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Applying Toulmin's Argumentation Framework to Explanations in a Reform Oriented Mathematics Class

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APPLYING TOULMIN'S ARGUMENTATION FRAMEWORK TO EXPLANATIONS IN A REFORM-ORIENTED MATHEMATICS CLASS

by

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ABSTRACT

APPLYING TOULMIN'S ARGUMENTATION FRAMEWORK
TO EXPLANATIONS IN A REFORM-ORIENTED
MATHEMATICS CLASS

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This study looks at conceptual explanations given in a reform-oriented mathematics class for preservice secondary mathematics teachers and extends Toulmin's argumentation framework to account for some of the complexities of the explanations given by these students. This study explains the complexities that arose in applying Toulmin's framework to explanations and extends the framework by accounting for the features of conceptual explanations. The complexities of these explanations are that they are made up of multiple arguments that build on each other to reach a final conclusion and that they are also dependant upon the social aspects of the class in which they are situated. Recognizing that some statements have dual

purposes in the explanation and that there are varying levels of justification used in the explanations helped to account for the first complexity of explanations. The classification of class conventions helps to account for the social influences on explanations. This study differs from other studies that use Toulmin's framework to analyze formal proofs or to identify taken-as-shared understanding in a classroom. This study instead focuses on using the framework to analyze the components of explanations and to provide insight into the structure of conceptually oriented explanations. This study contributes to the existing research by extending Toulmin's argumentation framework to account for how social influences help determine the appropriate components of an explanation.

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Chapter 1—Introduction

Recent changes in mathematics education are altering the emphasis and goals of mathematics curricula. These reform curricula are becoming increasingly well known. Their emphasis on explanation, justification, and communication of mathematical ideas differs from the emphasis on developing skills in quickly carrying out procedures in traditional mathematics curriculum. Instead of focusing on mastering procedures and getting a correct numerical answer, the goal of reform curricula is to help students better understand the mathematics they are learning. As a part of this difference, students are often required to give more that just a correct numerical response. They are asked to explain how they got their answer and why the answer is correct.

Explanations are important

There are many reasons why explanations are an important part of reform-oriented mathematics classes. The first reason is that when students give explanations, their understanding of mathematical concepts increases (Hiebert, et al., 1997). By explaining solution methods, students are communicating their mathematical ideas to others. When students explain, their ideas need to be clear enough that others will understand. In order to construct an explanation, students may need to think again about what they have done and evaluate it for correctness and soundness of logic. This clarifying process helps students' understanding to grow and solidify. Having students explain themselves is beneficial to the students' understanding.

Explanations are also an important part of reform-oriented mathematics classes because they give more evidence than a numerical answer of what a student understands. Seeing numerical calculations worked out by a student only shows that student's procedural fluency and gives no indication of the student's conceptual understanding (National Research Council, 2001). Having students give explanations was first motivated by studies that showed that students who could give correct numerical answers did not necessarily understand the mathematical concepts and principles. They solved the problems using other incorrect ideas they had come up with. For example, Erlwanger (1973) reported that Benny could score very well on his assessments, but had incorrect ideas of fractions and fraction operations. Similarly, Sowder (1988) wrote about students who found correct answers to word problems through various methods that were based on incorrect conceptions of how to use given information to determine a proper operation. These studies and others like them have demonstrated that a correct numerical answer is not a good measure of understanding. An explanation of their solution sheds more light on how they arrive at their answer and what they understand.

Another reason that explanations are important is that teachers often use them as a form of assessment. Many teachers believe that explanations reveal their students' understanding of a topic (Miller, 1992) and therefore use them to assess how much their students understand. However, explanations will only be an effective assessment tool when students know how to explain the things they have learned. Students need to know how to give explanations so their teachers' assessments of their understanding will be accurate. Otherwise, students will not be allowed to progress in mathematics.

A fourth reason why explanations are important is that through them students can participate legitimately in mathematical activity. Communicating what one understands through explanation and justification is an essential part of participating in mathematics (Hiebert, et al, 1997; NCTM, 2000). The ability to express one's mathematical understanding is often an important part of becoming a member of a mathematical community. When students are asked to give explanations as part of the mathematics curriculum, they are being prepared to become a fully participating member of a mathematics community.

Explanation needs to be taught

These reasons show why explanations are an important part of reform mathematics classes. However, students may not realize the importance of mathematically justifying themselves if it is not pointed out to them that they need to do so (Smith, 1996). Even if they do realize the importance of justification, research shows that many students are not able to give good mathematical explanations on their own. Students' ability to clearly explain thinking and justify solutions is something that has to be developed (Spillane & Zeuli, 1999). Due to the conventions that are specific to mathematics, good mathematical explanations do not necessarily come inherently to those who can express themselves well in other subjects or understand mathematics well. Teachers cannot assume that all students will naturally pick up the skills they need to master in order to give quality explanations because these language skills do not always develop naturally (Morgan, 2001). As mathematics teachers "model important mathematical actions," they must also create a discourse of justification and explanation where students learn to reason with the ideas being taught (Smith, 1996, p. 397). Since some students will not be able to give good conceptual explanations on their own, they need to be explicitly taught how to give good explanations.

Students need to learn how to give good explanations and there are many well-known mathematics education researchers who are calling on teachers to instruct their students in explanations. For example, Wood (1999, 2001) claims that it is the teacher's responsibility to make it clear to the students what is expected in an explanation. The teacher's expectation for explanations to be thorough and clear has a direct effect on how detailed the students' explanations are. The National Council of Teachers of Mathematics has emphasized that it is the mathematics teacher's responsibility to help students learn to communicate mathematically (NCTM, 2000).

It is reasonable that mathematics teachers should be the ones that provide the instruction students need to give good explanations. Mathematics teachers are more qualified to give this instruction than any of the students' other teachers because they have the mathematical knowledge that is necessary to teach explanations. Mathematics teachers also know more than other teachers about the mathematical discourse practices necessary to know what should be in an explanation and to teach students to give mathematical explanations (Morgan, 2001). Mathematics teachers are already giving mathematics instruction to their students. Therefore, it would be the most natural for these teachers to give instruction on how to explain the mathematical concepts the students are learning. Since mathematics teachers are the most qualified to provide students with the instruction they need to give good conceptual explanations, mathematics teachers have a responsibility to provide it.

In order for teachers to instruct their students on how to give a good explanation, they first need to know themselves what constitutes a good explanation. This includes what structure it should have and what components should be included. Research has found that teachers find it difficult to explicitly describe what they expect students to put into a good explanation. They can usually use mathematical vocabulary, definitions, and notation easily, but find it more difficult to clearly lay out how these things should be put together in an acceptable explanation (Morgan, 2001). If teachers are going to be able to instruct their students in explanations, they first need to know more themselves.

Teachers also need to know how to evaluate the explanations that are given. It is reasonable to assume that if teachers are not clear what should go in an explanation, it will be difficult for them to explicitly determine which explanations adequately explain reasoning. It can be tricky to point out exactly where flaws are located and the reason for them, especially when each explanation has a different feel to it depending on students' understanding and personality. It would be helpful to have a model for explanations that could be applied generally, without having to take the specific student or content of the class into account.

Research on the details of explanations

An analysis of the literature on explanations only gives broad descriptions of the components of good explanations. Researchers offer many accounts of teachers leading class discussions where students give an answer or partial explanation and then the teacher asks the students to explain how they got that answer, why their answer is correct, or why the things they said make sense (Spillane & Zeuli, 1999; Wood, 1999; Wood, Cobb, &Yackel, 1991). The teachers' comments about what has been left out implicitly tells the students that these are important parts of an explanation. However, there is little detailed analysis of the sequence of statements students make and how these statements function in the explanation. No one has pointed out specifically how these factor into composing a good conceptual explanation. Knowing how to analyze explanations with this amount of detail would allow teachers to know more clearly what should be in an explanation. Teachers could pass along these ideas about the structure of explanations onto their students. This would be effective instruction on explanations.

Currently, the descriptions of instruction on explanations are also very broad and general. For example, Wood (1999) describes a teacher asking her students to explain their reasoning when their explanations are challenged, instead of just re-explaining the steps they took. By questioning the students' explanations, the teacher is letting the students know that something important was left out of their explanation and needs to be included; however, no explicit details of how to make sure they know when to include details are given. Spillane and Zeuli (1999) give an example of a teacher who has her students explain their problem solutions, even though her instruction focuses only on the mathematics. She has to "continuously [press]" students to give thorough and well-justified explanations by asking how they knew what they had done was correct and how they knew the things they said (p. 10). Except for this encouragement to give thorough explanations, there did not seem to be any other instruction on explanations. This general instruction is insufficient to empower students to give good explanations independent of teachers' constant prodding. Explicit instruction from their mathematics teachers on the components of good explanations can help students know how to give well-justified explanations.

In the field of mathematics education, there does not appear to be a widely accepted and known structure for mathematical explanations. Beyond general questions that teachers use to urge their students to give more information, there seems to be a lack of vision about what teachers can explicitly teach their students about the structure and components of an explanation so that students can independently create satisfactory explanations. Therefore it would be worthwhile to find out if there is a way to identify the structure of a conceptually oriented explanation that would be accepted in a reform mathematics class. Such a structure would be helpful to teachers and students because it would enable both groups to have a better understanding of what should go into a conceptual explanation. This would help teachers give explicit instruction on explanations and help students know what is expected of them so they can give explanations that accurately reflect what they understand.

Conclusion

Student explanations are a fundamental part of reform mathematics classes. Instruction in these classes is geared toward having students give explanations and explanations are prevalently used as part of instruction, homework, and assessment. However, little explicit instruction is given on explanations. This may be the case because teachers are not exactly sure what should be in them. Knowing more about the structure of explanations would be beneficial to teachers and researchers. The purpose of this study is to investigate the structure of mathematical explanations in a reform-oriented classroom and to identify common components and structural features. Such a study would not only provide teachers with information they could teach their students, but would also make a contribution to the research literature on mathematical explanations and argumentation.

Chapter 2—Conceptual Framework

In order to study the structure of conceptual explanations, I looked outside of the field of mathematics education to a general theory of argumentation developed by Toulmin (1969). It has previously been applied to explanations from reform-oriented classrooms by mathematics education researchers because it has potential to uncover important characteristics of these explanations. It is a general structure of argumentation and recognizes specific components of arguments that would also be expected to be part of the explanations. In this chapter, I will first describe Toulmin's argumentation framework. Second, I will discuss the contributions and shortcomings of others' attempts to apply Toulmin's framework to mathematics. Lastly, I will explain some of tools the students in the class where I did my research may have used to explain their reasoning.

Toulmin's argument structure

There are four main components to Toulmin's argumentation structure. Each type of statement can be identified by its function—the role it plays in the argument—and its form. These are laid out in Table 1. The first is the *claim*. This is the statement for which the argument provides justification. Establishing the claim is the reason the argument is being given. It is something that is not considered obvious; hence, an argument is given to prove its truthfulness. The statement that provides the foundation for the claim is

	Toulmin-Form	Toulmin-Function
Data	Fact Statement	Fact that justifies the claim
Warrants	General Principle	Connect data statement to claim, establish the step as legitimate
Backing	Fact Statement	Justify warrant, establishes authority of the warrant; does not need to be made explicit
Claim	Statement to be supported	Conclusion that is being established

Table 1 The Form and Function of Each Part of Toulmin's Argumentation Structure

called *data*. This statement of fact is a specific piece of information that is explicitly given in the argument. It is necessary to have data in the argument because without it, the claim is left without any support, and there is no argument at all (Toulmin, 1969). Toulmin portrays data as something that can be accepted without question. However, he acknowledges that it is possible that there could be a problem if the factuality of the data is doubted. Toulmin addresses this problem quickly with one sentence. He states that a side argument, a lemma, could be produced to establish its truthfulness, after which one could progress with the current argument.

Even if the factuality of the data statement is not questioned, its applicability to the claim could be. One could wonder how it supports the claim, why it gives the claim any validity. The statement that would be given to establish applicability is called a *warrant*. The warrant is a general statement or principle that shows how the datum is related to the claim. It must be general, for if another specific piece of information was given, it would just be more data and would not establish general applicability of the datum to the claim. The warrants must "[certify] the soundness of *all* arguments of the appropriate type," (Toulmin, 1969, p. 100). If the warrant is challenged, then *backing* can be produced to establish the warrant. The backing is another statement of fact that justifies the warrant and establishes its authority. It does not need to be made explicit, but could be if the warrant was challenged. This structure is laid out as Toulmin shows in his paper in Figure 1. These statements all fill unique roles. They make it possible to distinguish different types of justification, such as general principles verses specific information, as opposed to more justification of the same type.

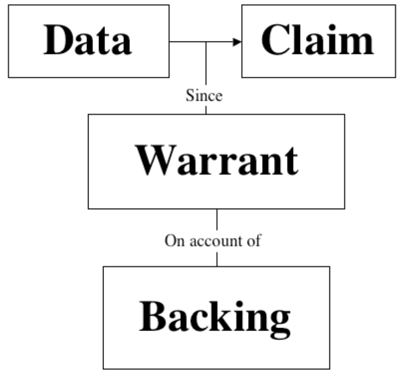

Figure 1 Toulmin's Argumentation Structure in general

To better illustrate Toulmin's components of argumentation, consider the following argument that Toulmin uses to demonstrate his framework. The purpose of the argument is to establish the claim that Harry is a British subject (see Figure 2). The datum that establishes this claim is the fact that Harry was born in Bermuda. However, one could ask what Harry being born in Bermuda has to do with Harry being a British subject. At this point, the warrant could be given: that a man born in Bermuda will generally be a British subject. This is a general principle that establishes the applicability of the datum that Harry was born in Bermuda to the claim that Harry is a British subject. Even with the applicability established, the truthfulness of the warrant could still be questioned. If that was the case, the actual statutes of law and other legal provisions that verify that those born in Bermuda will generally be British subjects could be provided as backing to the warrant.

Toulmin's examples of arguments do not shed light on how his argument structure would apply to mathematical explanations. All the examples that Toulmin gives are just as simple as this one— all the components are given in only a few sentences. There is only one sentence for data, one sentence for a claim, and one sentence for a

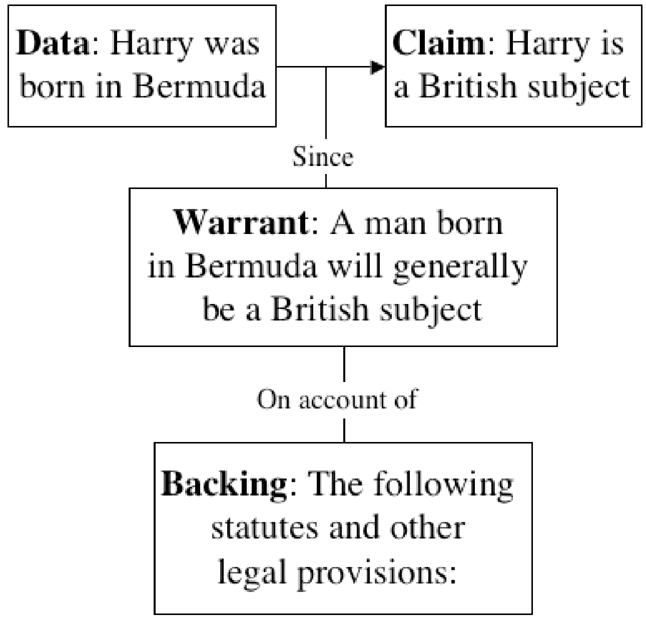

Figure 2 Toulmin's Structure laid out with an example

warrant. The claims deal with straightforward situations like Harry’s hair not being black, based on the fact that someone has seen it and it is red, or that Peterson is not a Roman Catholic, based on the fact that he is a Swede. His theories about argumentation hold very well with these cases. However, mathematical explanations are not always this simple and straightforward. Explanations often consist of multiple arguments that build on each other to reach a final conclusion. Toulmin acknowledges the field dependency of what are considered acceptable warrants and backings, meaning that warrants and backings will be different depending on the situation and the background of the argument according to the specialty of each field (Toulmin, 1969). He does not, however, discuss the field dependency of knowing what is acceptable as data to back up claims, even though the field where an argument is being given certainly would affect the data as well.

These differences between mathematical explanations and Toulmin's arguments show that the explanations do not easily fit in his argument structure.

Applications of Toulmin's framework to mathematics

Many researchers have used Toulmin's framework as the basis for their research. These researchers have applied Toulmin's framework in different ways and their applications have strengths and weaknesses. However, their work fails to apply his framework in a way that informs the field of mathematics education about the structure of arguments. To show this, I first summarize Aberdein's work, which has focused on applying Toulmin's framework to formal mathematical proofs. I then examine the work of three different research groups who have applied Toulmin's framework to the mathematical justifications and explanations given in reform-oriented classrooms.

Aberdein (2005) attempted to show that Toulmin's scheme of arguments could be applied to formal mathematical proofs. First, he recognized that in proofs there are often multiple arguments that build on each other to reach the final result, which Toulmin did not address. Claims from initial arguments are used as data in further arguments. In the proofs that Aberdein lays out using Toulmin's structure, the arguments are linked together through these statements that are first justified as claims and then used as data to justify other statements. Second, in his analysis of proofs, determining which statements are data and determining what to put as warrants and backing in the proofs seems to be unproblematic. Some of the warrants and backings are missing in the original proof he examines in his paper, so Aberdein adds the necessary warrants and backings to complete the argument. The warrants and backing he includes are widely recognized as accepted facts and mathematical conventions. Aberdein does not address if the choice of warrants and backing reflect the intentions of the original author. It seems that as long as acceptable warrants and backing can be determined by finding a mathematical principle that links data and claim together, they are acceptable.

The differences between the social contexts in which proofs and conceptually oriented explanations are situated result in only one of Aberdein's contributions being useful to the study of conceptually oriented explanations. The first point made above, that mathematical arguments build on each other to reach the final conclusion, still holds true in explanations. There are still multiple steps that have to be completed to reach the final result. However, the second point made above about warrants and backing is problematic in explanations. The differences between the purpose and social context of proofs and explanations makes determining the warrants in an explanation much more problematic than in a proof for two reasons. First, even if a correct, conceptual mathematical principle that links the datum to the claim can be produced by the reader, there is no way of knowing if that is the warrant actually intended by the author. It is possible that the author made the conclusion based on a much less conceptual principle, or even an incorrect principle that still resulted in the correct claim. Second, which principles can be used as warrants depends on the classroom it is given in. Classes have their own social features that determine what is part of an acceptable explanation. Thus there is not a global set of acceptable warrants that can be used in any explanation. This is a different context than the published proofs that Aberdein is analyzing, where the warrants could be inferred to be the logical mathematical principle that would link the datum to the claim. Aberdein does not account for these social features that affect how Toulmin's framework can be applied to mathematical arguments.

Unlike Aberdein, some mathematics educators have tried to extend Toulmin's framework while allowing for the social context of arguments. For example, Krummheuer (1995) looked at argumentation in the interaction of students who were negotiating the meaning of number facts and explaining their answers to problems. However, he dismissed the idea that statements in these interchanges could be classified as elements of Toulmin's argumentation structure by recognizable features of their form. He claimed that the functionality of a statement depends on what role it plays in the interchange, how it fits in with other statements. As he analyzed a short interchange between two students and a researcher, he classified certain data-like specific statements as warrants because the students used the statements to establish the applicability of the data to the claim. However, they were not general principles and therefore did not fit Toulmin's criteria for warrants. When Toulmin's framework is skewed this way, the general principles that connect the fact statements that justify the argument are ignored. Fact statements alone do not build arguments unless there are general principles linking them together. If more facts are needed to justify an argument after it has been given, then the original argument needed more data to begin with. The warrant was not the problem. Toulmin makes the distinction between these different parts of an argument. By dismissing part of his criteria, Krummheuer's application of his framework does not fully capture the reasoning behind an explanation.

A second group of researchers have worked together using Toulmin's framework to examine the effect of the social context on how the requirements for a well-justified explanation change over time. Yackel (2002) pointed out that what is acceptable is dependent on what constitutes public knowledge in that class. Cobb, Wood, Yackel, and

McNeal (1992) found that what are acceptable as data, warrants, and backing evolve in a classroom as concepts become taken-as-shared, meaning that they are used as if they are commonly understood and accepted by everyone in the class. Yackel (2001) and Stephan and Rasmussen (2002) found in their research that ideas have become taken-as-shared when warrants and/or backing are no longer given in students' explanations and it appears that the ideas can stand alone with no further justification. For example, if a statement that previously was a claim that needed to be justified is used as a data statement with nothing to support it, then they would consider the ideas in that statement as having become taken-as-shared. Yackel (2001) said that "what constitutes data, warrant, and backing is not predetermined, but is negotiated by the participants as they interact" (p. 17). This negotiation leads to acceptable data, warrants, and backing evolving over time. Although this research provides valuable insight into the process by which what needs to be explained changes over time, it does not provide specific guidelines for what constitutes a good explanation at a particular moment in time. Thus, it does not adequately describe the structure of good explanations, the focus of this research.

Maher, Powell, Weber, and Lee (2006) are a third group of mathematics education researchers who have applied Toulmin's framework to mathematical explanations. Their work raises two important issues related to the structure of explanations. First, it is unclear from their work whether warrants need to be explicitly included in acceptable explanations. While evaluating students' arguments, they found that acceptable warrants were rarely given. However, the researchers did not take a stance on whether the explanations were adequate without the warrants. Instead, to determine the adequacy of the arguments, they attempted to infer what general principles the students were relying on to link the data to their claims. Second, their findings raise questions about the acceptability of data statements. In one of the examples they gave, students argued about the acceptability of the data that one student gave in her argument. The argument was eventually rejected because the other students would not accept the statement as valid data. Although their paper raises interesting issues about warrants and data that need to be accounted for in an argumentation theory, the authors do not propose solutions to these problems. More specifically, they do not indicate when explicit warrants are required or which statements can be taken as data and why.

Even though many researchers have used Toulmin's framework, none have described what the structure of a good explanation is. Toulmin's framework has been applied to mathematical arguments in many different ways, but none that look at the components of a good conceptual explanation. If Toulmin's structure of arguments could be extended to describe some of the key features of an explanation, then the extended framework could help in distinguishing good explanations. Therefore, I will focus my research on the question: How can Toulmin's structure of arguments be extended to describe explanations in a reform-oriented mathematics class?

Class conventions as tools for constructing explanations

In the class studied, the students were given tools to use in justifying their explanations. These tools consisted of the concepts, definitions and terminology applicable to the concepts, and acceptable practices that could be used to reason with the concepts. I will refer to this collection of common concepts, definitions, terminology, and practices as *class conventions*. The students were provided with these tools to help them to reason about quantities and operations and to solve problems. Because these conventions undoubtedly influenced students' explanations, an understanding of the tools and meanings they were using is important in analyzing the data. In this section, I discuss some of the key conventions students could use to reason, explain, and justify.

The students in this class were studying fractions by analyzing fraction quantities and operations. The class conventions dealt with a conceptual understanding of fractions that is different than how fractions are often taught. Whole number language that is often used to describe fractions as one number out of another number, for instance, where 3/4 is seen as three out of four objects, was not accepted in this class. Instead a part-whole interpretation of fractions was used. In a part-whole interpretation, the notation for a fraction corresponds to a whole that has been cut up into equal-sized pieces (Lamon, 1999; Tzur, 2000). The number in the denominator shows how many pieces there are. If the pieces were not of equal size, then knowing how many there are would not give any information about how big they are. Though fractions are often thought of as the number of pieces, like the three out of four objects mentioned above (Mack, 1990), it was negotiated in this class that a fraction described the size of the pieces. By establishing a fraction as a certain amount of the whole that was made up of a number of pieces of a particular size, the image of how a fraction relates to a whole becomes clearer.

These conventions were established at the beginning of the unit on fractions so students could build meaning with them as the course progressed. Only a few of the conventions that were negotiated in the class will be relevant to the results reported in this paper, and only those will be described in this section. The relevant conventions are those that deal with unit fractions, non-unit fractions, and simplifying fractions.

Unit fractions. Partitioning and iterating are the two different ways students were taught to define a unit fraction. Using the partitioning definition, a unit fraction comes from dividing up the unit whole into a certain number of smaller equal sized pieces and taking one of those pieces. For example, if a whole is divided into five equal parts, then each resulting smaller pieces is one-fifth of the whole (Lamon, 1999). In contrast, the iterating definition of a unit fraction starts with a smaller piece that represents the unit fraction. The piece has to be copied a certain number of times to make the larger whole. The number of times it needs to be copied is what fraction of the whole it is. For example, a piece of size one-sixth will need to be copied six times to make the whole (Siebert & Gaskin, 2006; Tzur, 2000). Partitioning and iterating were the only two meanings of unit fractions that were accepted in this class.

Non-unit fractions. The acceptable way to define non-unit fractions in this class was based on the definition of unit fractions described above. Once iterating or partitioning has been used to define what the unit fraction is, combining more than one copy of the unit fraction can create a non-unit fraction. For example, once partitioning or iterating has been used to define one-sixth, five-sixths is defined by putting five of the one-sixth pieces together (Behr, Harel, Post, & Lesh, 1992; Siebert & Gaskin, 2006). An acceptable way of explaining what a non-unit fraction, for example 3/4, means from a partitioning perspective is to take a whole, cut it into four equal pieces. It follows that each piece is 1/4. If three of these 1/4 pieces are put together, then the result is 3/4. From an iterating point of view, if a piece is copied four times to make a whole, then that piece is 1/4. Again, three 1/4 pieces together make 3/4.

Simplifying fractions. The ideas behind the accepted way to simplify fractions in this class are based on drawings of a whole being cut into equal sized pieces, with each piece representing the size of the unit fraction. Then the number of unit-fraction pieces in the fraction is shaded in. In order to simplify the fraction, the total number of pieces in the whole are grouped together into larger, but still equal sized pieces. If the number of shaded pieces can also be grouped into these larger pieces with none left over and all groups completely shaded, then the fraction can be simplified. For example, a fraction of 8/10 can be simplified to 4/5. Figure 3 shows a picture of 8/10. There are ten total pieces and these can be put into groups of two to form five groups. Then the eight shaded pieces can also be put into groups of two to form four shaded groups. The darkened lines in Figure 5 show the pieces in their new groups. Since the ten total pieces and the eight shaded pieces can be grouped into the same size new groups, 8/10 can be simplified to 4/5. In Figure 4, 8/10 can be seen as the eight 1/10 pieces colored in. Four-fifths can be seen as four of the tall 1/5 pieces colored in.

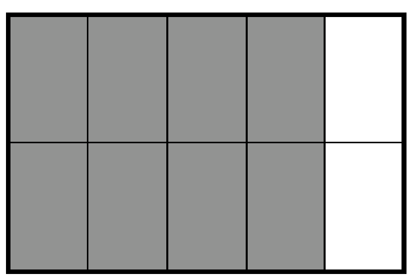

Figure 3 A drawing of 8/10

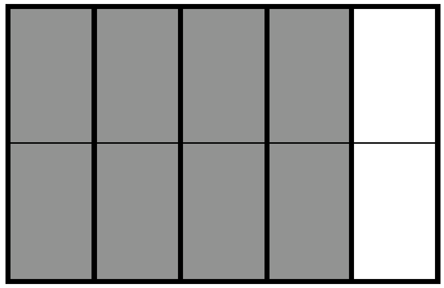

Figure 4 A drawing showing how 4/5 can be seen in 8/10

With the background presented in this chapter, I will give the details of the study used to answer my research question in the next chapter. Studying explanations given in a reform-oriented mathematics class gave the opportunity to analyze how Toulmin's framework could be extended to account for the characteristics of these explanations.

Chapter 3—Research Methods

In this chapter, I describe the qualitative study I completed to answer my research question. After collecting mathematical explanations from pre-service secondary mathematics teachers, I looked for how Toulmin's structure of arguments applied to the explanations the participants gave and determined how the framework needed to be extended to better fit the structure of these mathematical explanations. The participants in my study and the setting where it took place are described first, along with the reasons for my choices. I then explain my data collection methods and the data I collected. Finally, I describe how I analyzed my data and how this analysis answered my research questions.

Participants

I worked with Brigham Young University students who were just beginning the Mathematics Education major and were enrolled in MthEd 117, Critical Review of School Mathematics, during Fall Semester 2006. I had six participants, which provided enough of a variety of explanations to analyze and compare. To select the participants, I explained the study to the whole class of 25 students and asked for interested volunteers. From those volunteers I selected six participants with varied mathematical and educational backgrounds based on an information sheet they filled out for their instructor. This was based on their prior experiences with mathematics, their description of mathematics and mathematical activity, and what they felt was the most important thing they still needed to learn before starting to teach. All the participants shared that they enjoyed doing mathematics. Their prior experience ranged from succeeding in math competitions in high school to struggling with upper level math classes. While some felt that mathematics was logical and challenging, others saw mathematics as using numbers and symbols to solve problems based on data, while others thought that creativity in mathematics is just as important as precision. Before starting to teach, some of the participants felt they needed to know how other people learn. Other participants wanted experience and practice teaching those who did not already know the material and wanted to know different ways to explain what they already knew. In contrast, two of the participants felt they needed a deeper understanding of mathematics they were already familiar with. Only one of the two males in the class volunteered, resulting in five female participants and one male participant.

Involving MthEd 117 students was advantageous to my study. All the students in this class had successfully completed at least two calculus classes before enrolling in this course. Therefore, it was reasonable to assume that they all had a strong mathematical background and a positive attitude toward doing mathematics. In this class the students were required to learn to give explanations that were acceptable in a reform-oriented mathematics class. They learned how to give written and verbal explanations that explained their reasoning from explicit instruction from their teacher. Therefore the students in this class had opportunities to learn to write good conceptual explanations and had opportunities to improve. Additionally, it was ideal to work with college students because they were generally more expressive and communicative as to what they are thinking and learning. It is likely that they put more thought into their explanations than younger students might have, resulting in explanations that are more clear and organized.

Setting

My study took place within the context of a MthEd 117 Critical Review of School Mathematics class. This was a mathematical content course that presents a conceptual approach to the mathematics future secondary teachers will likely teach in the schools. This class focused on increasing students' understanding of fractions, integers, and algebra. All in-class activities and homework were inquiry-based. Students spent a lot of their time in class working on mathematics problems and presenting their ideas and findings to their peers and the whole class.

In this course, the students were expected to give detailed mathematical explanations both verbally and in writing. In these explanations procedural answers were not sufficient. The meaning of all quantities had to be explained. Students were expected to explain and justify all the processes they used in their explanations. If they could not explain and justify their procedures, they were not allowed to use them.

This course was a good place to look at mathematical explanations. During this course, the students had frequent instruction on how to give good conceptual explanations and received feedback on their explanations from the professor, both in class and on their written assignments. They also were developing an increased understanding of the subject matter. Due to these experiences, the participants quickly improved in their written and verbal explanations. This gave them the opportunity to be able to give well-developed explanations. It was advantageous to collect well-developed explanations as data because this would allow for a thorough analysis of the structures of explanations and the components of argumentation in them. Thus it was beneficial to look at the explanations of students taking this course.

The first unit in the course, which focused on fractions, was an ideal time to examine mathematical explanations. The students were familiar with the subject matter of fractions from previous math experiences. They could already correctly perform fraction operations and work successfully with fraction quantities. The students' familiarity with fractions assured them they could get the right answer and this helped them be comfortable expressing their thinking in the explanations they gave because they did not have to worry about being wrong when they explained themselves to their peers. During this unit, the class worked on developing conceptual meanings and images for fractions and fraction operations. As they did this, there were many opportunities for students to explain these concepts in depth to the teacher and to their peers. The many different aspects of fractions provided the students with many different concepts to explain. The material was difficult enough that it took two months for pre-service mathematics teachers to work through and explain it all. Even though the students were already familiar with fractions, most were open to learning more about them. They could see how these conceptual meanings were helping them to gain a richer understanding of fractions and they realized that it would help them teach fractions in the future and clear up students' misunderstandings on this subject.

Data collection

Data collection was limited to the fraction unit, which lasted two months. During this time, I attended the class, observed the instruction and the interactions between the teacher and the students, and took field notes. Being in the class helped me know what the classroom norms were, what the teacher was expecting of the students, and how the teacher and students were developing conceptual meanings for fractions. I was able to refer back to my field notes to remember what concepts and issues had been addressed in class. My class observations allowed me to be informed for the interviews that I conducted for this study.

I interviewed each of the six students three times during the fraction unit—at the beginning, in the middle, and at the end. In the interviews, the students were asked to explain problems both in writing and verbally. All interviews were video recorded and all written work from the interviews was kept for analysis. The layout for all three interviews was the same. Before the interview, the students were requested to have completed a certain homework assignment. In that homework assignment, students were required to give written solutions and explanations for mathematical problems. In the interview, the student was asked to explain one of the homework problems to me verbally. They did not reference their written homework. They were not asked any questions during their explanation. After they finished, I gave them a different problem that was similar to the one just discussed. I asked them to solve the problem and explain it to me verbally after they were sure of their solution. I then asked them questions about their response, eliciting some understanding that they may not have previously expressed. These questions often dealt with parts of the explanation that I felt were incomplete, or where clarification would result in a better explanation. After they responded, I asked them to write an explanation of this problem. At the third interview, I then asked the students follow-up questions about their explanations. I asked them what they thought the key elements of a good explanation were and what they do to ensure they include all those aspects in their explanations. I also asked them if they thought their experience of learning to give conceptual explanations in this class affected the way they explained themselves in other math classes.

This layout for the interviews helped me answer my research question because I was interested in looking at how Toulmin's argumentation structure fit with explanations from an inquiry-based mathematics class. In inquiry-based mathematics classes, students often engage in both written and verbal explanations. By having the students give both written and verbal explanations in the interviews, I could gather data for both types of explanations. This allowed me to perform my analysis on both types of explanations and see if Toulmin's framework applied the same way.

The other aspects of the interviews helped assure that certain factors would not confound the results. Having the students explain two problems helped me know that their explanations were not rooted to one particular problem. This increased the likelihood that patterns in the explanations did not occur due to some unique characteristic of the problem. Switching the order of written and verbal explanations reduced the possibility that trends could have come up from always writing an explanation before giving a verbal explanation or visa versa. By questioning their verbal explanation on the second problem, but not on the first, I could first see what they would give as an explanation with no interference from me, but also see if there were things they understood that they did not include for some reason.

Analysis

Coding was used to apply Toulmin's structure of arguments to the explanations collected in the data. After all the interviews, I transcribed the verbal explanations and typed up each written explanation to make it easier to code and analyze. All the explanations had pictures associated with them. However, I did not refer to these as I coded because I found the explanations to be descriptive enough of the pictures that I did not need to see them during the analysis. It was a norm of the class that students had to explain what they saw in their picture. Therefore most of their explanations were clear enough without actually seeing the picture. I did not make any distinction between written and verbal explanations as I coded and analyzed them, but I knew if there were major differences I could spotlight them as the analysis went on.

The initial codes used to analyze the data corresponded to the main elements of Toulmin's structure of arguments—data, claim, warrant, and backing. For example, if there was a statement that was not supported by anything else in the explanation and it was used to justify another statement, then I coded it as data. Statements that had other statements backing them up I coded as claims. In order to make the coding process simpler, I put each sentence or statement on its own line and numbered the lines. After coding all the lines that seemed to be data, warrant, backing, or claim, I also linked the lines that made up an argument as shown in Table 2. For example, on the data lines, I put the line number of the claim it supports. On the claim lines, I put the line number(s) of the data that back it up. Also, key pieces of the argument were missing in some cases. When I first recognized they were not there, I noted that something was missing and moved on. In order to organize what I thought the author was trying to say, I later filled in the places that were missing pieces of the argument with statements that fit with my best guess of where the author intended the flow of the argument to go.

Some statements did not fit into any of Toulmin's four categories. I labeled these statements with a name that described what role they played in the explanation. For

Numbered Line from Explanation	Type of Statement
1 This picture represents 9/20.	DATA 2
2 I could get 1/10 pieces	CLAIM 1,3
3 by combining twentieths into sets of two	DATA 2

Table 2 Example of part of an explanation broken into numbered lines and function linked

example, the statement "To get hundredths from twentieths" was labeled as an *advanced organizer*. The statements that went in this category of being advanced organizers seemed to be there to give direction on the general flow of the explanation and to help the audience know where the explanation was going. Though they help with the explanation as a whole, they did not play a role in the logic of the arguments themselves. Therefore, if they were taken out of the explanation, the arguments would not be weakened. I was not sure if these other types of statements would be significant to my final analysis, but labeled them so I could look back and see if there were any significant patterns as the analysis continued. I did not notice any significant patterns and did not use them as I was extending Toulmin's framework because they did not seem to affect the logic of the actual arguments.

With multiple arguments in an explanation, it was confusing to follow all the numbered lines and see the links between the statements. I laid out the statements of the explanation in a flow chart, with different shapes representing data, claims, warrants, and backing to make the organization more visually clear. I then connected each of the shapes with arrows corresponding to how the arguments fit together. In Figure 5, a small portion of a flow chart is shown. The arrows are pointing right to show that the boxes on the left are data that lead to the claim in the circle on the right. Other elements of the

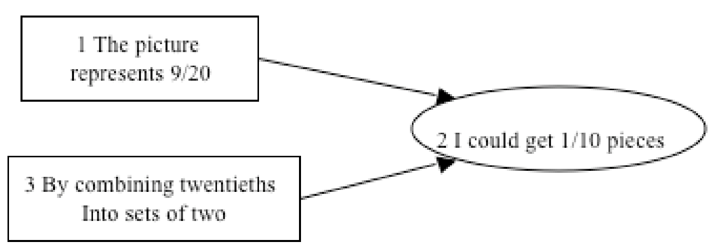

Figure 5 Example of data laid out in a flow chart

explanations were put in places on the flow chart that best represented their role in the argument. For example, when there was a warrant in the flow chart, it would be between the data and the claim that it links together on the line that connects the data to the claim.

Through the analysis of how the statements in the explanation link together in arguments, I was able to pinpoint what aspects needed to be modified and added to Toulmin's framework to extend his argument structure to apply to these explanations from a reform oriented mathematics class. I realized that the codes did not accurately portray the complexities of arguments in the explanations so I had to modify the codes and what some of the codes meant in order to extend Toulmin's framework. These modifications allowed the coding to show how arguments build on each other to make up an explanation and how to determine legitimate use of Toulmin's argument structure. A description of the process of how these codes were changed would be meaningless without a concurrent discussion of the results from extending Toulmin's framework to better fit explanations from a reform oriented mathematics class. Therefore further details of how the codes were modified will be given in Chapter 4.

In summary, the data collected in this study led to an interesting analysis of applying Toulmin's structure of arguments to explanations from a reform oriented mathematics class. Through coding, I was able to find features of these explanations that fit Toulmin's framework and others that were not accounted for. Analyzing the data led to extending the framework to account for the features of these explanations.

Chapter 4—Results

Applying Toulmin's argument structure to explanations from an inquiry-based mathematics class allowed me to modify my initial codes to extend Toulmin's framework. Extending Toulmin's framework by adding some elements allows his argument structure to better reflect the characteristics of explanations from a reform-oriented mathematics class. In this chapter, I will outline my findings and conclusions based on my data analysis. I will give a description of the data that I will refer to in this chapter and then I will discuss four problems that arose during my analysis. These problems are first, identifying and determining the legitimacy of data statements; second, determining the legitimacy of claim statements; third, determining classifications of conventions; and finally, uncovering multiple levels of backing. After identifying each problem I will explain how I dealt with it by extending Toulmin's framework.

In this chapter, a single written explanation is used to illustrate the results that arose from the data analysis. I have chosen to discuss only one explanation for three reasons. First, this explanation is representative of many of the explanations in my data set. Second, all of the major shortcomings in Toulmin's framework can be illustrated with this explanation. Third, the analysis of these issues is complex even when the discussion is limited to only one explanation. Inserting excerpts from additional explanations would complicate the discussion of the results even further. There is no need to make the results more complicated by including more than one explanation when one explanation is adequate.

Description of explanation

The explanation selected to illustrate the results was given by Sarah (name has been changed) during the second interview, approximately half way through the six-week fractions unit. This was her first written explanation from the second interview, so it is a rewriting of a problem she wrote out for her homework before arriving to the interview. The problem Sarah solved in this explanation was how to convert 9/20 to a decimal using a picture. When this idea was introduced in class the day before the interview, the picture used was a rectangle representing the whole, divided up into smaller equal pieces with some pieces shaded to represent the fraction to be converted to a decimal. Sarah used this picture to start off her problem. Her picture and explanation are shown in Figure 6. She first explained that she would solve the problem by looking for how many hundredths the fraction is equivalent to because the fraction could not be simplified to tenths. She explained that all the twentieths could be regrouped into groups of two, but the nine shaded pieces could not be put into sets of two evenly, so to convert 9/20 to a decimal, she had to work with hundredths. Next, to get hundredths from twentieths, she cut each twentieth piece into five equal pieces. She said she knew that doing this would yield

6. This picture represents 9/20. I could get 1/10 pieces by combining twentieths into sets of two but I can't put 9 twentieths into sets of 2 twentieths so I will have to work with hundredths instead. To get hundredths from twentieths, I cut each twentieth into 5 equal pieces. We can see that each 1/20 part contains 5 hundredths because it would take 100 of these smaller pieces to make the whole. Since 9/20 makes 45 of these smaller pieces, 9/20 is equivalent to 45/100 of the whole. 45/100 can be written in decimal form as 0.45.

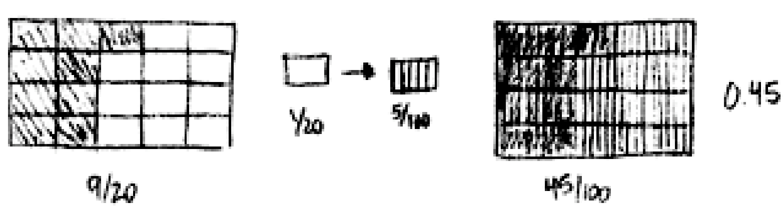

Figure 6 Sarah's Written Explanation and picture of converting 9/20 to a decimal using a picture.

hundredth pieces because it would take one hundred of these new smaller pieces to make up the whole. She then concluded that since 9/20 makes forty-five of the new smaller pieces, 9/20 is equivalent to 45/100, and 45/100 can be written as 0.45.

The first step I took in applying Toulmin's framework to this explanation was to determine which category of Toulmin's argument structure each of the statements in the explanation fit into. I looked at the role each statement played in the argument to determine how Sarah used the statement in her explanation. By coding Sarah's explanation this way, I realized that there were no warrants or backing in this explanation. The statements were either data, claims, or did not fit into one of Toulmin's four categories. Table 3 shows Sarah's explanation in a table with the numbered lines linked to corresponding parts of the argument. Deciding how to code the statements was not always easy. I encountered some difficulties in determining how to code the statements that fit into Toulmin's argument structure.

Numbered Line from Explanation	**Type of Statement**
1 This picture represents 9/20.	DATA 2
2 I could get 1/10 pieces	CLAIM 1,3/DATA 5
3 by combining twentieths into sets of two	DATA 2
4 but I can't put 9 twentieths into sets of 2 twentieths	DATA 5
5 so I will have to work with hundredths instead.	CLAIM 2,4
6 To get hundredths from twentieths,	ADV. ORGANIZER
7 I cut each twentieth into 5 equal pieces.	DATA 8
8 We can see that each 1/20 part contains 5 hundredths	CLAIM 7,9
9 because it would take 100 of these smaller pieces to make the whole.	DATA 8
10 Since 9/20 makes 45 of these smaller pieces,	DATA 11
11 9/20 is equivalent to 45/100 of the whole.	CLAIM 10/DATA 13
12 45/100 can be written in decimal form as 0.45.	DATA 13
13 So 9/20 is equivalent to 0.45	Missing CLAIM 11, 12

Table 3 Sarah's Explanation laid out in Table Format

Problem 1: Identifying and determining the legitimacy of data statements

Determining whether the statements were data or claims was difficult. Some statements were backed up by previous data statements and thus seemed to be claims, but then they were used as data to back up claims later in the explanation. As explained in Chapter 2, mathematical proofs and explanations often have several arguments in them that build on each other with a claim acting as data for further arguments, so this was not a surprising finding. Nevertheless, if I had chosen to label these statements as just data or just claim, it would not have captured the full picture of the role that statement played in the argument. I decided to label these statements "claim/data." This label gives a full description of these statements' function. First they function as a claim, and then later they function as data. This label also helps the audience to know why a statement can be accepted as data. If there is a question about the legitimacy of a statement when it is used as data, knowing it was previously a claim will allow the audience to refer back to the previous argument and use that argument to determine if the statement is acceptable data. Extending Toulmin's framework to allow some statements the option of being both claim and data took into account that mathematical explanations are composed of multiple arguments that build on each other to reach the end result. This extension made it easier to distinguish which statements fit Toulmin's framework and their function in the argument, and is shown in Figure 7.

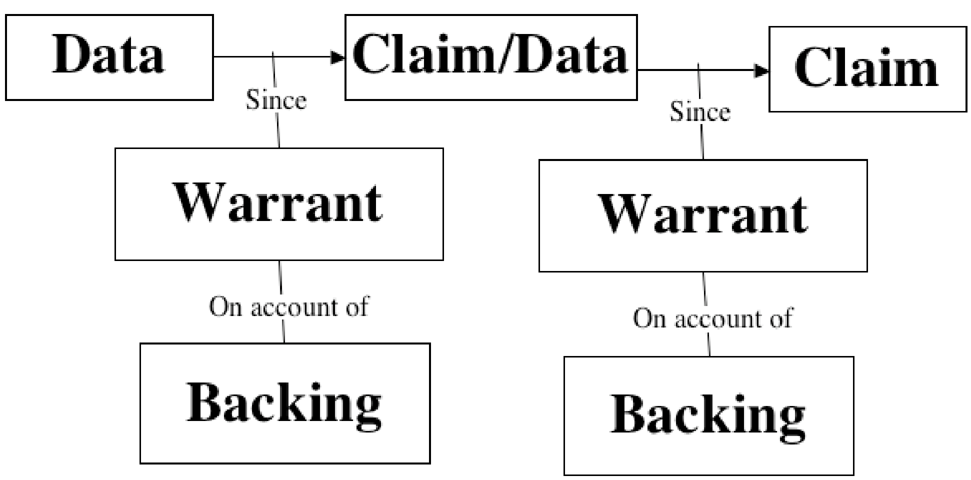

Figure 7 Toulmin's framework with two arguments linked by a claim/data statement

This minor extension of Toulmin's framework did not address all of the issues related to data and claim statements. A much more difficult problem was determining whether Sarah's use of a data statement was legitimate or if she should have treated that statement as a claim to be further supported by additional data statements. According to Toulmin's framework, a statement can legitimately be used as data if it can be accepted with no further justification. It was difficult to determine which statements could be accepted with no further justification because the data statements had to be in accordance with the terminology, definitions, and conceptual understanding that were acceptable in this class. These features are part of the class conventions discussed in Chapter 2. Since the conventions were the acceptable ways of defining and talking about quantities, a statement could be accepted as data with no further justification if it was based on and supported by one of the class conventions.

For example, I had to determine if the first statement in Sarah's explanation, "This picture represents 9/20," was acceptable as data. Sarah's statement could only be accepted as data if labeling the quantity represented in a picture with no explanation was in accordance with the class conventions. The conventions of the class, however, did not give her the ability to just declare what a picture represents. Rather, she needed to include the argument that her picture represented 9/20 because the whole was broken in twenty equal pieces, so each one was one-twentieth, and there were nine shaded pieces. This argument follows the convention established in class of justifying a picture of a fraction. She did not give this argument, so someone—particularly the instructor—may question that it is a legitimate data statement. Had she included the necessary leading argument, there could not be any question from anyone in the class that this could be used as a data statement.

Therefore, having a class-accepted convention that supports the data is the key to establishing the legitimacy of the data statement. I will call a convention that is used to support a data statement a *basis* (the plural of which is bases). As long as the basis is a class accepted convention, then the data can be accepted. If there is not an accepted convention acting as a basis, then the data statement can be questioned. In these cases, another argument would need to be given to establish the acceptability of the data. If such an argument is needed, however, then the statement must be classified as a claim/data. It should have been a claim with other data to back it up instead of being data to be accepted out right. Once established as acceptable, this claim/data statement can now be used as data for further arguments.

Recognizing the need for a basis to support the legitimate use of a data statement is an extension of Toulmin's framework and is shown in Figure 8. He does not

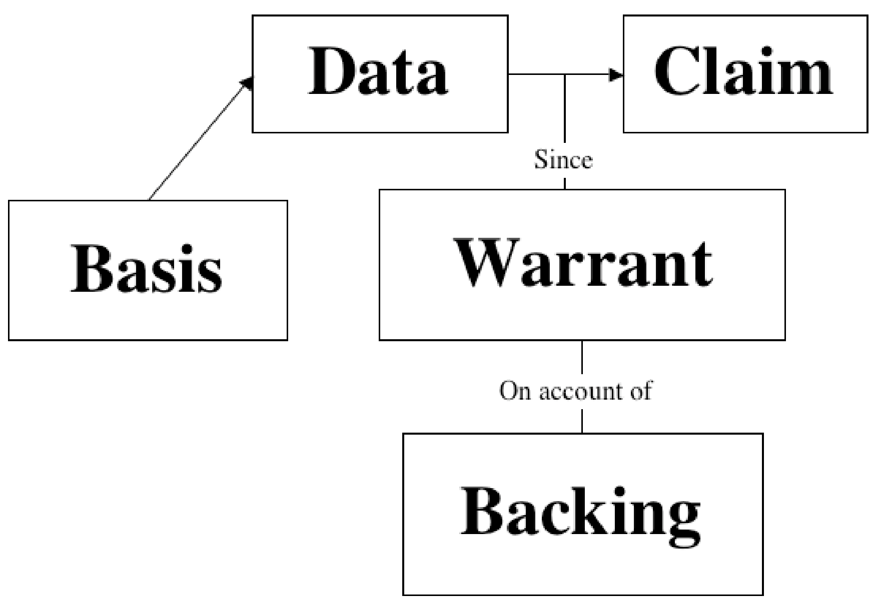

Figure 8 Toulmin's argument structure with basis

sufficiently address how the acceptability of the data is dependant on the social context in which the argument is being given. Being able to recognize a statement as data seems trivial to him, but as illustrated, it is actually quite complex. By knowing the acceptable conventions, it is possible to determine if a statement can be used legitimately as data.

Problem 2: Determining the legitimacy of claim statements

Data statements were not the only statements I needed to evaluate. I also needed to determine how to evaluate if the author's use of claim statements was legitimate. There are two parts of Toulmin's framework that he conveys as having a direct effect on the legitimacy of the claim: the data behind the claim and the warrant connecting the data to the claim. In order to evaluate the legitimacy of the claim, I had to consider both of these. I have already addressed how the data can be established as acceptable, but the applicability of the data to the claim is addressed by the warrant. As discussed in Chapter 2, warrants are general principles that link the data to the claim. I found that the general principles that are acceptable in this class are the class conventions. These conventions determine which terminology, definitions, and usages are acceptable. The conventions are principles that allow conclusions to be made based on certain prerequisite information. The data statement is the prerequisite information and the convention in the warrant links this information to the claim that can now be made. If the data is acceptable and fits the prerequisite information of a class convention that is used as a warrant, then the claim that follows will also be acceptable. Therefore by determining if the warrant was based on a class convention that fits with the data, I could determine if claims were legitimate.

Toulmin does not account for the influence of the social context on the process of determining if claim statements are legitimate. I extended his framework again by establishing that class conventions are used not only in the bases, but also in the warrants. By noting that the convention used as a warrant plays a crucial role in establishing the legitimacy of claim statements, I extended Toulmin's theory to show how the social context dictates which claim statements are deemed legitimate.

Problem 3: Determining classifications of conventions

Unfortunately, extending Toulmin's framework by recognizing that bases and warrants incorporate conventions that determine the legitimacy of data and claim statements was useful in theory, but hard to see in Sarah's explanation. This was due to the fact that she did not give any explicit warrants or bases in her explanation. In fact, the vast majority of explanations that were given by students did not contain any explicit warrants or bases, even the explanations that received full credit and were held up as models to the other students. Therefore, it was not possible to rely on students' explicit bases and warrants to determine whether data and claim statements were legitimate, because the students usually did not say which conventions they were using to support their data and claim statements.

In spite of this, as I analyzed the arguments, I found that some were well justified and easy to follow while some were not. It seemed like there was a difference in the underlying warrants that were in use, even though they were not given. In the arguments that were well justified, it was often obvious to me which convention was used as the underlying warrant. Sometimes an argument was so obvious, it almost seemed like it did not even need to be justified. In the arguments that did not seem well justified, I could not come up with a class convention that would have worked as a warrant. This difference that I found between the warrants in the arguments led me to investigate the different types of conventions to see what made the difference between arguments. I found that there were four classifications of conventions: explicit, implicit, unaccepted, and incorrect.

Explicit Conventions. There were some arguments in Sarah's explanation where it was obvious which convention was being used as the warrant, even though the warrant was not stated. I noticed that all these conventions were class-accepted conventions—those that had been addressed explicitly and taught in class. When they were taught in class, the teacher made it clear that these were acceptable principles to build an explanation on. I decided to categorize all these conventions as *explicit conventions*. For example, partitioning and iterating were two of the many conventions that were explicitly taught in the MthEd 117 class. From the first day in class, the teacher made it clear that partitioning and iterating were the two acceptable ways to define a unit fraction in this class. He explained the benefits of using these two part-whole oriented conventions instead of whole-number language. The teacher repeatedly stressed the importance of using these two ideas in class and in comments on homework, and insisted that one of these conventions be used to justify or explain all fraction values. Furthermore, students were penalized in homework and tests if they did not use these and other explicit conventions as the foundation for their arguments.

I found that there were certain patterns of information given in the data and claims that made it obvious that an explicit convention was being used, even though the warrant was not stated. I decided to call these patterns *templates*. There is a template that corresponds to each explicit convention. All the students have to do is put the appropriate data into the template, and then the claim immediately follows because of how closely the completed template resembles the explicit convention. When a template is used, there is no question which convention is being used; in fact, it would actually be redundant to state the warrant. As data and claims are fitted into templates based on explicit conventions, the legitimacy of the claim is because the template makes it obvious which explicit convention is being used as the warrant.

As an example, consider the template for the iterating convention that is used in Sarah's explanation. The iterating template involves a specific image of copying pieces a certain number of times to make the whole and determining the size of each piece based on the number of copies. The template has a verb that implies a copying or duplicating action, it describes how the piece was copied enough times to make the whole, and then it states the size of one piece based on the number of copies. One possible version of this template would be "___ copies of the piece will make a whole. Therefore, each piece is one-___th of the whole." The number of copies needed to make the whole is put into the blanks. This yields a legitimate argument of what size the piece is. In Sarah's explanation, after cutting the twentieths into five equal pieces, she claimed that each twentieth contained five hundredths. The data she gave for that claim is "because it would take 100 of these smaller pieces to make the whole." This fulfills the prerequisite of iterating because the phrasing she used implies that if she were to take one of those new smaller pieces and iterate it one hundred times, she would get the whole. Therefore, she could make the claim that she knew the size of the smaller piece was one one-hundredth because of the iterating convention. She never said she was using iterating, but it is clear from what she did say that she was using it.

Implicit conventions. After I identified the well-justified arguments that involved templates and their accompanying explicit conventions, there still remained arguments that seemed well justified but for which no obvious convention was being used as a warrant. This helped me realize that there must be other class-accepted conventions that give well-justified arguments besides the explicit ones. I realized that there were *implicit conventions* in use in these arguments. I discovered that these conventions were the shared practices and reasonings that never had to be addressed in class because it was tacitly understood that they could be used even though they have not been explicitly addressed. Implicit conventions were used regularly in class and on assignments and they were never marked wrong on graded assignments or questioned by the teacher or other students during in-class explanations. In contrast to explicit conventions, there were no templates for these implicit conventions because they had not been explicitly negotiated in class, where the templates were established.

The existence of implicit conventions is nor surprising, because it was impossible for every acceptable idea to be made explicit. Some ideas remained unspoken and served as a foundation for the explicit conventions to be built on. It was not problematic that these conventions were not discussed because they did not violate the explicit conventions. Neither the teacher nor the students seemed aware they were using these implicit conventions because they were never discussed. After all, it was only after analyzing many explanations that I noticed some of the implicit conventions.

An example of an implicit convention can be found in Sarah's explanation. Sarah uses an implicit convention—specifically the convention that it is permissible to cut a piece into smaller, equal-sized pieces—as the basis for the data statement, "I cut each twentieth into 5 equal pieces" (Line 7). For several weeks, the teacher and fellow students had implicitly accepted that it is legitimate to cut a whole or a piece into any number of equal sized pieces. This practice was never explicitly condoned by the teacher or the students, but was nonetheless frequently used by everyone in the study.

Unaccepted conventions. Once I had identified all the arguments that were well justified, I started to look at the arguments that were not as easy to follow and seemed to be lacking justification. It was not clear which conventions were used in the warrants of these arguments. There were no templates to follow, so I knew they were not using explicit conventions. There was no classroom practice or common definitions that could have been used as a warrant, so they were not using implicit conventions. In this class, acceptably justified arguments used explicit and implicit conventions, and these arguments did not seem to use those conventions. Therefore these arguments would not have been considered acceptably justified in this class. Though there is no way to know exactly which convention they were using, I concluded that the students were using an *unaccepted convention* as the warrant in these arguments. These unaccepted conventions did not obviously contradict widely held mathematical beliefs because the students acted like they had used a legitimate convention in their underlying warrant when they used these conventions in their arguments. These conventions may even be accepted in other mathematics classes. However, an explanation using an unaccepted convention would not have been acceptable in this class. In fact, the students would have been penalized for using these conventions on graded assignments.

There are two types of unaccepted conventions. First, a convention may be unaccepted because it violates the explicit conventions negotiated in this class. Second, an unaccepted convention may lump together more than one acceptable or unaccepted convention and is therefore missing the description of some of the mathematical actions that were taken. It is likely that these conventions exist because of students' mathematical experiences—they may have been established during the students' previous mathematical experience and could have been legitimate in other mathematics classes they have taken. Some unaccepted conventions could have been created during this class to correspond with previous experiences and their unacceptability has not been realized yet.

It is difficult to determine which unaccepted convention is being used because there is no template to follow. An example from Sarah's explanation illustrates this difficulty. She uses the data, "By combining twentieths into sets of two" for the claim "I could get 1/10 pieces." The convention she uses in her warrant may have been something like "When pieces are regrouped, they can be called by a new name that comes from how many new pieces there are." This argument is laid out in Figure 9a. However, this violates the explicit convention in the class that partitioning or iterating must be used to define the size of fraction parts. On the other hand, it is possible that she could have combined more than one explicit convention to make an unaccepted convention. For example, she may have lumped an accepted division convention with the partitioning convention. In order for these conventions to be used acceptably, she would need to add a claim/data statement in between, "I get ten equal sets." This statement would be a claim, backed by the statement "By combining twentieths into sets of two" as data and the division convention as a warrant. Then it would act as data, with the partitioning convention as a warrant to get the claim "I could get 1/10 pieces." Figure 9b shows this argument compared to the previous argument. If Sarah had used these conventions in separate warrants and followed a template, then we would know what convention the warrant is using. However, with both conventions being used together, there is an unaccepted convention and no template to follow.

Incorrect conventions. It is possible students could use a convention that is incorrect or use an acceptable convention incorrectly. I did not find any incorrect conventions in the data and this is likely due to the students' strong mathematical background. It would be unlikely for an incorrect convention to be used by these students, but it may happen in other classes with students who are not as familiar with the

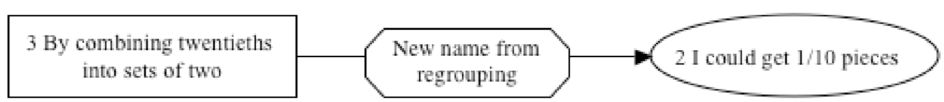

Figure 9a Sarah's Argument in Flow Chart Form with one possible warrant: The rectangle on the left is the data, the octagon in the middle is the warrant, and the circle on the right is the claim. The arrow shows that the data leads to the claim, because of the warrant.

Figure 9b Sarah's Argument in Flow Chart Form with another possible warrant: The circle in the middle has been added as a claim/data statement. It would be the claim from the rectangle on the left, then used as data for the circle on the right. There are two warrants now—one for each argument.

concepts they are explaining. Incorrect conventions would not be accepted in any mathematical discourse community and would violate a mathematical fact that is widely held across mathematical communities. Students would be penalized for using them and they could lead to incorrect claims. Using an acceptable convention incorrectly would yield an invalid argument because the warrant would not imply the claim from the data.

Classifying conventions based on how they show up in the explanations and whether they make up an acceptable argument is an extension of Toulmin's framework. When arguments are based on explicit and implicit conventions, the argument will be acceptably justified. However, if an argument is based on unaccepted and incorrect conventions, the argument will seem questionable. Recognizing an explicit convention from its template makes it obvious which convention is being used as a warrant, even when it is not explicitly stated. Implicit conventions have never been discussed in class, but the way they are used and never objected to makes it clear they are acceptable. When unaccepted conventions are used in arguments, those arguments will lack acceptable justification. When incorrect conventions are used, the claims from the argument will be questionable because they are not based on mathematically correct justification.

Problem 4: Backing is not always simple

The fourth part of Toulmin's argument structure that I have not yet discussed in detail is the backing statements that confirm warrants. As I worked with the conventions in the warrants, I realized that the backing statements behind the warrants had varying levels of complexity. The backing for some warrants would just be "because we say so." For example, the backing of partitioning is just that simple. It cannot be broken down to any deeper concepts. When partitioning is used as a warrant, it can be used because that is how it has been established in class. The convention the warrant is based off of is simple and therefore its backing is simple as well. I decided to call the warrants with this simple backing *basic warrants*. There are some basic warrants that are based on explicit conventions and some that are based on unaccepted conventions. The example of a basic warrant that is based on an explicit convention is partitioning, as described above. In contrast, an example of a basic warrant that is based on an unaccepted convention is the "out of" convention: it many mathematics class it is acceptable to define a fraction of 3/4 as three out of four things and the backing for this warrant is simply "because we say so."

The backing for some warrants is not as simple. Some conventions are built on other conventions and are made up of multiple components. I will call these *advanced warrants*. The backing of advanced warrants is more complex than the backing of basic warrants. It would consist of another data, warrant, and claim sequence. If the warrant used on this next level were basic, then the advanced warrant has been broken down as far as it can be. This is shown in Figure 10 using the same diagram that was used in

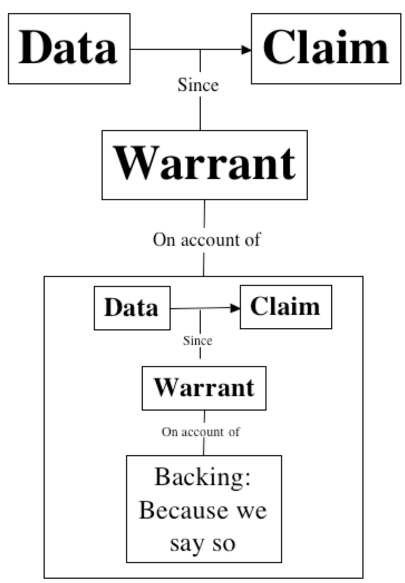

Figure 10 Diagram of Advanced Warrant

Chapter 2, with another data, warrant, backing, and claim sequence where the backing goes. If the warrant used on that second level is also advanced, then its backing would be another data, warrant, and claim sequence. This could continue until there is a basic warrant used. Toulmin does not acknowledge that there are warrants of differing complexity. However, an example of one explicit convention shows that they do exist.

An example of an advanced warrant would be a warrant using the simplifying fraction convention. The warrant would be “You can simplify a fraction if and only if you can group the shaded parts and the non shaded parts of the fraction evenly into same sized groups.” The backing is more complex than “because we say so.” In fact, it would have to do with not knowing how to interpret the fraction if this was not the case. The backing could be another data, warrant, and claim sequence. It could be “If there is a group made up of shaded and non-shaded pieces, then you do not know how many groups are shaded,” where “If there is a group made up of shaded and non-shaded pieces” is the data and “then you do not know how many groups are shaded” is the claim. There is another sequence that needs to go here as well: “If in the new grouping, all shaded and non-shaded are grouped together, but not all the pieces in the whole are the same size, then you do not know what to call the new pieces.” The backing for the warrants for both of these sequences is “because we say so,” so there is no need to break it down any further. This example is illustrated in Figure 11.

I realized that there were differences in the backing of warrants while analyzing Toulmin’s argument structure. His structure points out the different types of justification behind a claim. Rather than simply pulling back the layers of justification—looking at the ideas that back up the initial justification—he differentiates between data, warrants, and

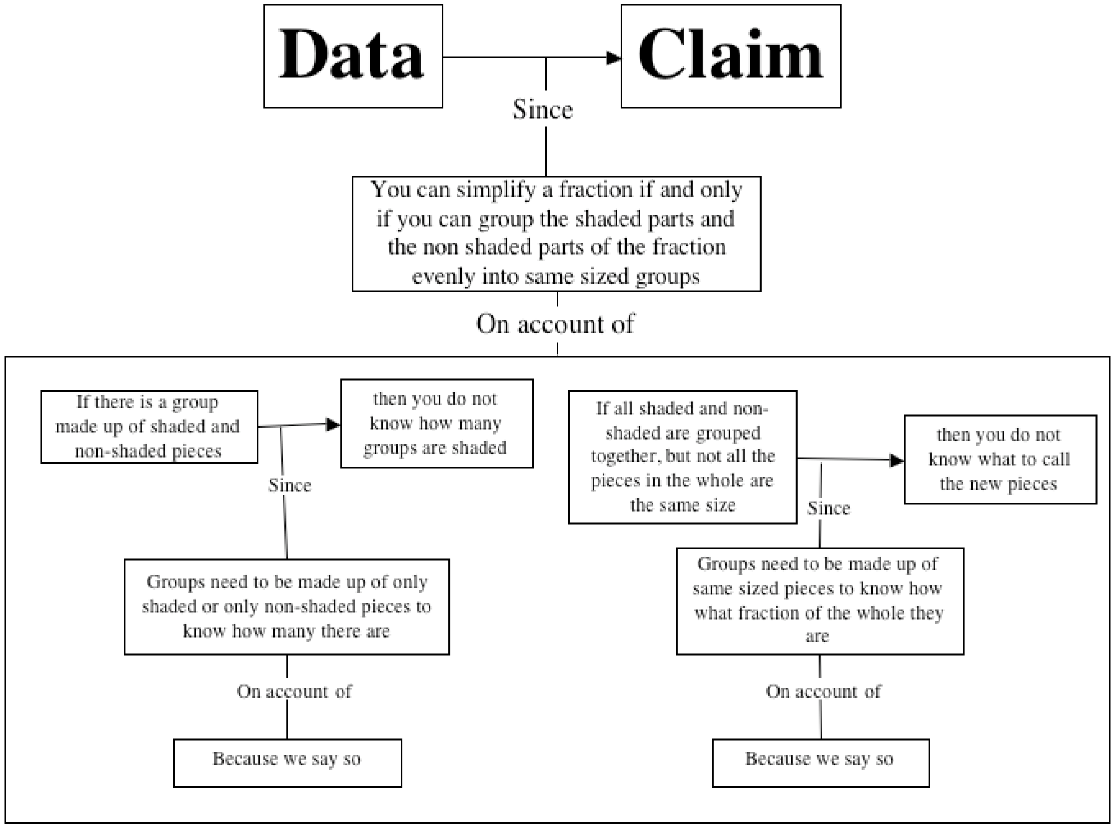

Figure 11 Diagram of Advanced Warrant with Arguments in the Backing

backing. However, he does not address the complexity of backing. Though he mentions that what counts as legitimate backing is field dependant, he does not distinguish between basic and advanced warrants based on simple or more complex backing. He does not acknowledge that sometimes the backing is not simple and could be unpacked more itself. Building up advanced warrants coincides with how students work to establish mathematical meanings. They learn new concepts that can later be used as conventions for the foundation of more new ideas.

Conclusion

This chapter has explained the results of my study that answered my research question. Adding some elements that allow Toulmin's argument structure to reflect some of the characteristics of explanations from a reform-oriented mathematics class extended his framework. Toulmin’s argument structure with all of these extensions is shown in Figure 12. First, there are not only data statements and claim statements, but also claim/data statements that have different roles in separate arguments. The basis that supports the data is what allows it to be accepted without further justification. Second, claims are legitimate when a class convention is used as the warrant and has appropriate data. Third, there are conventions in use in the bases and warrants, but they are not usually made explicit. Recognizing the different classifications of conventions extended Toulmin’s framework. Bases and warrants are acceptable if they use explicit and implicit conventions, but they will be questionable if other conventions are used. Lastly, recognizing that some warrants are advanced and might have more than one layer of backing is an extension of Toulmin’s framework.

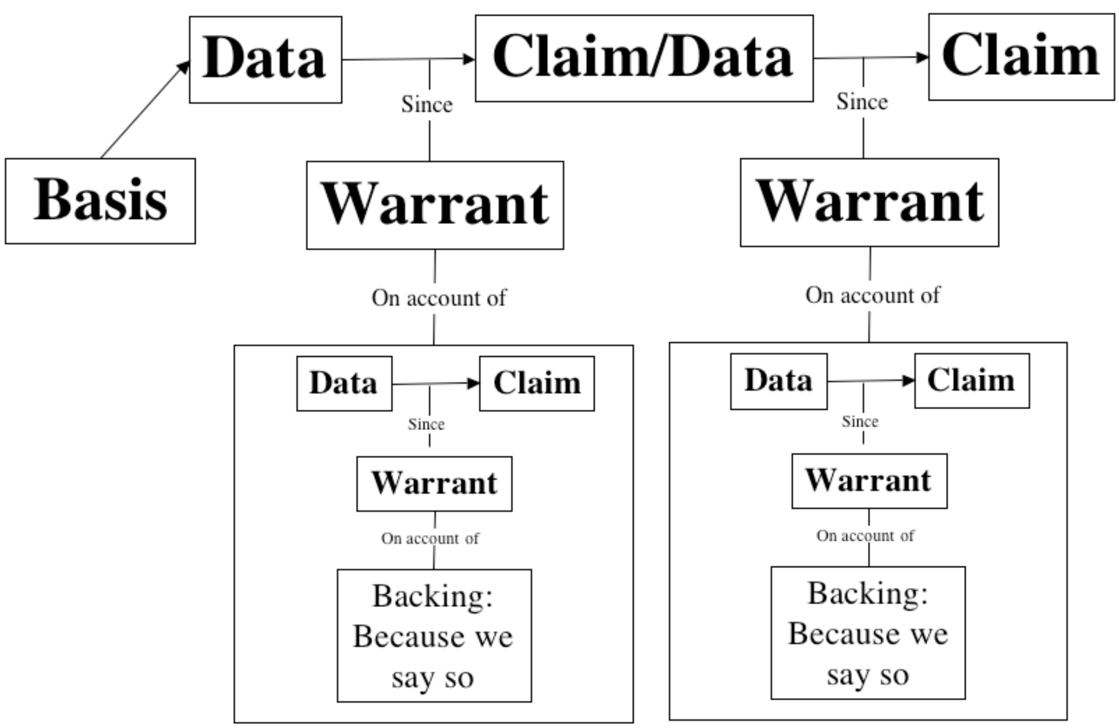

Figure 12 Toulmin's argument structure with all extensions shown

Chapter 5—Conclusion

The purpose of this study was to identify common components and structures of conceptual explanations. In the previous chapter, I extended Toulmin's framework to better describe these explanations. In this chapter, I will first discuss the contributions of my extensions of Toulmin's framework. Following the contributions, I will discuss the limitations and implications of my study.

Contributions

Toulmin's framework is too simple to account for all the aspects of explanations and my extensions helped to account for the complexities that were not accounted for by his framework. The first complexity of explanations that Toulmin does not account for is that mathematical arguments build on each other to reach the final conclusion. I resolved this complexity through extending his framework. My decision to extend the framework by labeling some statements as claim/data statements was consistent with the findings of other researchers who have noted how claims can become data for later arguments (Aberdein, 2005; Yackel, 2002). This decision also extends their results by giving these statements a name that fully describes their function in the argument. The second complexity that Toulmin does not account for is the varying levels of justification involved in backing. I dealt with this complexity by distinguishing the difference between basic and advanced warrants to show how the backing is not always simple. Sometimes the backing can be broken down to many more levels. The idea of advanced warrants is consistent with building up complex ideas in mathematics. By combining pieces of new knowledge to create more advanced justification, further arguments can be simpler because they rely on principles that have been grouped together.

Toulmin's original framework does not account for the social influences on explanations so I extended it to do so in a way that holds true to his original framework. First, I used Toulmin's criteria for the form and function of data, warrants, backings, and claims, unlike Krummheuer (1995) who dismissed the form of the statements to be unimportant in determining an application of Toulmin's argument structure. My extension of his framework allowed me to show how social influences affect explanations. Second, I showed that the class conventions are the warrants that the arguments are based on. I showed that the legitimacy of data and claim statements is dependent upon the class conventions used in the bases and warrant. This means that acceptable warrants in Toulmin's framework come from the classroom and that data statements cannot be used as data without taking the social context into account. Also, recognizing that there are templates that correspond with explicit conventions makes it clear when arguments are well justified. These templates result from the social interactions in class when they are established. All of the extensions of Toulmin's framework are based on conventions. Conventions result from classroom interactions and instruction, and therefore, explanations are dependant upon these conventions.

Limitations

Although I have accounted for many of the statements in the explanations by extending Toulmin's framework, there were still statements that did not fit in the framework. Some of these statements were advanced organizers that would help the audience follow the direction of the explanation. There were other statements that did not fit into Toulmin's categories and were not advanced organizers that still need to be accounted for. I did not investigate the specific features of these other statements because they usually did not have a direct impact on the arguments in the explanation. My extensions of Toulmin's framework enabled me to capture the essential parts of the logical arguments; however, these other statements still have an important role in how the students express what they were thinking about the mathematics. These statements may be unique to explanations and not to formal proofs. It is possible that another framework is needed besides Toulmin's to account for all the statements in an explanation. More research on other frameworks for explanations would be very informative to the field.

Another limitation of my study is that these findings are limited to this specific conceptually oriented mathematics class. The explanations analyzed in this study are from a class where students' prior understanding is being reexamined and they are working with new conceptual understandings of subjects they are already familiar with. In classes where students are learning things for the first time, the students' explanations may be different. It is more likely that they might use incorrect conventions or use correct conventions incorrectly because they are not familiar with the subject matter. They may also be less likely to use unaccepted conventions because they do not have prior experiences developing understanding of the subject matter that are not accepted in their class. Similar studies in other mathematics classes could add more understanding of the complexities and social aspects of argumentation in other situations.

Implications

This study highlights the influence of the social context on the requirements for a good explanation. This influence is important to recognize because it affects how explanations are critiqued. Since each class has its own unique context, this context has to be taken into consideration when evaluating explanations. If teachers acknowledge the context they create in their classes, they can recognize the conventions that have been established for students to base their explanations on. Then they can help students know how to give good explanations by using these conventions correctly. The findings of this study allow teachers to understand these ideas about conventions and explanations in their instruction so they can address them in their classes.

Teachers need to spend more time addressing bases and warrants with their students. Teachers need to make it explicit how conventions should be used in the bases and warrants. They also need to help the students realize how the bases and warrants affect the strength of their arguments. Teachers can help students recognize how to use an explicit convention by explaining how templates make their warrants obvious in their arguments. By teaching students about templates, there is a risk that students will learn to use the templates to say the right words in their explanations in order to sound like they understand the concepts, even if they do not. However, teachers can carefully address this issue to work around this risk. Regardless, students will need to know when it is appropriate to use the templates in their explanations. If students realize there is not a template to follow to make their claim, this can bring to their attention that they may be using an unaccepted convention. By giving students the opportunity to distinguish between the warrants they are using to justify their claims, they can know whether their explanations will be acceptable in their class.

Acknowledging that well justified arguments use templates of explicit conventions makes it easier to find the unaccepted and incorrect conventions students may be using. If there is not a template being used in an argument, and the warrant would not correspond with an accepted classroom practice or definition, then teachers and students can identify that there is most likely an unaccepted or incorrect convention being used in the warrant. This can make it easier to identify where students may not be using proper justification. Even though the warrants do not need to be given explicitly, it should be clear to their audience what the underlying justification is.

Conclusion

This study has helped in finding the structure of a good explanation by extending Toulmin's framework to account for the complexities and social aspects of explanations. However more research is needed to continue to grasp what the general structure of an explanation would look like. Analyzing the explanations in a variety of mathematics classes with this and other frameworks will help to nail down the specifics of what constitutes a good explanation. This in turn will help teachers give their students explicit instruction on explanations.

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