof: The first step of the induction proof is to call that TPs of orders 1 and $2, \sin \left(\kappa_{1,1} x\right), \cos \left(\kappa_{2,1} x\right)$, and $P_{1} \sin \kappa_{1,1} x+P_{2} \cos \kappa_{2,1} x$, have each infinitely many positive zeros for any $P_{1,2}=$ const, $P_{1}^{2}+$ $P_{2}^{2} \neq 0$, and $\kappa_{q, 1} \neq 0, q=1,2$. The latter statement is easily extended to the case when $P_{1,2}=P_{1,2}(x)$ are any two functions satisfying the condition of the theorem. Next, assume that TP $\mathcal{G}_{2 M-2}(x)$ in the form (6) of order $2(M-1)$ has a positive zero $x_{M-1}^{*}$ and consider TPs of order $2 M-1$

$$
\begin{align*}
& \mathcal{G}_{1,2 M-1}(x)=P_{1}^{(M)}(x) \sin \left(\kappa_{1, M} x\right)+\mathcal{G}_{2 M-2}(x),  \tag{12}\\
& \mathcal{G}_{2,2 M-1}(x)=P_{2}^{(M)}(x) \cos \left(\kappa_{2, M} x\right)+\mathcal{G}_{2 M-2}(x) . \tag{13}
\end{align*}
$$

Applying Lemmas 4.3 and 5.1 with ( $a, b$ ) being an interval formed by two neighboring zeros of $\sin \left(\kappa_{1, M} x\right)$ or $\cos \left(\kappa_{2, M} x\right)$ and choosing $\kappa_{1, M}$ or $\kappa_{2, M}$ in line with Lemma 4.3 with respect to $x_{M-1}^{*} \in$ $(a, b)$ we prove the statement of the theorem.

Statement of Theorem 5.2 is valid for cosine TP (2) considered by Littlewood in the form stronger than that for non-periodic TPs and GTPs. Namely, one can estimate the number of zeros on halfperiod ( $0, \pi$ ] and establish the existence of infinitely many positive zeros of TP (2).

Corollary 5.3: $T P(2)$ of order $M \geq 2$ where $n_{1}$ and $n_{2}, n_{2}>n_{1}$, are two aribtrary (different) integers, has at least $2 n_{2}$ zeros on half-period $(0, \pi]$ and infinitely many positive zeros.

Proof: The first step of the induction proof is to call that TPs (2) of orders 1 and $2, L_{1}(x)=\cos n_{1} x$ and $L_{2}(x)=\cos n_{21} x+\cos n_{22} x=2 \cos \left(\left(n_{22}+n_{21}\right) / 2\right) x \cos \left(\left(n_{22}-n_{21}\right) / 2\right) x$, have each infinitely many positive zeros

$$
\begin{equation*}
x_{n_{1}}^{(1, k)}=\frac{(2 k-1) \pi}{2 n_{1}}, \quad x_{n^{+}}^{(2, l)}=\frac{(2 l-1) \pi}{n^{+}}, \quad x_{n^{-}}^{(3, m)}=\frac{(2 m-1) \pi}{n^{-}}, \quad k, l, m=1,2, \ldots, \tag{14}
\end{equation*}
$$

for any integers $n_{1}, n_{21}$, and $n_{22}, n_{22}>n_{21}$, with $n^{ \pm}=n_{22} \pm n_{21}$; in particular, $L_{1}(x)$ has $n_{1}$ zeros and $L_{2}(x)$ has $n_{22}=n^{+} / 2+n^{-} / 2$ (when parity of $n^{+}$and $n^{-}$is the same) or $n_{22}+1=\left(n^{+}+1\right) / 2+$ $\left(n^{-}+1\right) / 2$ (parity of $n^{+}$and $n^{-}$is different) zeros on half-period ( $\left.0, \pi\right]$. Thus, zeros of $L_{2}(x)$ form a finite set of distinst points on the interval $(0, \pi)$. To show details of the proof of the induction step, consider $L_{3}(x)=\cos n_{3} x+L_{2}(x)$. According to Lemma 4.3, one can choose $n_{3}$ such that intervals $\Delta_{n_{3}}^{(s)}=\left(x_{n_{3}}^{(1, s)}, x_{n_{3}}^{(1, s+1)}\right), s=1,2, \ldots, n_{22}^{e o}$ (where $n_{22}^{e o}=n_{22}$ or $n_{22}^{e o}=n_{22}+1$ ) between two neighboring zeros of $\cos n_{3} x$ contain each one zero of $L_{2}(x)$. This means, in line with Lemma 5.1, that $L_{3}(x)$ has a zero on each $\Delta_{n_{3}}^{(s)}$. Consider now $L_{M}(x)=\cos n_{M} x+L_{M-1}(x)$ and assume that $\left\{x_{M}^{(p)}\right\}_{p=1}^{p=L_{M-1}}$ are $\mathrm{L}_{M-1}$ zeros of $L_{M-1}(x)$ on $(0, \pi)$. According to Lemma 4.3, one can choose $n_{M}$ such that intervals between two neighboring zeros of $\cos n_{M} x$ contain each one zero of $L_{M-1}(x)$. Therefore, Lemma 5.1 implies that the corollary is proved.

Figures 2 and 3 illustrate the proofs and display examples of TPs (2) of different orders. Comparing Figures 2 and 4 one observes a clear similarity between CPs and TPs with respect to the statements of Theorems 5.2 and 5.4 and Lemma 4.3.

### 5.2. GCPs

According to Corollary 4.2, Theorem 5.2 can be applied to GTP (8). However it is reasonable to give an independent proof which provides sufficient conditions of the existence of zeros of both GTPs (8) and GCPs (9).

Theorem 5.4: There are 'frequency' vector sets $\Theta_{T}^{(M)}=\cup_{m=1}^{M} \bar{\kappa}_{T}^{(m)}$ and $\Theta_{G}^{(M)}=\cup_{m=1}^{M} \bar{\kappa}_{G}^{(m)}$, where $\bar{\kappa}_{T}^{(m)}=\left(\kappa_{1, T}^{(m)}, \ldots, \kappa_{N_{m}, T}^{(m)}\right)$ and $\bar{\kappa}_{G}^{(m)}=\left(\kappa_{1, G}^{(1)}, \ldots, \kappa_{N_{m}, G}^{\left(N_{m}\right)}\right)(m=1,2, \ldots, M)$ have real components, such that GTP (8) and GCP (9) of order $(N, M)\left(N=\sum_{m=1}^{M} N_{m}\right)$ where $Q^{(m)}(x)$ and $P^{(m)}(x)(m=$ $1,2 \ldots, M)$ are arbitrary constants or bounded continuous functions that do not vanish on the half-line $x>0$ have each a positive zero.

Proof: First, note that at $M=1$, GTP (8) and GCP (9) of order ( $1, N_{1}$ ) are single products having each (as a function of $x$ ) infinitely many positive zeros forming countable sets $T_{N_{1}}^{(1)}=\cup_{n=0}^{N_{1}} t_{n}^{(1)}$ and $Z_{N_{1}}^{(1)}=\cup_{n=0}^{N_{1}} Z_{n}^{(1)}$ being unions of the infinite sets $t_{n}^{(1)}$ and $Z_{n}^{(1)}$ of the zeros $\tau_{n, k}^{(1)}$ and $z_{n, k}^{(1)}(k=1,2, \ldots)$ of all $\mathcal{U}_{1,2}\left(\kappa_{n, T}^{(1)} x\right)$ and $\mathcal{I}_{n}^{(1)}$ that enter these products.

Second, for any $M=2,3 \ldots$, and $m=1,2 \ldots, M$, the products $\mathcal{U}^{\left(m, N_{m}\right)}(x)$ and $\mathcal{W}^{\left(m, N_{m}\right)}(x)$ comprising $N_{m}$ factors have each infinitely many positive zeros forming countable sets $T_{N_{m}}^{(m)}=\cup_{n=0}^{N_{m}} t_{n}^{(m)}$


Figure 2. TPs $L_{3}(x)=\cos 3 x+\cos 7 x+\cos 20 x \quad\left(M=3, \quad n_{22}=7, \quad n_{21}=3, \quad n_{3}=20\right)$ with $L_{2}(x)=\cos 3 x+\cos 7 x=$ $2 \cos 2 x \cos 5 x, x \in(0, \pi) ; L_{2}(x)$ (o) has 7 zeros located together with 7 zeros of $L_{3}(x)$ (*) between pairs of neighboring zeros of $\cos 20 x$.


Figure 3. $\operatorname{TPs} L_{9}(x)=\sum_{m=1}^{9} \cos n_{m} X$ (lower curve with greater number of oscillations) with $\bar{\kappa}_{M}=\left(n_{1}, \ldots, n_{9}\right)=(7,13,2$, $15,8,4,11,17,6), L_{8}(x)=L_{9}(x)-\cos 6 x$ (upper curve with greater number of oscillations), and $\cos 6 x, x \in(0, \pi) ; z e r o s$ of TPs alternate and $L_{9}(x)$ and $L_{8}(x)$ have each 11 zeros located between pairs of neighboring zeros of $\cos 6 x$.
and $Z_{N_{m}}^{(m)}=\cup_{n=0}^{N_{m}} Z_{n}^{(m)}$ being unions of the infinite sets $t_{n}^{(m)}$ and $Z_{n}^{(m)}$ of the zeros $\tau_{n, k}^{(m)}$ and $z_{n, k}^{(m)}$ $(k=1,2, \ldots)$ of all $\mathcal{U}_{1,2}\left(\kappa_{n, T}^{(m)} x\right)$ and $\mathcal{I}_{n}^{(m)}$. Elements of $t_{n}^{(m)}$ and $Z_{n}^{(m)}$ (and of $T_{N_{m}}^{(m)}$ and $\left.Z_{N_{m}}^{(m)}\right)$ depend on the parameter vectors $\bar{\kappa}_{T}^{(m)}$ and $\bar{\kappa}_{G}^{(m)}(m=1,2 \ldots, M)$.


Figure 4. $\operatorname{CPs}_{3}(x)=J_{0}(3 x)+J_{0}(7 x)+J_{0}(20 x)$ with $C_{2}(x)=J_{0}(3 x)+J_{0}(7 x), x \in(0, \pi) ; C_{2}(x)$ (0) has 4 zeros located together with 6 zeros of $C_{3}(x)\left(^{*}\right)$ between pairs of neighboring zeros of $J_{0}(20 x)$.

Perform the next step of induction and represent GTP (8) and GCP (9) as

$$
\begin{align*}
& \mathcal{T}_{N, M}(x)=Q^{(M)}\left(k^{(M)} x\right) \mathcal{U}^{\left(M, N_{M}\right)}(x)+\mathcal{T}_{N^{\prime}, M-1}(x)  \tag{15}\\
& \mathcal{G}_{N, M}(x)=P^{(M)}\left(k^{(M)} x\right) \mathcal{W}^{\left(M, N_{M}\right)}(x)+\mathcal{G}_{N^{\prime}, M-1}(x) \tag{16}
\end{align*}
$$

where $N^{\prime}=\sum_{m=1}^{M-1} N_{m}, \mathcal{U}^{\left(M, N_{M}\right)}(x)$ and $\mathcal{W}^{\left(M, N_{M}\right)}(x)$ vanish at the endpoints of the intervals $u_{n, k}(M)=(a, b)=\left(\tau_{n, k}^{(M)}, \tau_{n, k+1}^{(M)}\right)$ and $I_{n, k}(M)=(a, b)=\left(z_{n, k}^{(M)}, z_{n, k+1}^{(M)}\right)\left(n=1,2 \ldots, N_{M}\right)$ between every two their neighboring zeros.

Assume that the parameter vector sets $\Theta_{T}^{(M-1)}$ and $\Theta_{G}^{(M-1)}$ in (15) and (16) are such that GTP $\mathcal{T}_{N^{\prime}, M-1}(x)$ and GCP $\mathcal{G}_{N^{\prime}, M-1}(x)$ of order $\left(M-1, N^{\prime}\right)$ have each a zero $x_{T}^{*}>0$ or $x_{G}^{*}>0$. Choose $\Theta_{T}^{(M-1)}$ and $\Theta_{G}^{(M-1)}$ according to Lemma 4.3 with $(a, b)$ being intervals $u_{n^{\prime}, k^{\prime}}(M-1)$ or $I_{n^{\prime \prime}, k^{\prime \prime}}(M-1)$ formed by two neighboring zeros of $\mathcal{U}^{\left(M, N_{M}\right)}(x)$ or $\mathcal{W}^{\left(M, N_{M}\right)}(x)$ with index pairs ( $\left.n^{\prime}, k^{\prime}\right)$ or $\left(n^{\prime \prime}, k^{\prime \prime}\right)$ such that $x_{T}^{*} \in(a, b)$ or $x_{G}^{*} \in(a, b)$. Next, use Lemma 5.1 to conclude that GTP (8) and GCP (9) of order $(M, N)$ have each a zero on these intervals proving thus the statement of the theorem.

## 6. Application to DEs in electromagnetic field theory

### 6.1. Waveguides with circularly symmetric layered media

When electromagnetic wave propagation is considered [20,24] in waveguides having circular symmetry, like DWs or GLs formed by several concentric layers of media, all the field components of symmetric and non-symmetric waves are expressed [21,24] via a potential function $\phi(r)$ which is sought generally as a linear combination of cylindrical functions of order $m=0,1,2, \ldots$. This fact enables one to reduce finally the determination of the wave propagation constants to the solution of DEs involving CPs and GCPs. Further stages of the mathematical model are described in the next section.

### 6.1.1. Sturm-Liouville problems

Analysis of running (normal) waves in waveguides filled with circularly symmetric layered media is reduced $[21,22,25]$ to the non-selfadjoint Sturm-Liouville problems for the Bessel equation with piecewise constant coefficients on the semi-axis

$$
\begin{align*}
& \mathcal{L} \phi \equiv \frac{1}{r} \frac{d}{d r}\left(r \frac{d \phi}{d r}\right)+\widetilde{k}^{2} \phi=0, r>0,\left.\quad[\phi]\right|_{r=a}=\left.\left[\alpha_{1}(\gamma) \frac{d \phi}{d r}\right]\right|_{r=a}=0,  \tag{17}\\
& \phi \in C^{1}[0,+\infty) \cap C^{2}(0, a) \cap C^{2}(a,+\infty), \quad \phi(r) \rightarrow 0, r \rightarrow \infty, \quad(\mathrm{DW}), \\
& \mathcal{L} \phi=0, \quad r>a, \quad \phi(a)=0,\left.[\phi]\right|_{r=b}=\left.\left[\alpha_{2}(\gamma) \frac{d \phi}{d r}\right]\right|_{r=b}=0,  \tag{18}\\
& \phi \in C^{1}[a,+\infty) \cap C^{2}(a, b) \cap C^{2}(b,+\infty), \quad \phi(r) \rightarrow 0, r \rightarrow \infty, \quad(\mathrm{GL}) \tag{19}
\end{align*}
$$

here

$$
\widetilde{k}^{2}=\left\{\begin{array}{l}
k_{0}^{2}-\beta^{2}, r>a, \\
\epsilon k_{0}^{2}-\beta^{2}, r<a
\end{array} \quad \text { (DW), } \widetilde{k}^{2}=\left\{\begin{array}{l}
k_{0}^{2}-\beta^{2}, \quad r>b, \\
\epsilon k_{0}^{2}-\beta^{2}, \quad a<r<b
\end{array} \quad\right. \text { (GL), }\right.
$$

$\beta$ is the wave propagation constant (spectral parameter of the problem), $\epsilon$ and $k_{0}$ are, respectively (real or complex) permittivity of the media and wavenumber of vacuum (a real parameter), and $\alpha_{1,2}(\gamma)$ with $\gamma=\beta / k_{0}$ are given expicitly for a particular structure.

Finding eiganvalues of (17) and (18) (propagation constants of running waves) is reduced [20-26] to the solution of functional equations w.r.t. spectral parameter $\gamma$ or $\lambda=\gamma^{2}$ usually called DEs. When real spectrum of problems (17) and (18) is considered, the quantities are sought as real-valued functions of real spectral parameter $\gamma$ (or $\lambda$ ) varying on a certain interval $\gamma \in\left(\gamma_{1}, \gamma_{2}\right)$.

For multi-layered DWs or GLs when the number $M>1$ of dielectric layers (that is, discontinuities of the coefficient in the differential equations entering (17) and (18)) may be arbitrary, transmission conditions in (17) and (18) comprise $2 M$ jump relations with different $\alpha_{1,2}^{(j)}(\gamma)$ at the layer boundaries $r=a_{j}, a_{j+1}>a_{j}>0$, for DW or $r=b_{j}, b_{j+1}>b_{j}>a>0(j=1,2, \ldots M-1)$, for GL.

The general results of the classical Sturm-Liouville theory concerning the existence and distribution of (real or complex) spectrum for this type of problems are not applicable because the boundary (transmission) conditions depend on the spectral parameter; this dependence which is specified by concrete functions obtained explicitly governs the presence or absence of spectrum.

The explicit expressions for the DEs obtained in [22-24] for multi-layered DWs and GLs (open or shielded) or structures formed by plane-parallel layers of media [27,28] show that these DEs involve GCPs where weight coefficients $P^{(m)}(x)$ and $Q^{(m)}(x)$ are constants or bounded continuous functions for $x>0$ determined explicitly and $\kappa_{n, T}^{(m)}$ and $\kappa_{n, G}^{(m)}$ forming the parameter vectors $\bar{\kappa}_{T}^{(m)}=\left(\kappa_{1, T}^{(m)}, \ldots, \kappa_{N_{m}, T}^{(m)}\right)$ and $\bar{\kappa}_{G}^{(m)}=\left(\kappa_{n, G}^{(1)}, \ldots, \kappa_{n, G}^{\left(N_{m}\right)}\right), m=1,2, \ldots, M$ are real quantities expressed in terms of parameters of the particular structure (Figure 5).

### 6.1.2. Single-layer DW and GL

The DEs for single-layer DW and GL can be represented [21-23,25,26] in the general form involving GCPs (9)

$$
\begin{gather*}
F_{d}(x) \equiv P_{D}(w) J_{1}(x)+x J_{0}(x)=0 \quad(\mathrm{DW})  \tag{20}\\
F_{g}(x) \equiv P_{D}(q w) \Phi_{1}(x)-q x \Phi_{0}(x)=0 \quad(\mathrm{GL}) \tag{21}
\end{gather*}
$$

where $J_{m}, Y_{m}$ and $K_{m}(m=0,1)$ are the Bessel, Neumann, and Macdonald functions, $P_{D}(w)=\epsilon$ $\left(w K_{0}(w) / K_{1}(w)\right), \Phi_{0}=J_{0}(q x) Y_{0}(x)-J_{0}(x) Y_{0}(q x)$ and $\Phi_{1}=J_{0}(x) Y_{1}(q x)-J_{1}(q x) Y_{0}(x)$ are Bessel cross-products [19], and the waveguide geometric and material parameters $x=\kappa \sqrt{\epsilon-\gamma^{2}}, \gamma=\frac{\beta}{k_{0}}$,


Figure 5. Zeros of a GCP $F_{g}$ (curve with highest oscillation) given by (21) situated between neighboring zeros of $\Phi_{0}$ (curve with a negative starting value) and $\Phi_{1}$ (curve with a positive starting value).
$\kappa=k_{0} a, u=\kappa \sqrt{\epsilon-1}\left(\beta, \epsilon\right.$, and $k_{0}$ are, respectively, the wave progation constant, permittivity, and free-space wavenumber), $w=\sqrt{u^{2}-x^{2}}=\kappa \sqrt{\gamma^{2}-1}$, and $q=(b / a)>1$ ( $a$ and $b$ are characteristic dimensions of DW and GL).

For a GCP $F_{g}$ in (21) we can apply this reasoning and Lemma 5.1 by setting $f_{1}(x)=-q x \Phi_{0}(x)$ and $f_{2}(x)=\Phi_{1}(x)$. Then $F_{g}(x)$ has a zero between every two neighboring zeros $h_{k}^{(1)}$ and $h_{k+1}^{(1)}$ of $\Phi_{1}(x)$ as soon as $q>1$ is such that a zero of $\Phi_{0}(x)$ belongs to the interval $\left(h_{k}^{(1)} h_{k+1}^{(1)}\right)$. The latter condition can be satisfied because zeros $h_{k}^{(j)}=h_{k}^{(j)}(q)$ of $\Phi_{j}(x)$ alternate for different $j=1,2$ and form sequences of points decreasing with respect to $q[21,22,26]$. The conclusion concerning the existence and location of the zeros of $F_{g}$ between neighboring alternating zeros of $\Phi_{0}(x)$ and $\Phi_{1}(x)$ is perfectly illustrated by Figure 7.

Straightforward analysis of (20) and (21) demonstrates that functions $F_{d}$ and $F_{g}$ entering DEs have distinct common features: they are sums of (products of) cylindrical functions $J_{m}$ and $\Phi_{m}$ each having infinitely many alternating simple positive zeros. The latter yields an immediate proof (illustrated by Figure 6 and verified below) of the (sufficient conditions) providing the existence of real roots of the DEs located between zeros of $J_{m}$ and $\Phi_{m}(m=0,1)$. The existence, localization, and number of the DE roots are governed actually by a number of zeros of $J_{m}$ or $\Phi_{m}$ that are inside the domain $x \in(0, u)$ of $F_{d}$ and $F_{g}$; that is, by the value of parameter $u$.

### 6.1.3. Multi-Iayered waveguides

For $M$-layer DWs or GLs with $M>1$ DE takes the form [29] of the determinant equation

$$
\begin{equation*}
\mathrm{F}_{M}(\gamma) \equiv \operatorname{det} A(\gamma)=0, \quad A=\left\|a_{p q}(\gamma)\right\|_{p, q=1}^{2 M} \tag{22}
\end{equation*}
$$

where $\zeta(\gamma)$ is a one-to-one function of the spectral parameter considered on a certain interval $\gamma \in$ $\left(\gamma_{1}, \gamma_{2}\right)$ and $a_{p q}(\gamma)=\mathcal{I}^{(p q)}\left(\kappa_{G}^{(p, q)} \zeta(\gamma)\right)$ are cylindrical functions, either of different order or type or with different parameter factors $\kappa_{G}^{(p, q)}(p, q=1,2, \ldots 2 M)$.


Figure 6. Example for $\mathrm{GCP} F_{d}(x)$ in $\mathrm{DE}(20)$ : plots of $J_{0}(x)$ (upper curve with no oscillations), $J_{1}(x)$ (lower curve with no oscillations), and $F_{d}(x)$ (curve with one oscillations) at $\epsilon=5$ and $\kappa=2(u=\kappa \sqrt{\epsilon-1}=4)$ displaying a zero of $F_{d}(x)$ between neighboring zeros of $J_{1}(x)$ and neighboring zeros of $J_{0}(x)$ and $J_{1}(x)$.


Figure 7. An example showing alternating zeros of $\Phi_{0}(x)$ (curve starts at -0.3 ), $\Phi_{1}(x)$ (curve starts above 0.2 ), and $\mathcal{G}_{2,4}^{(G)}=$ $P_{G} \Phi_{0}-\Phi_{1}$ (curve starts above 0).

Thus, solution to Equation (22) reduces to the determination of zeros of a GCP (9) obtained as a result of calculation of the determinant. The GCP involves products $\Pi_{n=0}^{N_{p, q}} \mathcal{I}_{n}^{(p, q)}\left(\kappa_{n, G}^{(p, q)} \zeta(\gamma)\right)$ of the cylindrical functions entering (22).
$A$ is a block-diagonal matrix obtained explicitly in [29] which yields the possibility of recursive computation of its determinant and obtaining formulas similar to (15)

$$
\begin{equation*}
\mathrm{F}_{M}=\mathcal{I}^{(11)}\left(\kappa_{G}^{(1,1)} \zeta\right) \mathrm{F}_{M-1}^{1}-\mathcal{I}^{(12)}\left(\kappa_{G}^{(1,2)} \zeta\right) \mathrm{F}_{M-1}^{2} \tag{23}
\end{equation*}
$$

using low-order minors $\mathrm{F}_{M-1}^{1,2}$ and cofactors so that at the first (and each subsequent) recursive step the expression for $\mathrm{F}_{M}(\gamma)$ contains two terms. This enables one to apply the procedure outlined in the proof of Theorem 5.4: Assume that parameter $\kappa_{G}^{(1,1)}$ in (23) is such that an interval $u_{k^{\prime}}^{(11)}$ formed by two neighboring zeros of $\mathcal{I}^{(11)}\left(\kappa_{G}^{(1,1)} \zeta\right)$ with index $k^{\prime}$ contains one particular zero of $\mathcal{I}^{(12)}\left(\kappa_{G}^{(1,2)} \zeta\right)$. Such a choice of parameter is possible because each $\mathcal{I}^{(1 p)}\left(\kappa_{G}^{(1, p)} \zeta\right), p=1,2$, has infinitely many zeros forming almost periodical sequences and each zero (as well as the distance between any two neighboring zeros) is a monotonically decreasing function of $\kappa_{G}^{(1, p)}$.

Next, we use Lemma 5.1 to conclude that $\mathrm{F}_{M}$ in (23) has a zero on the interval $u_{k^{\prime}}^{(11)}$

### 6.2. Plane-parallel layered guiding structures

Determining resonant states and eigenfrequencies of plane-parallel layered dielectrics in free space, between parallel perfectly conducting planes, or in a waveguide of rectangular cross section is considered in terms of non-selfadjoint Sturm-Liouville problems (17) for the equation $\mathcal{L} \phi \equiv-\phi^{\prime \prime}+\widetilde{k}^{2} \phi=0$ on the line with piecewise constant coefficients; eigenfunctions $\phi(r)$ are sought as a linear combination of trigonometric functions
and the problem in question is reduced $[27,28]$ to DEs involving complex-valued TPs

$$
\begin{equation*}
\mathcal{G}_{2}(z)=\mathcal{G}_{2}^{(M)}(z)=\sum_{q=1}^{2} P_{q}(z) \mathcal{U}_{q}\left(\kappa_{q} z\right) \tag{24}
\end{equation*}
$$

here $z$ is a real or complex variable associated with one particular layer in an $M$-layer structure (e.g. $z=\kappa \sqrt{\epsilon_{s}-\gamma^{2}}$ with $s$ being the index of the layer with permittivity $\left.\epsilon_{s}, 1 \leq s \leq M\right)$ and $P_{1,2}(z)$ are complex-valued functions depending on all the problem parameters. In particular, for the single-layer structure (comprising one dielectric slab), we have [27]

$$
\begin{equation*}
\mathcal{G}_{2}(z)=\mathcal{G}_{2}^{(1)}(z)=P_{1}(z) \sin (t z)+\cos (t z), \quad P_{1}(z)=i Z(C, z)=\frac{i}{2}\left(\frac{z}{C}+\frac{C}{z}\right) \tag{25}
\end{equation*}
$$

where $t>0$ and $C>0$ are (real) parameters. In [27], it is proved that $\mathcal{G}_{2}^{(1)}(z)$ is an entire even function, the $\mathrm{DE} \mathcal{G}_{2}^{(1)}(z)=0$ has no real zeros and has infinitely many complex zeros located in pairs in the first and third quadrants in the complex $z$-plane. This result is extended to $M$-layer plane-parallel structures. Explicit (tedious) expressions for $P_{1,2}(z)$ in the cases of two and three layers may be found in [27].

## 7. Conclusion

We have developed the theory of TPs and proposed a generalization of the notion of TP that can be applied to the analysis of CPs.

The proposed method is based on the introduction and analysis of GTPs and GCPs. The technique can be applied in electromagnetics using the explicit forms of DEs expressed as weighted sums of products of trigonometric and cylindrical functions that describe eigenoscillations and normal waves in layered structures. The approach enables one to complete rigorous proofs of existence and determine domains of localization of the DE roots and validate iterative numerical solution techniques.

The obtained results complete mathematical theory of DEs for multi-layered waveguides possessing circular or plane-parallel symmetry and can be extended to more general structures as well as to determination of complex waves.

## Disclosure statement

No potential conflict of interest was reported by the author.

## References

[1] Milovanovic GV, Mitrinovic DS, Rassias ThM. Topics in polynomials: extremal problems, inequalities, zeros. Singapore: World Scientific; 1994.
[2] Littlewood JE. Some problems in real and complex analysis, heath mathematical monographs. Lexington (MA); 1968.
[3] Davis PJ. Interpolation and approximation. New York: Blaisdell; 1968.
[4] Dumitrescu DA. Positive trigonometric polynomials and signal processing applications. Springer; 2007.
[5] Chanane B. Computing the spectrum of non-selfadjoint Sturm Liouville problems with parameter-dependent boundary conditions. J Comp Appl Maths. 2007;206(1):229-237.
[6] Littlewood JE. On the real roots of real trigonometrical polynomials (II). J Lond Math Soc. 1964;39:511-552.
[7] Littlewood JE. The real zeros and value distributions of real trigonometrical polynomials. J Lond Math Soc. 1966;41:336-342.
[8] Erdelyi T. Polynomials with littlewood-type coefficient constraints. In: Chui CK, Schumaker LL, Stoeckler J, editors. Approximation theory X. Nashville, TN: Vanderbilt University Press; 2001. p. 1-40.
[9] Borwein P, Erdelyi T. Trigonometric polynomials with many real zeros and a littlewood-type problem. Proc Am Math Soc. 2000;129(3):725-730.
[10] Lanczos C. Trigonometric interpolation of empirical and analytical functions. J Math Phys. 1938;17: 123-199.
[11] Oliveira-Pinto F. Experiments with a class of generalized trigonometric polynomials. Calcolo. 1984;21(3): 253-268.
[12] Polya G, Szegö G. Aufgaben und Lehrsätze aus der Analysis. 3rd ed. Berlin: Springer; 1964.
[13] Borwein P, Erdelyi T, Ferguson R, et al. On the zeros of cosine polynomials: solution to a problem of littlewood. Ann Math. 2008;167:1109-1117.
[14] Runovsky VK. On real zeros of an even trigonometric polynomial. Mat Zametki. 1988;43(3):334-336.
[15] Watson GN. Theory of bessel functions. 2nd ed. New York: Cambridge University Press and Macmillan Co.; 1948.
[16] Jahnke E, Emde F. Tables of functions. 4th ed. New York: Dover Publications; 1945.
[17] Abramowitz M, Stegun IA., editors. Handbook of mathematical functions with formulas, graphs, and mathematical tables. New York: Dover; 1972.
[18] NIST Digital Library of Mathematical Functions. Release 1.0.20 of 2018-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds.
[19] Ek JM, Thielman HP, Huebschman EC. On the zeros of cross-product bessel functions. J Math Mech. 1966;16(5):447-452.
[20] Snyder AW, Love J. Optical waveguide theory. Springer; 1983.
[21] Shestopalov Y, Kuzmina E, Samokhin A. On a mathematical theory of open metal-dielectric waveguides. FERMAT. 2014;5.
[22] Shestopalov Y, Kuzmina E. Symmetric surface complex waves in Goubau line. Cogent Eng. 2018;5(1): 1-16.
[23] Shestopalov Y, Kuzmina E. Complex waves in multi-layered metal-dielectric waveguides. Proc. 2018 Int. Conf. Electromagnetics in Advanced Application (ICEAA), 20th Edition, Cartagena de Indias, Colombia: September 10-14, 2018. p. 122-125.
[24] Weinstein LA. Open resonators and open waveguides. Golem Press; 1969.
[25] Shestopalov Y. Complex waves in a dielectric waveguide. Wave Motion. 2018;57:16-19.
[26] Ismail MEH, Muldoon M. Monotonicity of the zeros of a cross-product of bessel functions. SIAM J Math Anal. 1978;9(4):759-767.
[27] Shestopalov Y. Resonant states in waveguide transmission problems. PIER B. 2015;64(1):119-143.
[28] Shestopalov Y. Singularities of the transmission coefficient and anomalous scattering by a dielectric slab. J Math Phys. 2018;59(3):033507.
[29] Karchevskii EM, Beilina L, Spiridonov AO, et al. Reconstruction of dielectric constants of multi-layered optical fibers using propagation constants measurements, arXiv: $1512.06764 v 1$ [math.NA], Dec. 2015.

## Appendix

The Fejer kernel

$$
\begin{equation*}
F_{n}(x)=\frac{1}{n} \sum_{k=0}^{n-1} D_{k}(x)=\frac{1}{n}\left(\frac{\sin \frac{n x}{2}}{\sin \frac{x}{2}}\right)^{2}, \tag{A1}
\end{equation*}
$$

the Dirichlet kernel

$$
\begin{equation*}
D_{n}(x)=\sum_{k=-n}^{n} \mathrm{e}^{i k x}=1+2 \sum_{k=1}^{n} \cos (k x)=\frac{\sin \left[x\left(n+\frac{1}{2}\right)\right]}{\sin \frac{x}{2}} \tag{A2}
\end{equation*}
$$

and other trigonometric sums like

$$
\begin{equation*}
E_{N}(x)=\sum_{k=0}^{N-1} \mathrm{e}^{i k x}=C_{N}(x)+i S_{N}(x)=\mathrm{e}^{i(1 / 2)(N-1) x} \frac{\sin \frac{N x}{2}}{\sin \frac{x}{2}}, \tag{A3}
\end{equation*}
$$

a cosine TP with all $P_{1}^{(k)}(x)=0, P_{2}^{(k)}(x)=1$, and the parameter vector $\bar{\kappa}_{2, N-1}=(1, \ldots, N-1)$

$$
\begin{equation*}
C_{N}(x)=\sum_{k=0}^{N-1} \cos (k x)=\frac{\sin \frac{N x}{2}}{\sin \frac{x}{2}} \cos \frac{(N-1) x}{2} \tag{A4}
\end{equation*}
$$

a sine TP with all $P_{1}^{(k)}(x)=1, P_{2}^{(k)}(x)=0$, and the parameter vector $\bar{\kappa}_{1, N-1}=(1, \ldots, N-1)$

$$
\begin{equation*}
S_{N}(x)=\sum_{k=1}^{N-1} \sin (k x)=\frac{\sin \frac{N x}{2}}{\sin \frac{x}{2}} \sin \frac{(N-1) x}{2} \tag{A5}
\end{equation*}
$$

a TP with all $P_{1}^{(k)}(x)=P_{2}^{(k)}(x)=1$

$$
\begin{align*}
T_{N}(x) & =\sum_{k=0}^{N-1}(\sin k x+\cos k x)=\sum_{q=1}^{2} \sum_{k=0}^{N-1} \mathcal{U}_{q}(k x) \\
& =S_{N}(x)+C_{N}(x)=\sqrt{2} \sin \left[\frac{(N-1) x}{2}+\frac{\pi}{4}\right] \frac{\sin \frac{N x}{2}}{\sin \frac{x}{2}} \tag{A6}
\end{align*}
$$

'shifted' TPs (A4), (A5) with all $P_{1}^{(k)}(x)=-\sin a$ and $P_{2}^{(k)}(x)=\cos a$

$$
\begin{align*}
\tilde{C}_{N}(x ; a) & =\sum_{k=0}^{N-1} \cos (a+k x)=\sum_{q=1}^{2} \sum_{k=0}^{N-1}(-1)^{q-1} \mathcal{U}_{3-q}(a) \mathcal{U}_{3-q}(k x) \\
& =C_{N}(x) \cos a-S_{N}(x) \sin a=\cos \left[a+\frac{(N-1) x}{2}\right] \frac{\sin \frac{N x}{2}}{\sin \frac{x}{2}} \tag{A7}
\end{align*}
$$

and all $P_{1}^{(k)}(x)=\cos a$ and $P_{2}^{(k)}(x)=\sin a$

$$
\begin{align*}
\tilde{S}_{N}(x ; a) & =\sum_{k=0}^{N-1} \sin (a+k x)=\sum_{q=1}^{2} \sum_{k=0}^{N-1} \mathcal{U}_{q}(a) \mathcal{U}_{3-q}(k x) \\
& =C_{N}(x) \sin a+S_{N}(x) \cos a=\sin \left[a+\frac{(N-1) x}{2}\right] \frac{\sin \frac{N x}{2}}{\sin \frac{x}{2}} \tag{A8}
\end{align*}
$$

are examples of TPs which can be calculated in the closed form involving only products of trigonometric functions. This enables one to obtain all zeros of TPs (A1)-(A6)

$$
\begin{aligned}
x_{C, N}^{(1, m)} & =(2 m-1) \frac{\pi}{N-1}, m=1,2, \ldots ; x_{C, N}^{(2, l)}=2 l \frac{\pi}{N}, x_{S, N}^{(2, l)}=x_{C, N}^{(2, l)}, l=1,2, \ldots \\
x_{S, N}^{(1, n)} & =2 n \frac{\pi}{N-1}, n=1,2, \ldots ; x_{T, N}^{(1, s)}=\frac{2 \pi}{N-1}\left(s-\frac{1}{4}\right), s=1,2, \ldots \\
x_{T, N}^{(2, p)} & =x_{C, N}^{(2, p)}, p=1,2, \ldots
\end{aligned}
$$

(limiting ourselves only to positive zeros, $n=1,2, \ldots$ ).

Zeros

$$
\begin{align*}
x_{c}^{\left(1, m^{\prime}\right)}(k) & =\frac{\left(2 m^{\prime}-1\right) \pi}{2 k}, k=1, \ldots N-1, \quad m^{\prime}=1,2, \ldots  \tag{A9}\\
x_{s}^{\left(2, l^{\prime}\right)}(k) & =\frac{l^{\prime} \pi}{k}, \quad k=1,2, \ldots N-1, \quad l^{\prime}=1,2, \ldots \tag{A10}
\end{align*}
$$

of all intermediate terms $\cos k x$ and $\sin k x$ in TPs (A1)-(A6) alternate with $x_{C, N}^{(1, m)}, x_{C, N}^{(2, l)}, x_{S, N}^{(1, n)}, x_{T, N}^{(1, s)}$, and $x_{T, N}^{(2, p)}$. In fact, excluding, e.g. merging zeros of (A6) $x_{T, N}^{(1, s)}=x_{T, N}^{(2, p)}$ when

$$
\begin{equation*}
N^{\prime}(4 s-1)=l\left(4 N^{\prime}-1\right), \quad N=4 N^{\prime} \tag{A11}
\end{equation*}
$$

which is valid particularly for $p=s=N^{\prime}, N^{\prime}=1,2, \ldots$, we have that zeros of (A6) are all different if condition (A11) does not hold. Another valuable observation is that infinitely many zeros of $S_{N}, S_{N-1}, C_{N}, C_{N-1}$, and $T_{N}, T_{N-1}(N=$ $2,3, \ldots$ ) alternate.

Also, the distances $d_{T, N}^{(1)}=x_{T, N}^{(1, s+1)}-x_{T, N}^{(1, s)}=(2 \pi /(N-1))$ and $d_{T, N}^{(2)}=x_{T, N}^{(2, p+1)}-x_{T, N}^{(1, p)}=(2 \pi / N)$ between neigboring zeros of $C_{N}(x), S_{N}(x)$, and $T_{N}(x)$ are less than the distance $d_{c s, k}^{(12)}=x_{s, k}^{\left(2, m^{\prime}\right)}-x_{c, k}^{\left(1, m^{\prime}\right)}=(\pi / 2 k)$ between neigboring zeros of $\cos k x$ and $\sin k x(k=1, \ldots N-1)$ for $N>4 k+1$.


[^0]:    CONTACT Y. Shestopalov yuyshv@hig.se, shestop@hotmail.com

