MACROECONOMIC CONSEQUENCES OF UNCERTAIN SOCIAL SECURITY REFORM

by

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DISSERTATION ABSTRACT

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The U.S. social security system faces funding pressure due to the aging of the population. This dissertation examines the welfare cost of social security reform and social security policy uncertainty under rational expectations and under learning. I provide an overview of the U.S. social security system in Chapter I.

In Chapter II, I construct an analytically tractable two-period OLG model with capital, social security, and endogenous government debt. I demonstrate the existence of steady states depends on social security parameters. I demonstrate a saddle-node bifurcation of steady states numerically, and demonstrate a transcritical bifurcation analytically. I show that if a proposed social security reform is large enough, or if the probability of reform is high enough, the economy will converge to a steady state.

In Chapter III, I develop a three-period lifecycle model. The model is inherently forward looking, which allows for more interesting policy analysis. With three periods, the young worker's saving-consumption decision depends on her expectation of future capital. This forward looking allows analysis of multi-period uncertainty. Analysis in the three-period model suggests that policy uncertainty may have lasting consequences, even after reform is enacted.

In Chapter IV, I develop two theories of bounded rationality called life-cycle horizon learning and finite horizon life-cycle learning. In both models, agents use adaptive expectations to forecast future aggregates, such as wages and interest rates. This adaptive learning feature introduces cyclical dynamics along a transition path, which magnify the welfare cost of changes in policy and policy uncertainty. I model policy uncertainty as a stochastic process in which reform takes place in one of two periods as either a benefit cut or a tax increase. I find the welfare cost of this policy uncertainty is less than 0.25% of period consumption in a standard, rational expectations framework. The welfare cost of policy uncertainty is larger in the learning models; the worst-off cohort in the life-cycle horizon learning model would be willing to give up 1.98% of period consumption to avoid policy uncertainty.

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To my mother

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CHAPTER I

INTRODUCTION

The Old Age Survivor and Disability Insurance program (OASDI), commonly referred to as social security, is one of the largest transfer programs in the world, accounting for roughly twenty percent of U.S. federal government spending. OASDI currently provides retirement benefits to 44 million people and collects payroll taxes from 171 million workers. Social security's cost has exceeded its tax income in each year since 2010. For 2017, the program is expected to have a primary deficit of $59 billion. The Social Security Administration (SSA) projects this relationship to extend through the short and long-term. Similarly, the Congressional Budget Office (CBO) projects federal government deficits to grow from 2.9% of GDP in 2017 to 9.8% in 2047, driven in large part by entitlement spending. Concurrent with an increase in government deficits, the stock of federal debt is expected to reach 150% of GDP by 2047 (CBO, 2017).

The SSA has built up a trust fund to finance the difference between tax receipts and social security benefit payments; however, the fund is projected to be depleted by the year 2034. Following the depletion of the trust fund, the SSA will not be able to fully pay scheduled social security benefits. The SSA Board of Trustees project that an immediate and permanent payroll tax rate increase of 2.76 percentage points (12.4% to 15.16%) or an immediate and permanent benefit cut of 17% applied to all current and future beneficiaries would be needed to insure sufficient tax revenue to cover promised benefits over the long-term (SSA, 2017).

Given the projected shortfall between social security tax revenues and promised benefits, reform seems likely. However, it is not clear if or when the government will act. The uncertainty regarding when and how social security will be reformed is interesting and economically important. A survey conducted by the American Life Panel in 2011 found that 56% of respondents are "not too confident" or "not confident at all" that the social security system will be able to provide them the level of benefits currently promised. Wariness in the government's ability to pay benefits is decreasing in age, with 75% of respondents under the age of 40 reporting they are not confident they will receive benefits (47% "not too confident" and 28% "not confident at all").[1] Social security benefits are a major source of retirement income for many Americans, and uncertainty regarding future benefits could impact savings decisions. The economy wide uncertainty regarding social security deficits and debt accumulation is also interesting, as rapid debt growth could strain the economy (see, for example, Davig, Leeper, and Walker (2010)).

Social security reform uncertainty has been examined from the perspective of the government.[2] Additionally, a growing literature examines the welfare implications of uncertain social security reform from the perspective of the agent. However, little research has been done on the long-run macroeconomic consequences of social security policy uncertainty.

Bütler (1999) calibrates a partial equilibrium life-cycle model to the Swiss economy and examines the impact of expectations on social security policy. The underlying model does not have any policy uncertainty, but agents form (potentially incorrect) beliefs about future government policy. Bütler finds that the beliefs of agents have large impacts on the aggregate economy through

[1]The RAND American Life Panel is a nationally representative, probability-based panel of over 6000 members ages 18 and older who are regularly interviewed over the internet for research purposes. This question comes from the 2011 survey *Well-being ms179*. More information about the ALP is available at alpdata.rand.org.

[2]See Auerbach (2014) and Auerbach and Hassett (2007).

precautionary savings. The welfare cost of social security reform uncertainty is largest for the older cohorts in her model. Gomes, Kotlikoff, and Viceira (2012) builds a more complex life-cycle model with social security reform timing uncertainty. They find that agents would be willing to sacrifice up to 0.6% of life-time resources to have policy uncertainty resolved at age 35 rather than discovering that social security benefits have been cut at age 65.[3]

Recent work by Caliendo, Gorry, and Slavov (2015), Kitao (2016), and Nelson (2016), examine the welfare cost of timing and social security policy type uncertainty. Caliendo et al. (2015) build a continuous-time, partial equilibrium model that can accommodate rich timing and policy uncertainty. The (heterogenous) agents form expectations over a two-dimensional distribution of reform type and reform dates.[4] The welfare cost of policy uncertainty is small (about 0.01% of lifetime consumption), but varies across the income distribution with largest costs for low-income agents. Kitao (2016) considers a general equilibrium discrete-time model with benefit cut timing uncertainty, calibrated to the Japanese economy.[5] Kitao's model incorporates the changing population growth rate of Japan, and finds welfare costs of over two percent of life-time consumption for older generations. Nelson (2016) uses a smaller-scale general equilibrium discrete-time model to examine joint timing and policy uncertainty.

[3]Note, the agent's in the Gomes et al. model do not face actual policy timing uncertainty–benefit are cut with certainty at age 65–but, agents do not know their benefits level with certainty until a given age.

[4]Caliendo et al. (2015) consider a continuous distribution of reforms, governed by a parameter α that indicates the share of the reform that is a tax increase, with $(1-\alpha)$ of the reform is a benefit cut. Timing uncertainty is modeled using a continuous distribution of potential reform dates.

[5]The main analysis of Kitao's paper focuses on policy timing uncertainty. An extension of the model considers timing and policy type uncertainty with a given benefit cut occurring in either 2020, 2030, or 2040; or a smaller benefit cut occurring in 2040.

Pension reform either occurs early or late, and either takes the form of a benefit cut or a tax increase. Nelson focuses his analysis on varying the probability that reform takes a certain type. He find welfare costs of reform of similar magnitude to other authors.

Fiscal policy uncertainty more broadly, often motivated by a story about entitlement spending growth has also been studied in a variety of cases. Davig, Leeper, and Walker (2010, 2011) examine the long-run interactions of fiscal and monetary policy in a New Keynesian economy with growing government debt and uncertain fiscal policy. They find that hyperinflation is a possible, but unlikely outcome of growing entitlements, and that the government is likely to renege on promised future transfers. Davig and Leeper (2011b, 2011a) build similar models with fiscal policy uncertainty to examine the role of expectations and debt accumulation on the long-run path of the economy. Davig and Foerster (2014) examine the interaction of fiscal policy uncertainty and a fiscal cliff. The main weakness of these New Keynesian papers is the reliance on a representative, infinitely lived consumer to model household behavior. The authors have to abstract away from key details of a social security system (namely the transfer of wage income from the workers to retirees) because the models do not incorporate aging or retirement.

One natural alternative to the infinite horizon, representative agent models, is the overlapping generations (OLG) model, which explicitly models the aging and retirement of agents. OLG models provide a natural framework to study the impact of social security in an economy. Several authors have examined the desirability and feasibility of various social security reforms (for example, see Bouzahzah, De la Croix, and Docquier (2002) or Bohn (2009)). The implications of postponing reform have also been considered. R. Evans, Kotlikoff, and Phillips (2013) show that the economy breaks down if government debt becomes explosive. Similarly, Chalk (2000) demonstrates if government deficits are too large, government debt spirals out of control and convergence to a steady state is not possible.

The long-run implications of social security policy uncertainty have not been examined in an overlapping generations framework with rational agents (or with boundedly rational agents with model-consistent expectations). I begin this work in Chapter II of this disseration. I construct an analytically tractable two-period OLG model with capital and government debt. The government runs a social security transfer system that taxes the working generation to pay benefits to retirees. If retiree benefits exceed tax revenue, the government finances the short-fall with one period bonds. Thus, the government deficits are endogenous. I find that the existence of and convergence to steady states depends on the size of government deficits and the stock of government debt. I demonstrate two additional policy results with the two-period model. two-period First, if the economy is on an explosive trajectory, either because the existing stock of government debt is large relative to the capital stock or because the endogenous government deficits are too large, the government can change the social security system to put the economy on a stable path. Specifically, the government can increase social security taxes or decrease benefits to reduce the deficit and pay down the government debt. This intuitive result is important, because it highlights the government's ability to avoid catastrophic debt accumulation by making straight-forward (albeit likely unpopular) policy changes.

Second, if the economy is on an explosive path and agents believe that policy will change in the next period, these beliefs can be sufficient to put the economy on a stable path. Specifically, if the young generation believes that their social security benefits will be small, they increase their savings, which increases the aggregate capital stock, increases future wages, and increases future tax revenue. Using my simple two-period model, I can show that if the social security reform is large enough or if the probability of reform is high high enough, the economy can converge to a steady state, even if the government runs deficits (or accumulates debt) that would otherwise be explosive. This result highlights the role of expectations in the economy.

The main mechanism in the model is the precautionary savings of agents. Young agents who believe they may receive smaller pension benefits in the future, respond by increasing their savings today. The precautionary savings increases the capital stock. The general equilibrium feedback effects of higher capital (higher wages and lower interest rates) influence the social security deficit and thus government debt levels. Changes in capital and government debt alter the equilibrium paths of the economy.

In Chapter III, I extend my two-period model to a three-period model in which agents work for two-periods and then are retired for the third period. The appeal of a two-period model is that it can be solved analytically. The downside of a two-period model, is that agents only care about policy changes that happen in the next period (absent a bequest motive). This limits the range of policy experiments that can be considered. It is necessary to extend the model to three-periods in order to entertain reforms, or expected reforms, that are more than one period in the future or reforms that involve changes in taxes. The three-period model cannot be solved analytically, but it can be solved using standard computational methods (described in Appendix B).

The three-period model is inherently forward looking, which allows for more interesting policy analysis than the simple two-period model. With three-periods, the young worker's saving-consumption decision depends both on the predetermined capital and bonds for the next period, and also on her expectations over capital and bonds in the following period. This forward looking element of the model allows multi-period uncertainty analysis in the fashion of Davig et al. (2011) or Davig and Leeper (2011a).

The main policy implications of the two-period model can be replicated in the three-period model. In addition, the three-period model illustrates that certain types of policy uncertainty can dampen the precautionary savings motive of agents who expect reform in (or near) their lifetime. Additionally, analysis in the three-period model suggests that policy uncertainty may have lasting consequences, even after reform is enacted.

In Chapter IV, I explore the macroeconomic consequences of social security policy uncertainty in a multi-period lifecycle model. Agents' expectations about future social security policy drive agent-level decision making. These decisions, combined with realized government policy, determine macroeconomic output and prices. The way in which agents form expectations has a large impact on both the short-run responses to changes in policy (or possible changes in policy) and the long-run level of capital and output in the economy. I consider rational expectations and adaptive expectations. This chapter contributes to a growing literature that examines the response to anticipated fiscal policy in models with adaptive expectations (see G. W. Evans, Honkapohja, and Mitra (2009), Mitra, Evans, and Honkapohja (2013), Gasteiger and Zhang (2014), and Caprioli (2015)).

The rational expectations hypothesis is standard in macroeconomics and I develop a rational expectations model as the baseline for my analysis. I also relax the assumption of rational expectations and explore the consequences of agents using adaptive expectations in a model with an aging society and social security reform. One criticism of the rational expectations hypothesis is that it requires agents to possess more knowledge about the structure of the economy than any econometrician possesses about the actual economy. Models of adaptive learning relax that requirement, and allow for boundedly rational agents (Sargent (1993), G. W. Evans and Honkapohja (2001)).

In this chapter, I introduce two new frameworks for modeling bounded rationality that I call life-cycle horizon learning, and finite horizon life-cycle learning. In both models, agents use adaptive expectations to forecast future aggregates, such as wages and interest rates. The expectations are updated each period as agents acquire additional information. This adaptive learning feature introduces cyclical dynamics along a transition path, which magnifies the welfare cost of changes in policy and policy uncertainty. I model policy uncertainty as a stochastic process in which reform takes place in one of two-periods as either a benefit cut or a tax increase. I find the welfare cost of this policy uncertainty is equivalent to less than 0.3% of period consumption for the cohort of agents most harmed in a standard, rational expectations framework. The welfare cost of policy uncertainty is larger in the learning models; the worst-off cohort in the life-cycle horizon learning model would be willing to give up 1.98% of period consumption to avoid policy uncertainty, and would be willing to give up 0.8-1.2% of period consumption to avoid the cyclical dynamics introduced via learning.

In all chapters of this dissertation, I model policy uncertainty as policy risk. That is, I consider various stochastic processes for government policy parameters. The possible realizations of policy parameters and their relative probabilities are known to all agents in the model. I refer to this type of risk as "uncertainty." A growing body of economic and finance research has distinguished between risk (stochastic process with known probabilities) and uncertainty or ambiguity (process in which either probabilities or possible realizations are unknown). I will abstract from ambiguity in this dissertation; modeling the response to agents to social security policy risk provides an interesting baseline for thinking about the impact of government policy uncertainty on the macroeconomy. I may consider richer models of uncertainty or ambiguity in my future research.

CHAPTER II

POLICY UNCERTAINTY IN AN ANALYTICAL TWO-PERIOD OLG MODEL

Motivation

The need for pension reform in aging societies, like the United States, is well documented. Despite the looming need, uncertainty remains regarding the timing and structure of eventual reform. In this chapter, I construct an analytically tractable two-period OLG model with capital, government debt, and uncertain social security reform. I find that existence of and convergence to steady states depends on the size of endogenous government deficits and the stock of government debt. I demonstrate three key policy results with my model.

First, if the economy is on an unstable trajectory, either because the existing stock of government debt is large relative to the capital stock or because the endogenous government deficits are too large, the government can change the social security system to put the economy on a stable path. This intuitive result highlights the government's ability to avoid catastrophic debt accumulation by making straight-forward (albeit likely unpopular) policy changes. The model also highlights the tradeoff between the size and timing of reform. If the government acts sooner, it can make relatively small changes. If the government delays reform, a larger benefit cut (or tax increase) is needed to achieve stability.

Second, if the economy is on an unstable path and agents believe that policy will change in the next period, these beliefs can be sufficient to put the economy on a stable path. If the probability of reform is large enough, the economy can converge to a steady state, even if the government runs deficits (or accumulates debt) that might otherwise be explosive. If agents believe that benefits might be cut in the future, they engage in precautionary savings. This savings can increase the capital stock enough to delay the need for reform. This result parallels the results of Davig et al. and highlights the role of expectations in the economy.

Third, the optimal policy reform from the perspective of a benevolent planner depends on the length of the planner's time horizon, and the social discount factor used. If the planner looks many generations into the future, reforms that take place sooner will be socially preferable to later reforms. It also appears that agents are worse off in an economy that faces uncertain policy reform compared to an economy with known policy changes.

The remainder of the chapter is organized as follows. I begin by presenting the model in section II. The model has several interesting features, including a saddle-node bifurcation, which is discussed in section II. Phase diagrams are presented in section II. Several special cases are presented at the end of section II. Policy analysis is presented in section II. The paper ends with welfare analysis in section II.

Model

Household Problem

Households live for two periods and choose savings and consumption to maximize their lifetime utility. An agent who is young in period t, chooses savings (or assets), a_t, and consumption, $c_{1,t}$, for the first period, and retirement period consumption, $c_{2,t+1}$, to maximize utility, taking prices, and government social security policy (tax rate τ_t, and benefit z_{t+1}) as given. Agents supply labor inelastically in the first period and are retired during the second period. Agents receive the wage $A_t w_t$ where w_t is the wage in effective units for labor provided in period t. The gross real return on savings is given by R_{t+1}.

The labor force (and thus, the population) grows at rate n such that $L_t = (1+n)L_{t-1}$. Labor-augmenting technology grows at rate g such that $A_t = (1+g)A_{t-1}$.

The household's problem can be written as follows:

$$\max_{c_{1,t},\, a_t,\, c_{2,t+1}} u(c_{1,t}) + \beta E_t u(c_{2,t+1})$$

$$\text{s.t. } c_{1,t} + a_t = (1-\tau_t) w_t A_t$$

$$c_{2,t+1} = R_{t+1} a_t + z_{t+1}$$

Here the notation E_t is used to denote the time t conditional expectation. Optimization leads to the household first order condition:

$$u'(c_{1,t}) = \beta E_t R_{t+1} u'(c_{2,t+1}).$$

Assuming log-preferences, the first order condition implies the optimal savings choice:

$$a_t^* = \frac{\beta}{1+\beta}(1-\tau_t) w_t A_t - E_t\left(\frac{z_{t+1}}{(1+\beta)R_{t+1}}\right) \tag{2.1}$$

where a_t^* indicates the optimal choice.

Firms

Firms produce output using a Cobb-Douglass production with labor-augmenting technology (A_t). Aggregate output, Y_t is given by:

$$Y_t = F(K_t, L_t A_t) = K_t^{\alpha}(L_t A_t)^{1-\alpha}.$$

This production function can be written in per-effective worker or efficient terms as $y_t = f(k_t) = k_t^{\alpha}$ where $k_t = K_t/(A_t L_t)$ and $y_t = Y_t/(A_t L_t)$.

Firms are competitive which leads to interest rate and wage:

$$R_t = f'(k_t) + 1 - \delta \tag{2.2}$$

$$w_t = f(k_t) - k_t f'(k_t) \tag{2.3}$$

For convenience, I assume full depreciation ($\delta = 1$). This assumption allows me to solve the non-stochastic model analytically.

Government

The government runs a modified pay-as-you-go social security system funded via pay-roll taxes (rate τ_t) and the issuance of bonds B_t. Social security benefits are designed to replace a fraction, $\phi_t \in [0, 1]$, of an agent's wage. Social security benefits are wage-indexed, such that benefits in time period t are calculated as the replacement rate ϕ_t multiplied by the wage at time t:

$$z_t = \phi_t w_t A_t \tag{2.4}$$

The government issues one-period bonds, B_{t+1} (issued in period t, repaid in period $t + 1$) to satisfy the following flow budget constraint:

$$B_{t+1} = R_t B_t + \phi_t A_t w_t L_{t-1} - \tau_t A_t w_t L_t \tag{2.5}$$

where $\phi_t A_t w_t L_{t-1} - \tau_t A_t w_t L_t$ is the period t deficit (transfers minus revenues).

Markets

This economy has four markets: the labor market, the capital market, the bond market, and the goods market. In equilibrium, all markets clear. Agents supply labor inelastically. The wage rate adjusts to clear the labor market. Capital market clearing requires that the savings of all agents in period t is equal to aggregate capital and government debt in period $t+1$. Agents are indifferent between capital and debt, since both assets offer the same, safe return. The capital market clearing equation can be written as follows:

$$K_{t+1} + B_{t+1} = a_t^* L_t$$

Bond market clearing is given by the government's flow budget constraint 2.5, and the goods market clears by Walras law.

Along the balanced growth path, effective labor hours ($A_t L_t$), output (Y_t), capital (K_t), and bonds (B_t) all grow at rate $(1+n)(1+g)$. Therefore, it will be convenient to rewrite the market clearing equations in per-effective-hours terms by defining $b_t = B_t/(A_t L_t)$, and $k_t = K_t/(A_t L_t)$. I will refer to variables in per-effective-hours terms as "efficient."

The market clearing equations can be written in efficient terms as:

$$(1+n)(1+g)(k_{t+1} + b_{t+1}) = \frac{a_t^*}{A_t} \tag{2.6}$$

$$(1+n)(1+g)b_{t+1} = R_t b_t + \left(\frac{\phi}{1+n} - \tau\right) w_t \tag{2.7}$$

Equilibrium

Given initial conditions k_0, b_0, a_{-1}, an initial cohort of young and old, population growth rate n, and rate of technological progress g, a competitive equilibrium is a sequences of functions for the households $\{a_t\}_{t=0}^{\infty}$, production plans for the firm, $\{k_t\}_{t=1}^{\infty}$, government bonds $\{b_t\}_{t=1}^{\infty}$ factor prices $\{R_t, w_t\}_{t=0}^{\infty}$, government policy variables $\{\tau_t, \phi_t\}_{t=0}^{\infty}$, that satisfy the following conditions:

1. Given factor prices and government policy variables, individuals' decisions solve the household optimization problem (2.1)

2. Factor prices are derived competitively according to (2.2) and (2.3)

3. All markets clear according to (2.6), and (2.7)

The equilibrium path of the state variables, capital and bonds, can be characterized as a non-linear system of first order difference equations. These are called the transition equations. The equilibrium paths of the remaining variables are simply functions of the state variables and exogenous parameters.

The transition equation for bonds is found by plugging the competitive market prices into the government's flow budget constraint (2.7) and rearranging:

$$b_{t+1} = \frac{1}{(1+n)(1+g)}\left[\alpha k_t^{\alpha-1} b_t + \left(\frac{\phi}{1+n} - \tau\right)(1-\alpha)k_t^{\alpha}\right] \tag{2.8}$$

The equilibrium equations for capital is found by plugging the optimal savings decision into the capital market clearing equation, subbing in competitive prices and the social security benefit, and finally subbing in transition equation for bonds (2.8). This leads to the transition equation listed below.

$$\begin{aligned} E_t k_{t+1} = &\left[(1+n)(1+g) + \frac{\phi(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1} \\ &\left[\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)k_t^{\alpha} - \alpha k_t^{\alpha-1} b_t - \left(\frac{\phi}{1+n} - \tau\right)(1-\alpha)k_t^{\alpha}\right] \end{aligned} \tag{2.9}$$

The two transition equations (2.8) and (2.9) govern the dynamics of the system. The transition equations can be written compactly as follows:

$$E_t x_{t+1} = G(x_t) \tag{2.10}$$

where $x_t = (k_t, b_t)'$.

Saddle-node Bifurcation and Steady States

The steady state is the set $\{k, b\}$ for which the system (2.10) is constant. The non-stochastic steady state is a pair $\{k, b\}$ that solves:

$$(1+n)(1+g)b = \alpha k^{\alpha-1}b + \left(\frac{\phi}{1+n} - \tau\right)(1-\alpha)k^{\alpha} \tag{2.11}$$

$$(1+n)(1+g)(k+b) = \frac{\beta}{1+\beta}(1-\tau)(1-\alpha)k^{\alpha} - \frac{\phi(1+g)(1-\alpha)}{(1+\beta)\alpha}k \tag{2.12}$$

The steady state equations are both non-linear in k and can only be solved numerically. Both the existence and number of nontrivial steady states depend on the parameter values. The social security system deficit or surplus is determined endogenously. If the deficit becomes too large, the model no longer has a non-zero steady state. This relationship between steady states and the size of the deficit is a common feature in planar OLG models.[1] The key innovation in this model is that the deficit is endogenous, as described in equation (2.5). The deficit depends directly on the parameters τ, ϕ and n, as well as indirectly on the other parameters (α ,β, δ) through the steady state level of capital.

As an example, consider a parameterization of the model with two steady states. At each of these steady states capital is positive and the government holds a positive amount of bonds. As τ decreases, the government's deficit increases. At a threshold critical value τ_c, the government's deficit is as large as the economy can bear, and only one steady state of the model exists. If τ falls below the threshold τ_c, the deficit is unsustainably large and government debt becomes explosive; there are no steady states. Thus, as τ increases from zero to one, the number of steady states increases from zero (on the interval $\tau \in [0, \tau_c]$) to one (when $\tau = \tau_c$) to two

[1] See Azariadis (1993) Chapters 19 and 20, and De La Croix and Michael (2002) Chapter 4 section 3 for a discussion.

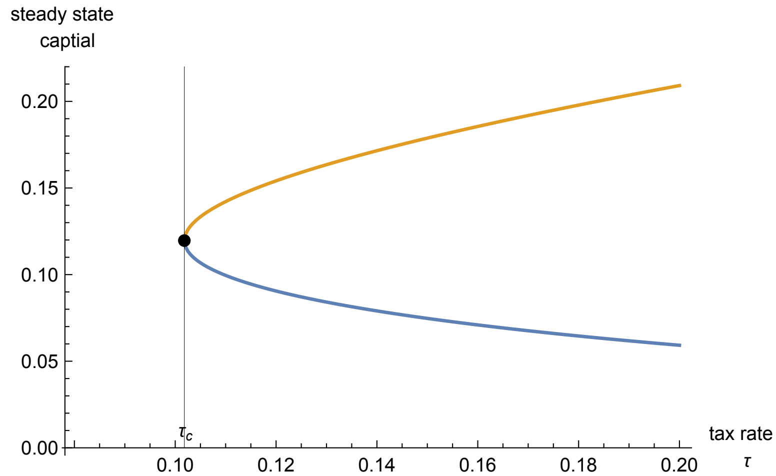


FIGURE 1. Saddle-node bifurcation diagram. For illustrative purposes, this graph plots steady state intensive capital for each tax rate τ. For $\tau < \tau_c$ no steady states exist; for $\tau > \tau_c$ two steady states exist. When $\tau = \tau_c$ one steady state exists.

(for $\tau > \tau_c$). This relationship is called a saddle-node bifurcation and is illustrated in Figure 1.

Similarly, the other parameters that directly affect the deficit (n and ϕ) can be treated as the bifurcation variable, holding all other variables constant. The diagram for n would look qualitatively the same as the diagram for τ presented above. The bifurcations for ϕ moves in the opposite direction as the τ bifurcation; that is, it is an inverse saddle-node bifurcation. At low values of the ϕ, below the bifurcation threshold, two steady states exist; and above the threshold, no steady states exist.

Formally, the saddle-node bifurcation can be described by the equation below:

$$x_{t+1} = G(x_t; \tau) \tag{2.13}$$

This is equation is exactly the same as the transition equation (2.10); the dependence of the steady state on the parameter τ is emphasized in equation (2.13) for notational ease. As τ varies, so will the fixed points of the system (2.13).

Let $D_xG(x_c; \tau_c)$ denote the Jacobian matrix of G evaluated at the steady state (x_c, τ_c). At the critical value τ_c, the Jacobian has a single eigenvalue equal to $+1$, with the other eigenvalue strictly less than one in modulus. Then, for a sufficiently small neighborhood of $(x_c; \tau_c)$, the system G has two hyperbolic steady states for $\tau > \tau_c$, exactly one steady state at τ_c, and no (non-trivial) steady states for $\tau < \tau_c$. In general, the steady states for this model do not have closed-form, analytical solutions, thus the eigenvalues of the Jacobian cannot be evaluated analytically. I use numerical methods to verify the saddle-node bifurcations for each of the relevant bifurcation parameters.

In a special case of the model when the government runs a balanced budget, the steady state equations can be solved analytically. In this special case, the model under goes a transcritical bifurcation rather than a saddle-node bifurcation. A transcritical bifurcation occurs when the stability properties of the fixed points of a system change as an underlying parameter changes. At a critical value of the parameter, a single steady state exists and a single eigenvalue of the Jacobian evaluated at that steady state is equal to $+1$. On either side of the critical value of the parameter, two steady states exist. The stability properties of the steady states switch as the parameter moves through the critical value. The transcritical bifurcation is demonstrated analytically in appendix A.

Stability

The stability of a steady state can be verified by examining the eigenvalues of the Jacobian matrix evaluated at the steady state. Capital and bonds are both predetermined in this model; steady states with two eigenvalues with modulus less than one are stable (the steady state is a sink). Given appropriate initial conditions, the model converges to the stable steady state.

A steady state with one or more eigenvalue greater than one in modulus is unstable, or explosive. Any deviation from the unstable steady state will either cause bonds to grow infinitely, or will converge to the stable steady state (if there are two steady states). At the saddle-node bifurcation (where there is only one steady state), one eigenvalue of the Jacobian will be equal to one and the other eigenvalue will be less than one in modulus.[2]

The eigenvalues and stability of steady states can be depicted graphically for a planar system, such as (2.10). The eigenvalues of a planar system are the solutions (λ) to the characteristic equation which can be written as

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

where λ_1 and λ_2 are the eigenvalues. Note if one of the eigenvalues of the Jacobian is equal to 1, that $p(1) = (1 - \lambda_1)(1 - \lambda_2)$ will be equal to zero. Thus, if we plot the eigenvalues in Trace-Determinant space (plotting the Trace on the horizontal axis, and the Determinant on the vertical axis), the saddle-node bifurcations appear on the segment of $p(1) = 0$ between (0,-1) and (2,1), which indicated by a black line in Figure 2.

[2]See Azariadis (1993) chapter 6 for a discussion of the stability of steady states in two-period OLG models. See Laitner (1990) for a discussion of stability in multi-period OLG models.

In the baseline parameterizations of the model with two steady states, the low capital steady state is explosive and the high capital steady state is stable.[3] The eigenvalues for an example case are plotted in Trace-Determinant space in Figure 2 (recall the product of the eigenvalues is equal to the determinant, and the sum of the eigenvalues is equal to the trace in a two-by-two system). The low-capital steady state is an (unstable) saddle and appears in the right-most region of the Trace-Determinant graph. The high-capital steady state is a sink and appears in the center.[4] I have numerically verified that the eigenvalues of the system approach $p(1) = 0$ along the relevant segment as the bifurcation variable approaches its threshold value.

Phase Diagrams

The dynamics of the system (2.10) can be displayed in (k_t, b_t) space using a phase diagram. The vector fields are determined by the following expressions:

$$b_{t+1} - b_t \tag{2.14}$$

$$k_{t+1} - k_t \tag{2.15}$$

The constant bond locus is the combination of capital and bonds for which bonds are unchanging and occurs when $b_{t+1} - b_t = 0$. Similarly, the constant capital locus plots the points for which $k_{t+1} - k_t = 0$. Using the transition equation for bonds (2.8), expression (2.14) implies changes in bonds are given by:

$$R(k_t)b_t + \left(\frac{\phi}{1+n} - \tau\right) w(k_t) - (1+n)(1+g)b_t \tag{2.16}$$

[3] I have verified the stability of steady states numerically.

[4] Chalk (2000) refers to the low capital steady state in a planar model as the saddle steady state, and refers to the high capital steady state as a sink. This dichotomy is also reflected in Azariadis (1993). I follow their word-choice in this paper and will refer to the high capital (stable) steady state as a sink. I'll also refer to the low capital, explosive steady state as a saddle.

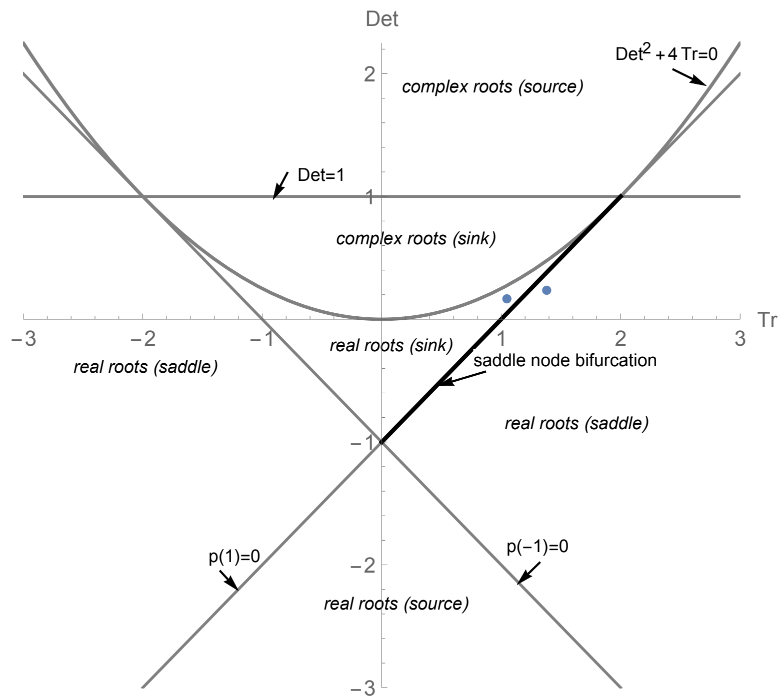


FIGURE 2. Stability of roots for a planar system. The eigenvalues for the standard case of the model are plotted (as Tr-Det pairs). As the policy parameter approaches threshold value for the bifurcation, the eigenvalues approach the line $p(1)$=0 which indicates the saddle-node bifurcation. When a saddle-node bifurcation occurs, the roots will fall along the segment $p(1)$=0 between (0,-1) and (2,1), which is indicated by a thicker line. This graph also includes the line $p(-1)$=0 which indicates a Flip bifurcation (along the segment (-2,1), (-1,0)), and the line $Det(D_xG)$= 0 along which the Hopf bifurcation occurs (along the segment (-2,1) and (1,2)). The graph also includes the parabola $Tr(D_xG)^2 - 4Det(D_xG)$=0. Eigenvalue pairs that fall above (inside) the parabola have complex roots; those that fall below (outside) the parabola have real roots. See Azariadis Figure 6.6 and 8.4 for further discussion.

Where $R(k_t) = \alpha k_t^{\alpha-1}$ and $w(k_t) = (1-\alpha)k_t^{\alpha}$ for notational ease. If we set (2.16) equal to zero, this gives the equation for the constant bond locus. The sign of this expression depends on the value of k_t. If the level of capital, k_t is less than the golden rule level of capital, k_{gr}, then the gross interest rate $R(k_t)$ is *greater* than the golden rule gross interest rate $R(k_{gr}) = (1+n)(1+g)$ and the sign of the denominator in (2.17) below is positive. When $k_t > k_{gr}$ is denominator of (2.17) is negative. The inequality in (2.17) below indicates when the change in bonds is positive (when $b_{t+1} - b_t > 0$). This inequality depends on the value of capital.

$$b_t \begin{cases} > \dfrac{-(\frac{\phi}{1+n} - \tau)w(k_t)}{R(k_t) - (1+n)(1+g)} & \text{if } k_t < k_{gr} \\ < \dfrac{-(\frac{\phi}{1+n} - \tau)w(k_t)}{R(k_t) - (1+n)(1+g)} & \text{if } k_t > k_{gr} \end{cases} \tag{2.17}$$

When k_t is equal to the golden rule level of capital, k_{gr}, the interest rate $R(k_{gr}) = (1+n)(1+g)$ and the denominator in (2.17) is zero.[5]

Using the transition equation for capital (2.9), expression (2.15) implies:

$$\psi^{-1}\left[\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)k_t^{\alpha} - \alpha k_t^{\alpha-1}b_t - \left(\frac{\phi}{1+n} - \tau\right)(1-\alpha)k_t^{\alpha}\right] - k_t \tag{2.18}$$

Where $\psi = (1+n)(1+g) + \phi(1-\alpha)(1+g)/(1+\beta)\alpha$. When (2.18) is equal to zero, this expression give the constant capital locus. Using the factor price notation discussed above, the change in capital is positive (i.e., $k_{t+1} - k_t > 0$) if

$$b_t < \frac{\frac{\beta}{1+\beta}(1-\tau)w(k_t) - (\frac{\phi}{1+n} - \tau)w(k_t) - \psi k_t}{R(k_t)} \tag{2.19}$$

Note that the direction of the inequality of (2.19) holds as long as the interest rate $R(k_t) > 0$, which will always be the case if $k_t \geq 0$.

Figure 3 displays the vector fields implied by (2.14) and (2.15), along with the corresponding constant capital and constant bond loci. The constant bond

[5]The derivation of the golden rule level of capital is given in Appendix A.

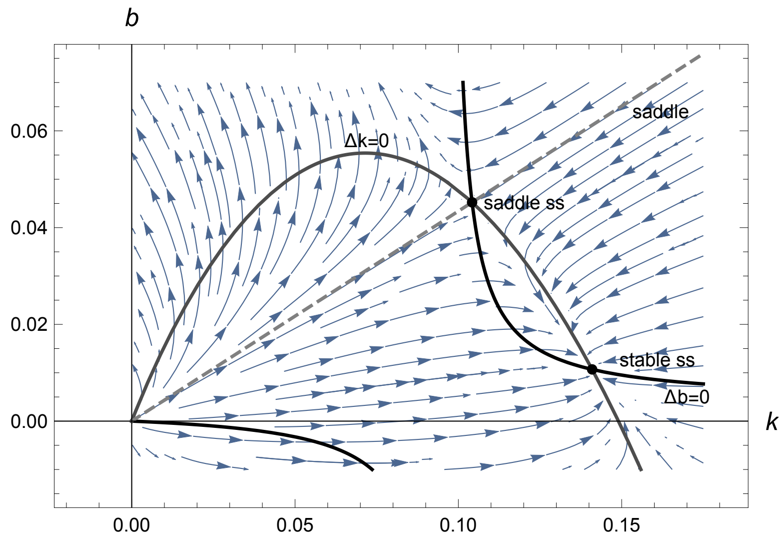


FIGURE 3. Phase diagram, with parameters chosen to show two non-trivial steady states. The constant capital loci is the solid gray line. The constant bond loci is depicted as a black line. The saddle path is indicated by a dashed line. The arrows represent the vector field for the dynamics of the system. The steady states occur at the intersections of the constant capital and constant bond loci.

locus has an asymptote at the golden rule level of capital. The graph depicts the dynamics for a set of parameters that permit two steady states. For this baseline case, I have chosen parameter values that generate steady states with positive values for both bonds and capital.[6]

As Figure 3 illustrates, two dynamically inefficient steady states can arise in this model because of the downward sloping constant capital locus ($\Delta k = 0$) and the downward sloping constant bond locus ($\Delta b = 0$). The constant capital locus is downward sloping (in the relevant region when $k > k_{gr}$) because of crowding out. Increased bond holdings by the government crowds out dynamically inefficient capital and reduces the capital stock. The constant bond locus is downward sloping (in the relevant region when $k > k_{gr}$) because of an interest rate effect. As the

[6]It is also possible to choose parameters such that one of the steady states has negative bonds (see section II).

capital stock decreases, the interest rate increases. This increases the government's debt burden and they must issue new bonds.

If the government changes policy in such a way that deficits increase (either by raising social security benefits, increasing ϕ, or by decreasing the social security tax rate τ), this shifts the constant bond locus out. As the bond locus shifts out, the two steady states move close together. At a certain threshold, the bond locus and capital locus are tangent, and only one steady state exists.[7] The threshold deficit level for a steady state to exist is endogenous. For deficit levels above this threshold, no steady state exists. This saddle-node bifurcation is depicted in Figure 4.

Dynamic Efficiency

I have assumed dynamic inefficiency and endogenous government deficits as the baseline case for this model. This results in steady state(s) with positive bonds and positive capital. In OLG models, debt is only valued if the interest rate is less than the population growth rate; that is, if the economy is dynamically inefficient. Some economists, notably Chalk (2000) and Azariadis, have argued that dynamic inefficiency is representative of the postwar U.S. experience.[8] I am agnostic about this assumption, and allow for dynamic efficiency in Appendix A (with a balanced social security budget) and in Appendix A (with endogenous government surpluses).

[7]Note, that there are several combinations of parameters that can generate this threshold deficit level.

[8] Martin and Ventura (2012) and Gali (2014) also consider dynamically inefficient OLG models with asset bubbles. In these models, agents over-accumulate capital as they save. Asset bubbles (or public pension with deficits), can be used to crowd out dynamically inefficient capital.

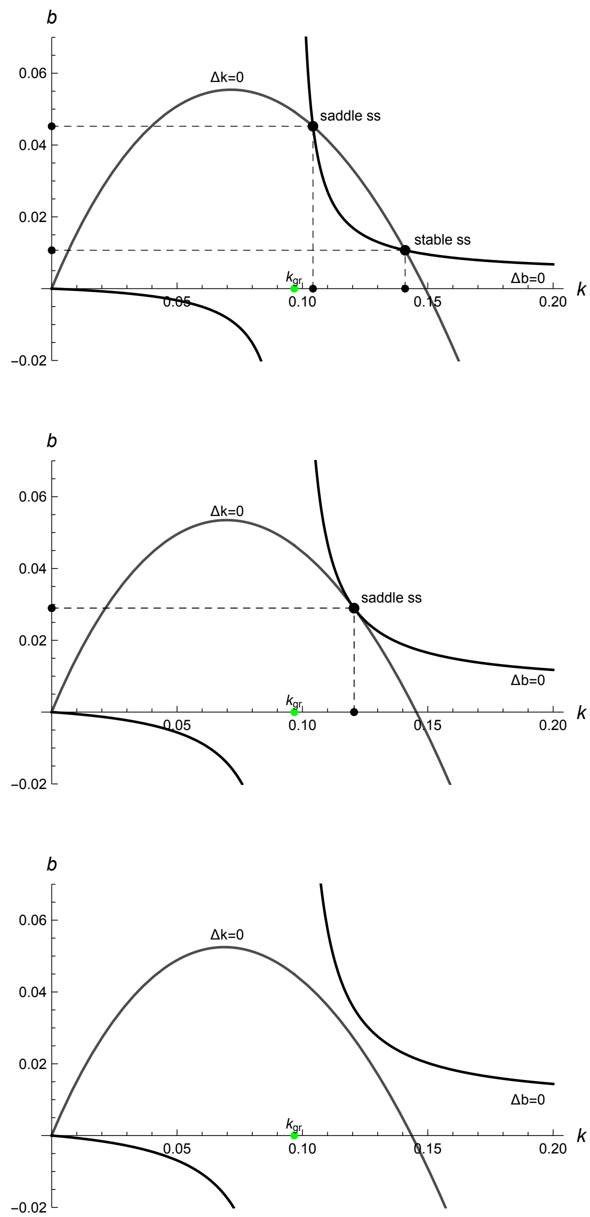


FIGURE 4. Phase diagrams for a dynamically inefficient economy running a deficit in the social security program. As the size of the deficit endogenously grows as an underlying parameter is changed (such as the tax rate being lowered), the constant capital locus shifts out (up and to the left). At a critical value of the parameter, only one steady state exists. When deficits are large enough, the capital locus shifts so that it no longer crosses the bond locus and no steady states exist. This is an illustration of the saddle-node bifurcation.

Explosive Paths and the Basin of Attraction

The parameters of the model govern both the number of steady states in the model, and the basin of attraction for the stable steady state. For the baseline case of the model (with dynamic inefficiency and government deficits), two non-trivial steady states exist as depicted in Figures 3. The basin of attraction for the stable, high capital steady state depends on the saddle path for the saddle steady state. Initial conditions that lie below (to the right of) the saddle path will converge to the stable steady state, and initial conditions above (to the left) of the saddle will not converge. For example, in Figure 5, initial conditions A lead to an explosive path where capital goes to zero in finite time and bonds approach infinity. The initial conditions B converge.

The size of the basin of attraction also depends on the parameters of the model. If the government runs a large (enough) deficit, no stable steady state exists and the basin of attraction disappears. Chalk (2000) emphasizes the existence and size of the basin of attraction is endogenous in a planar OLG model with deficits. Similarly, in my set-up, the possibility of converging to a steady state depends on government policy. Changes in policy alter the basin of attraction and eliminate (or add) initial conditions that otherwise might converge to a steady state. This has clear policy implications; there is not a single level of government debt that is too high and therefore explosive. Rather, the government's ability to carry a (large) debt burden depends on it current and past deficit policies. Running large deficits reduces the basin of attraction and reduces the feasible capital-bond pairings that are convergent in the long-run.

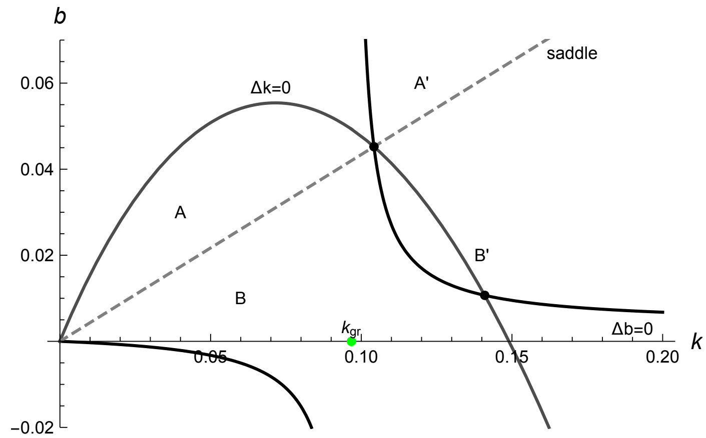


FIGURE 5. Parameters chosen such that two non-trivial steady states exist. Initial conditions above the saddle path, such as points A and A' will not converge to a steady state. Initial conditions below the saddle path, such as points B and B', will converge to the stable, high capital, steady state. The size of the government deficit (or surplus) determines the size of the basin of attraction. Smaller deficits (or larger surpluses below a certain threshold) create a larger basin of attraction.

Explosive Paths

Initial conditions that lie outside the basin of attraction cannot be on an equilibrium path. Similarly, for a given set of initial conditions, government policy may be such that the economy will not converge to a steady state (either because taxes are too low or benefit are too high). I will call paths with rapidly growing government debt "explosive." This naming convention reflects that fact that bonds will explode to infinity, if government policy is not changed.

If the economy is on an explosive path and policy is not corrected, capital will equal zero (or cross the zero threshold) in finite time. Generally, these explosive paths are ruled out of scalar OLG models by arguing that capital cannot go to zero. As capital approaches zero, the interest rate will approach infinity, and increase the incentive to save. This will increase capital and prevent capital from hitting zero. The argument against explosive paths that lead to zero capital in planar models is more nuanced. Although the interest rate increases as capital decreases, savings does not directly translate into capital. Capital can be crowded out by increased government borrowing along the explosive path. Thus, capital *can* approach zero in the planar model.

Suppose the economy is on an explosive path so that the path dictated by equation (2.10) leads to $K_T \leq 0$ for some T. Clearly, $K_T < 0$ is infeasible, so suppose $K_T = 0$. Then, $Y_T = 0$ and the government must default on its debt, i.e., the old at T will not receive either their promised returns on savings or promised benefits. Hence, the young at $T-1$ will instead consume all of their wage income at $T-1$. There will be no demand by the young at $T-1$ for the government debt held by the old at $T-1$ and bought by the young at $T-2$, etc. Hence, this path is infeasible. Ultimately, an explosive path with zero capital is not an equilibrium.

In the set-up of this model, policy ensures a rational expectations equilibrium for a given set of initial conditions (k_0, b_0) even though the paths initially follow an unstable trajectory in the sense that capital is falling towards zero and bonds are increasing. Later in section II, I will consider the case where agents anticipate the government will change policy before capital goes to zero. In this way, I'll be able to examine the dynamics of a path that starts out explosive, but ultimately converges to a steady state.

Parameterization

For the policy analysis that follows, the model will be parameterized to be dynamically inefficient and have two positive bond steady states. The baseline parameters are summarized in Table 1. Agents are assumed to enter the model at

Parameter		Baseline	Note
α	Capital share of output	0.22	Ensures two steady states
β	HH discount rate	0.86	Annual discount factor of 0.995
δ	depreciation rate	1	Allows analytical results
n	population growth rate	0.37	Annual population growth rate of 1.06% (average in US since 1960)
g	growth rate of technology	0.1	Annual wage growth of 0.3%
τ	social security payroll tax	0.106	Corresponds to the OASI portion of social security
ϕ	benefit replacement rate	0.147	Ensures two steady states

TABLE 1. Baseline parameterization for two-period model

age 25; each period lasts 30 years such that retirement occurs at age 55, and agents live to be 85.

Policy Experiments

Before considering the impacts of social security uncertainty on the long-run stability of the economy, it is useful to summarize the effects of announced, perfect-foresight changes to the pension program. Following the presentation of announced policy changes, I will consider uncertain tax changes and uncertain benefit changes.

Announced Policy Changes

If economy is on an unstable path with explosive government debt, either because the initial stock of debt was large relative to the initial capital stock or because the government is running large deficits, the government might be able to change policy in a way that will lead to steady state convergence. The reverse is also true; the government can derail the evolution of the economy by changing the social security system to include large deficits.

There is a natural trade-off between the size of government deficits and how long reform can be delayed. If the government runs a large deficit (or begins with a large initial stock of debt), they will need to make policy changes sooner than a government that runs a small deficit (or begins with a small stock of debt).

If the economy is on an explosive path and policy is not corrected, capital will equal zero (or cross the zero threshold) in finite time. When capital crosses zero (i.e., becomes negative) between periods T and $T+1$, then period T is the last possible date of reform, since it is the last date with positive capital. Finding the last possible date for reform is a simple numerical exercise. As an illustration, I provide six examples in table 2.[9]

An alternative way to view the trade-off between deficits and reforms needed to restore solvency, is to consider reform at a fixed date T, and then numerically compute the smallest benefit cut that could be used to ensure convergence to a steady state. This simply requires searching for the largest possible benefit rate (that is smaller than the initial benefit rate) such that the model converges. If the initial deficit (or stock of debt) is large, than a larger benefit cut is needed for a given T to ensure solvency. Table 3 illustrates the minimum benefit cut needed for four explosive paths.

[9]This exercise is similar to R. Evans et al. (2012). They construct a stochastic OLG model with a fixed transfer from the young to the old and run simulations to see how many periods it takes for the economy to shut down (i.e., zero capital). As expected, they find that larger government transfers result in more rapid economic shutdown.

	baseline	Explosive paths (high benefits)			Explosive paths (high initial debt)		
b_0	0.01	0.01	0.01	0.01	0.03	0.04	0.05
τ	0.106	0.106	0.106	0.106	0.106	0.106	0.106
ϕ	0.147	0.15	0.17	0.2	0.147	0.147	0.147
last reform date		1139	7	4	8	5	3

TABLE 2. The latest possible reform date for various explosive paths. The initial values of capital and bonds, and policy parameters are listed in the table. The other parameters are held constant at $k_0 = 0.1$ (initial capital value), $\tau = 0.106$, $\alpha = 0.22$, $\beta = 0.86$, $n = 0.37$ and $g = 0.1$. See table 1 for calibration details. The model has two steady states with initial conditions of (0.1, 0.01) when $\tau = 0.106$ and replacement rate of $\phi = 0.147$. The stable steady state associated with these baseline parameters is (0.1054, 0.0034).

	baseline	Explosive paths (high benefits)		Explosive paths (high initial debt)	
b_0	0.01	0.01	0.01	0.03	0.04
ϕ	0.147	0.15	0.17	0.147	0.147
given reform date		5	5	5	5
Min new benefit ϕ_2		0.149996	0.11101	0.12508	<0

TABLE 3. The smallest possible benefit increase for various explosive paths. The initial values of capital and bonds, and policy parameters are listed in the table. The other parameters are held constant at $k_0 = 0.1$ (initial capital value), $\tau = 0.106$, $\alpha = 0.22$, $\beta = 0.86$, $n = 0.37$ and $g = 0.1$. See table 1 for calibration details. The model has two steady states with initial conditions of (0.1, 0.01) when $\tau = 0.106$ and replacement rate of $\phi = 0.147$. The stable steady state associated with these baseline parameters is (0.1054, 0.0034).

Tax Rate Uncertainty

Suppose that taxes are stochastic and time varying, but that the benefit rate ϕ is constant. Changing the social security tax rate changes the endogenous deficits of the model and alters both debt and capital accumulation, as discussed above. Somewhat surprisingly, allowing taxes to change over time does not alter household behavior. In other words, the household's decision does not depend on the expectation of future tax rates. Stochastic changes in the tax rate affect the model exactly the same as the announced changes discussed in section II.

Young agents in period t make their saving consumption decision based on the expected rate of return R_{t+1} and their expected social security benefit z_{t+1} according to equation (2.1). The expectation drops out of the right hand side of the equation since second period consumption is predetermined (because the rate of return and the social security benefit are both predetermined). The savings decision of the young facing tax rate uncertainty, is the same as the savings decision of the young in the non-stochastic model. Similarly, the retired agents make the same decision under tax rate uncertainty as they do under certainty. This is because retired agents do not make any decisions; they simply consume their savings and social security benefit. Together, the household decisions in period t are not influenced by the tax rate in period $t + 1$. Any objective or subjective uncertainty about future tax rates will have the same (lack of) effect on the households. This result is sensitive to the model specification and could be changed if agents had a bequest motive, faced a labor-leisure choice, or if the model was extended to three periods.

The remainder of the paper will focus on policy changes that alter the benefit rate. Although tax changes do impact the model and can be used to correct explosive paths, they are somewhat less interesting than benefit changes, since there is no forward looking component with tax changes (in this simple set-up). The emphasis on benefit changes could also be viewed as political intolerance for higher taxes (although benefit cuts also seem political unpalatable).

Replacement Rate Uncertainty

Changes in the social security replacement rate, ϕ_t, change the size of government deficits, *and* alter young agents savings decisions. Specifically, the savings decision of the young is implied by the first order condition equation reprinted here:

$$u'(c_{1,t}) = \beta E_t R_{t+1} u'(c_{2,t+1}) \tag{2.20}$$

The agent's expectation of second period consumption $c_{2,t+1}$ (and therefore, the marginal utility of second period consumption), depend on her expectations about the replacement rate. The right hand side of 2.20 is the expectation of a non-linear function of ϕ_{t+1}. Suppose that the realization ϕ_{t+1} depends on ϕ_t, and is contained in the set Φ. Suppose further that the probability of realizing a particular value of ϕ_{t+1} given ϕ_t is described by $\pi(\phi_{t+1}|\phi_t)$. Then, the expected value in equation (2.20) can be written as:

$$u'(c_{1,t}) = \beta \sum_{\phi_{t+1} \in \Phi} \pi(\phi_{t+1}|\phi_t) R_{t+1} u'(c_{2,t+1}) \tag{2.21}$$

I will consider a simple stochastic processes for ϕ_{t+1}: the government lowers the social security benefit rate ϕ to a known level ϕ_{new} with a fixed probability p each period. In Appendix A I outline the process for reform that depends on the government's stock of debt.

Replacement Rate Uncertainty: Fixed Probability of Reform

Suppose the economy is on an explosive path such that benefits will need to be cut in the future (for simplicity, we will assume the government is unable or unwilling to change the tax rate). Suppose the current benefit rate ϕ is given by ϕ_{old}. The government can delay reform until period T, at which point it will cut benefits to rate ϕ_l (where the l indicates "low" or "last"). This last minute intervention is needed to prevent capital from going to zero, the government defaulting on its debt, and the economy unwinding in the preceding periods (as outlined earlier in the paper). If the government acts sooner than period T, the benefit cut need not be as large.

Suppose that there is fixed probability p that the government will cut benefits to known rate $\phi_{new} \geq \phi_l$ next period (up to period $T-1$). If the government hasn't cut benefits by period $T-1$ (that is, if $\phi_{T-1} = \phi_{old}$), everyone in the economy knows that benefits will be cut with certainty in period T. If the government cuts benefits at or before period $T-1$, call that period $\hat{T}$.

For all period $t < \hat{T} \leq T-1$, there is a fixed probability p that $\phi_{t+1} = \phi_{new}$. Thus, for periods $t < \hat{T}$, ϕ_t follows a two-state Markov switching process with an absorbing state. The transition matrix Π for the Markov process is given by:

$$\Pi = \begin{bmatrix} (1-p) & p \\ 0 & 1 \end{bmatrix}$$

Consider the following example. Suppose that the last possible date for reform is four periods in the future, $T = 4$. If reform has not happened by period $t = 3$, agents know that $\phi_4 = \phi_l$. If reform happens in period $t = 3$ or earlier, the new benefit level is ϕ_{new}. There are four possible paths for the evolution of the benefit rate given Table 4.

	Time period			
Probability	1	2	3	4
p	ϕ_{new}	ϕ_{new}	ϕ_{new}	ϕ_{new}
$(1-p)p$	ϕ_{old}	ϕ_{new}	ϕ_{new}	ϕ_{new}
$(1-p)^2p$	ϕ_{old}	ϕ_{old}	ϕ_{new}	ϕ_{new}
$(1-p)^3p$	ϕ_{old}	ϕ_{old}	ϕ_{old}	ϕ_l

TABLE 4. Illustrative example of possible time paths of the replacement rate. In this example, the last possible date for reform is $T = 4$, at which point the benefit rate is cut to ϕ_l if reform hasn't already happened. In the proceeding periods, there is a probability p that benefits get cut to ϕ_{new}, and a $(1 - p)$ chance that benefits remain unchanged at level ϕ_{old}. The table shows the four possible time paths. The left column shows the probability of each path.

Given this simple process for the evolution of ϕ_t, the household's first order condition (2.21) for periods before reform $t < \hat{T}$ is given by:

$$u'(c_{1,t}) = \beta\left[pR_{t+1}u'(c_{2,t+1}(\phi_{new})) + (1-p)R_{t+1}u'(c_{2,t+1}(\phi_{old}))\right]$$

Plugging in the equations for $c_{1,t}$ and $c_{2,t+1}$ and emphasizing that the social security benefit received z_{t+1} is a function of the realized replacement rate:

$$\begin{aligned} u'((1-\tau)w_tA_t - a_t) = \beta[&pR_{t+1}u'(R_{t+1}a_t + z_{t+1}(\phi_{new})) \\ &+ (1-p)R_{t+1}u'(R_{t+1}a_t + z_{t+1}(\phi_{old}))] \end{aligned} \tag{2.22}$$

$$\frac{1}{(1-\tau)w_tA_t - a_t} = \beta[\frac{pR_{t+1}}{R_{t+1}a_t + z_{t+1}(\phi_{new})} + \frac{(1-p)R_{t+1}}{R_{t+1}a_t + z_{t+1}(\phi_{old})}]$$

The optimal savings choice under uncertainty a_t^* is defined implicitly by equation (2.22). Let the first order condition be written as follows:

$$F = u'(c_{1,t}) - \beta\left[pR_{t+1}u'(c_{2,t+1}(\phi_{new})) + (1-p)R_{t+1}u'(c_{2,t+1}(\phi_{old}))\right] = 0$$

The implicit function theorem can be used to evaluate the the partial derivatives of the savings function. The savings function is increasing in p as shown in equation (2.23) below. The numerator and denominator are both positive, since the interest rate and consumption in both periods are positive as long as capital is greater than zero. If the probability of a benefit cut tomorrow were increased, the agents in the model would save more to off-set the benefit cut.

$$\frac{\partial a_t}{\partial p} = \frac{-\partial F/\partial p}{\partial F/\partial a_t} = \frac{\beta R_{t+1}\left(\frac{1}{c_{2,t+1}(\phi_{new})} + \frac{1}{c_{2,t+1}(\phi_{old})}\right)}{\frac{1}{c_{1,t}^2} + \beta R_{t+1}^2\left(\frac{p}{(c_{2,t+1}(\phi_{new}))^2} + \frac{1-p}{(c_{2,t+1}(\phi_{old}))^2}\right)} > 0 \tag{2.23}$$

The savings function is also increasing in the new benefit rate, ϕ_{new}, as show in equation (2.24). If the future benefit rate is large, the agents save less, since they anticipate larger benefits. If the future benefit is low, they save more.

$$\frac{\partial a_t}{\partial \phi_{new}} = \frac{-\partial F/\partial \phi_{new}}{\partial F/\partial a_t} = \frac{\frac{\beta R_{t+1} p}{(c_{2,t+1}(\phi_{new}))^2} A_{t+1} w_{t+1}}{\frac{1}{c_{1,t}^2} + \beta R_{t+1}^2\left(\frac{p}{(c_{2,t+1}(\phi_{new}))^2} + \frac{1-p}{(c_{2,t+1}(\phi_{old}))^2}\right)} > 0 \tag{2.24}$$

Note, that the future benefit rate influences savings even if the probability of reform, p is quite low. Agents change their behavior even if the threat of a benefit cut is small. The rate of change in savings in response to changes in the benefit rate decreases as the probability of reform increases. Similarly, the increase in savings due to increases in the probability of reform decreases as the as size of future benefits increases. This is evidenced by the cross-partial derivative in equation (2.25).

$$\frac{\partial^2 a_t}{\partial \phi_{new} \partial p} = \frac{-\frac{\beta R_{t+1} A_{t+1} w_{t+1}}{(c_{2,t+1}(\phi_{new}))^2} \beta R_{t+1}^2 \left(\frac{1}{(c_{2,t+1}(\phi_{new}))^2} - \frac{1}{(c_{2,t+1}(\phi_{old}))^2}\right)}{\left(\frac{1}{c_{1,t}^2} + \beta R_{t+1}^2\left(\frac{p}{(c_{2,t+1}(\phi_{new}))^2} + \frac{1-p}{(c_{2,t+1}(\phi_{old}))^2}\right)\right)^2} \tag{2.25}$$

The partial derivative is negative if $\phi_{new} < \phi_{old}$. This is because second period consumption is an increasing function of the benefit rate. If the new benefit rate is lower, new second period consumption is also lower. When this is true, the numerator of equation (2.25) is negative, since it is the product of a negative term and a positive term. The first term is negative because of the negative sign. The second (last) term of the numerator is positive (if $\phi_{new} < \phi_{old}$) as show below:

$$\begin{aligned} &\phi_{new} < \phi_{old} \\ \Longrightarrow\ & (c_{2,t+1}(\phi_{new}))^2 < (c_{2,t+1}(\phi_{old}))^2 \\ \Longrightarrow\ & \frac{1}{(c_{2,t+1}(\phi_{new}))^2} > \frac{1}{(c_{2,t+1}(\phi_{old}))^2} \\ \Longrightarrow\ & \frac{1}{(c_{2,t+1}(\phi_{new}))^2} - \frac{1}{(c_{2,t+1}(\phi_{old}))^2} > 0 \end{aligned}$$

The denominator of (2.25) is always positive. Thus the fraction is negative if $\phi_{new} < \phi_{old}$. These comparative static exercises confirm the basic intuition that agents will save more if there is a risk that they will receive smaller social security benefits when they retire.

The optimal savings choice under uncertainty a_t^* cannot be solved analytically; however, it can be easily approximated using a numeric solver. The numeric approximation of a_t can then be used with the market clearing conditions, and price equations, to numerically approximate the transition path for the economy. After reform takes place (either in period $\hat{T}$ or at the last possible minute in period T), the dynamics of the system are governed by the non-stochastic transition equations, since the uncertainty has been resolved.

Policy Timing Uncertainty: Uniform Reform

As a beginning example, consider an economy that faces uncertainty about the timing of a reform. Agents know that the government will cut benefits to a given amount ϕ_{new} in the future, they are just unsure about when. For simplicity, assume also that $\phi_{new} = \phi_l$. That means, that regardless when reform happens, benefits get cut to the level ϕ_l which the smallest benefit cut that could take place in the last period to put the economy on a stable path. Thus, the economy evolves to the same steady state, regardless of when the reform takes place. If reform takes place sooner, the economy evolves to the new steady state sooner. If reform is delayed, it takes longer to reach the steady state. Since reform requires social security benefits to be cut, agents would prefer the reform to be after their lifetime, all else equal.

Figure 6 shows the evolution of capital and bonds for an economy facing uncertain reform with constant probability p. The last possible date for reform, for the given parameterization of the model, is $T = 7$. The benefit cut required in period 7 is relatively large (the new benefit rate is lower than the tax rate, so the government runs a small surplus each period to pay down the debt it accumulated), so the economy evolves to a steady state with negative bonds. This is just a result of parameterization; it's also possible to have reforms that lead to steady states with positive bonds (especially if the government reforms sooner). The graphs include the time paths of each possible evolution of the benefit rate ϕ. The economy reaches the same steady state regardless of when the reform takes place. The graph also includes consumption and savings by the young, and the consumption of the old.

The final graph included in Figure 6, depicts the lifetime utility of each generation. Recall lifetime utility for an agent born in period t is defined as $Log(c_{1,t}) + \beta Log(c_{2,t+1})$. Agents who retire in the first period after reform (the first period that benefits are cut), have the lowest lifetime utility, compared to those born before or after the reform. These agents fare the worst, since they were hit with a benefit cut, and so their second period consumption is lower than they would have preferred. Agents who retire after the reform have higher lifetime utility that those in the moment of reform since the aggregate capital stock is higher, so wages and social security benefits are higher. The agents alive after the reform live in a world with low benefits, but higher capital; essentially a good economy with low benefits. In contrast, in the moment of reform, agents experience low benefits and low capital stock. Finally, if there are multiple periods before reform, each generation alive prior to the reform has lower utility than the previous generation, because the capital stock is falling. Agents have the highest utility in the first generation; this is because the capital stock is (relatively) high, and they escape the reform.

The government can achieve similar results implementing tax changes. As discussed in section II, tax changes do not alter behavior in advance; tax changes only alter contemporaneous choices by agents. Figure 7 depicts the time paths for the economy that faces a fixed probability p of tax reform in any period. The tax increase is the same, regardless of when it takes place. The last period for tax reform (given the initial conditions), is in period 7. Thus, if reform has not occurred by period 6, it occurs with certainty in period 7. The tax increase is very large, and leads to a steady state with negative bonds. Note, that the timing of tax reform is different than benefit reform. A tax reform that is announced in period t, changes the tax rate in period t and impacts the decisions during period t. A benefit reform that is announced in period t, takes place in period $t+1$, and impacts decisions during t.

Policy Uncertainty: History Dependent Reform

In the example above, the government made the same, relatively large, benefit cut regardless of when reform occurred. This example fails to capture the inherit trade-off between the minimum size of reform needed to ensure stability

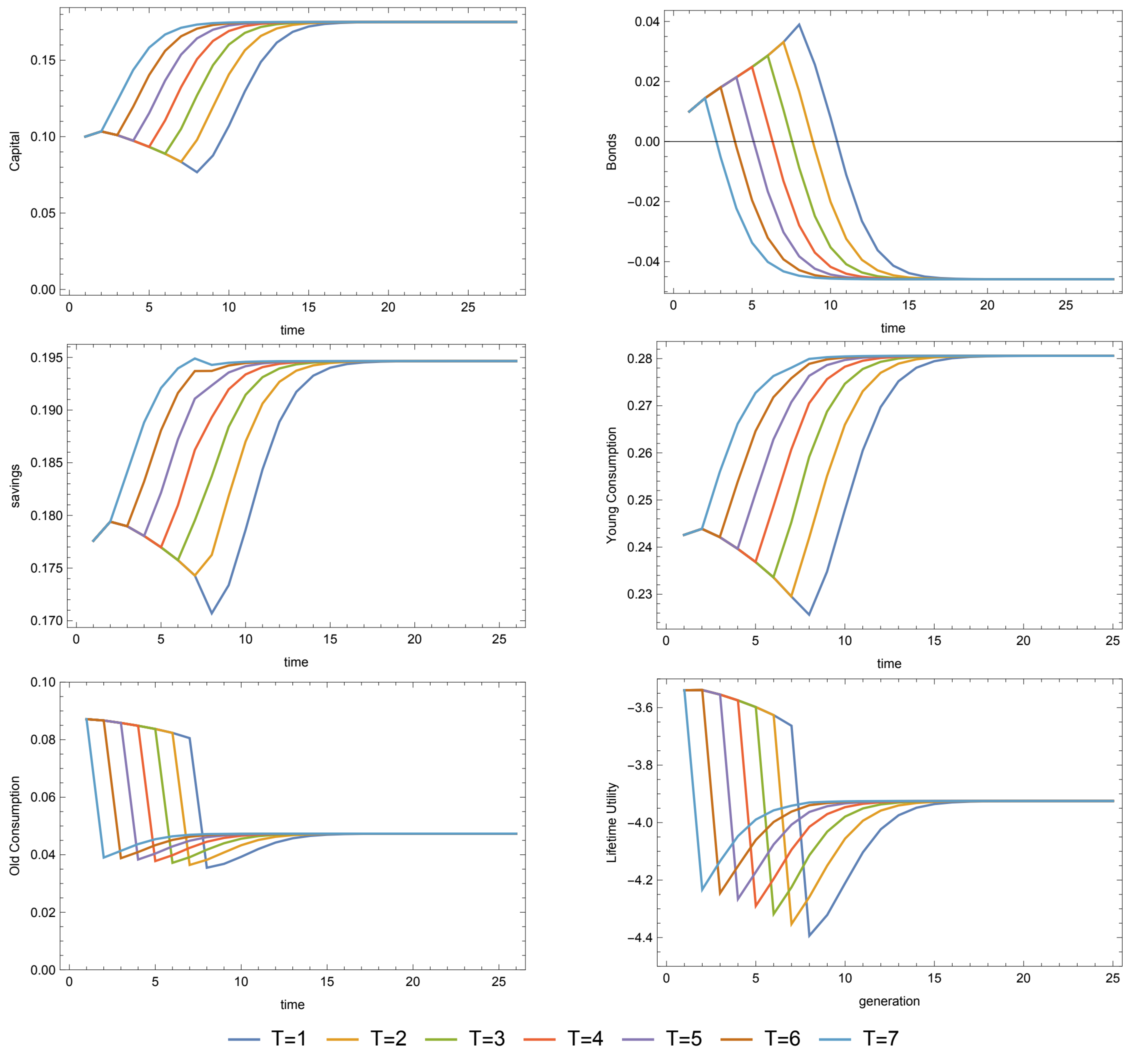


FIGURE 6. Possible time paths for economy given fixed probability p of benefit reform in each period. The reform is announced in period T. The top left shows the time paths for capital, top right is bonds, middle left is savings, middle right is young consumption, and bottom left is old consumption. The final plot (the lower right), graphs the lifetime utility of the various generations before, after, and during reform. For this example, the initial values of capital and bonds are $(0.1, 0.01)$, tax rate $\tau = 0.106$, the initial benefit replacement rate $\phi_{old} = 0.17$, and the new replacement rate is $\phi_{new} = \phi_l = 0.068$. The probability of reform next period is $p = 0.9$. The steady state values of capital and bonds are $(0.17, -0.04)$. The other parameters in the model $(\beta,\ n,\ g)$ have the same values as in all simulations in the paper.

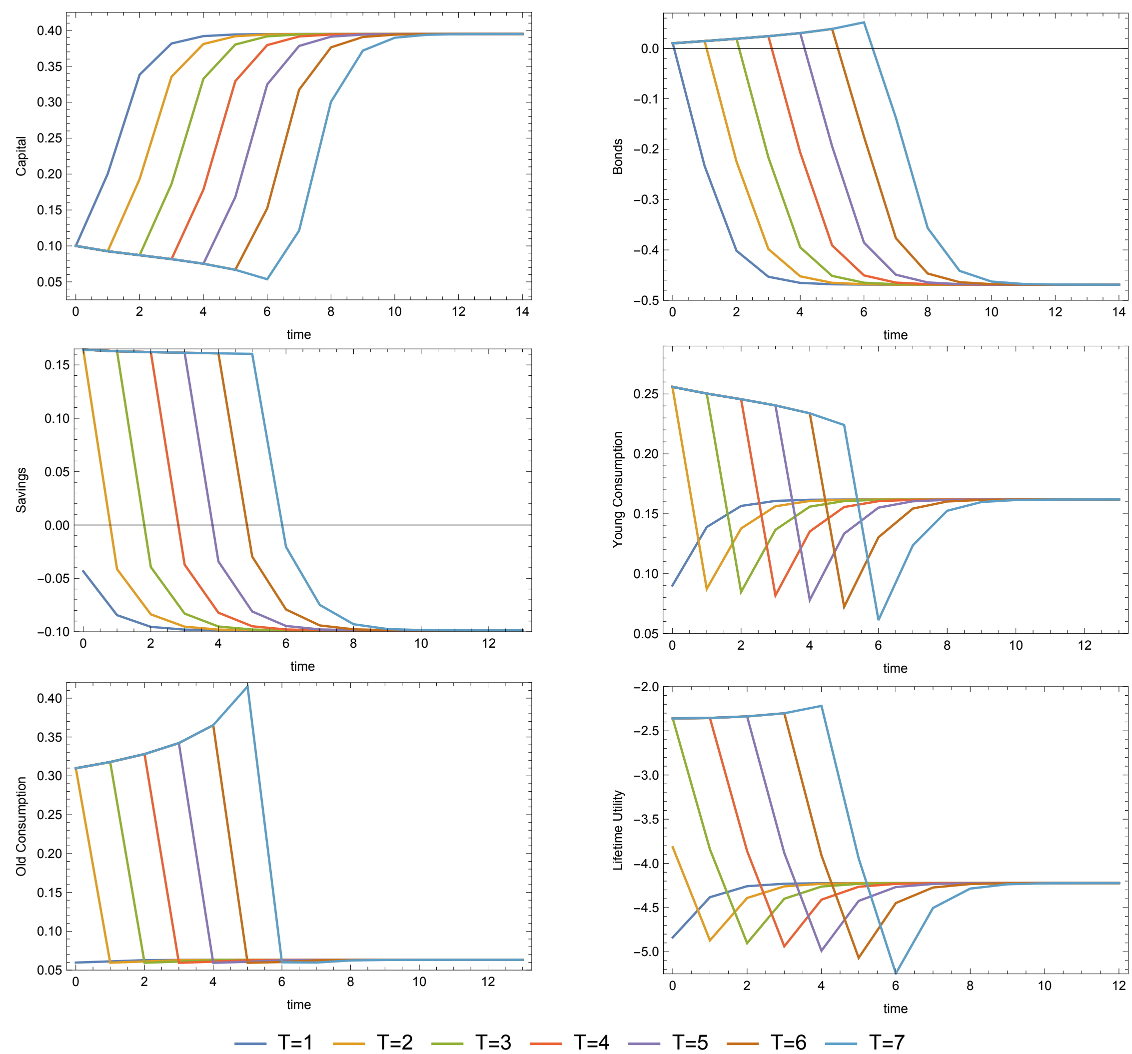


FIGURE 7. Possible time paths for economy given fixed probability p of tax reform in each period. The reform takes place in period T. The top left shows the time paths for capital, top right is bonds, middle left is savings, middle right is young consumption, and bottom left is old consumption. The final plot (the lower right), graphs the lifetime utility of the various generations before, after, and during reform. For this example, the initial values of capital and bonds are $(0.1, 0.01)$, th benefit replacement rate $\phi = 0.17$. The initial tax rate is 0.106, and the new tax rate is $\tau_{new} = 0.90$. The probability of reform next period is $p = 0.9$. The steady state values of capital and bonds are $(0.39, -0.46)$. The other parameters in the model (β, n, g) have the same values as in all simulations in the paper.

and the length of time before reform (discussed in section II). The model above can be modified slightly to allow for a more sophisticated policy process. Suppose, as before, that the economy is on an unstable trajectory and the last possible period for reform is period T. If the government waits to reform pensions until period $T-1$, they will cut the benefit rate to ϕ_l, effective in period T. However, if the government reforms before period $T-1$, they will make a smaller benefit cut. If the government acts, they will make the smallest possible benefit cut that ensures long-run stability, $\phi_{new,t}$. As in the model above, in period t, there is a probability p chance that the government changes the replacement rate in $t+1$ (for all periods before $T-1$). The forward-looking rational expectation agents can accurately predict what the benefit cut will be for any given reform date. The young make their saving-consumption decision based on the first order condition:

$$\frac{1}{(1-\tau)w_t A_t - a_t} = \beta[\frac{pR_{t+1}}{R_{t+1}a_t + z_{t+1}(\phi_{new,t})} + \frac{(1-p)R_{t+1}}{R_{t+1}a_t + z_{t+1}(\phi_{old})}] \tag{2.26}$$

Equation (2.26) is identical to equation (2.22), except that the new benefit rate $\phi_{new,t}$ is time indexed. The evolution of the economy under this type of policy reform can be solved recursively, moving forward. If reform happens in period $t = 1$, then benefits are cut to $\phi_{new,1}$ in the next period. The uncertainty is resolved, and the path of the economy is known. If reform does not happen, the state vector (k_1, b_1) can be used as an initial condition to solve for the minimum needed reform in period $t = 2$. This process can be repeated until the economy reaches period $T-1$.

Figure 8 and Table 5 show the possible time paths for an economy that starts with the same initial conditions as the economy in section II, but that faces time-dependent minimum reform $\phi_{new,t}$ with probability p. Table 5 shows the critical value of ϕ that the government must enact in the following period to ensure convergence to a steady state. The longer the government waits to reform, the larger benefit cut they must make ($\phi_{new,t}$ is smaller). The table also lists the steady state values of capital and bonds associated with each possible reform. Reforms that take place soon result in higher capital and lower accumulation of bonds. As reform is delayed, the capital stock is depleted and bonds are accumulated, resulting in lower capital, higher debt steady states. However, if reform is delayed long enough, the government must make such a large benefit cut that it starts running endogenous surpluses. This dramatic reduction in social security benefits drives up the savings rate and leads to a steady state with higher capital. The endogenous surpluses lead to a steady state with negative bonds. This happens if reform is delayed until period 6 or 7 in this numeric example.[10] Note that the government must make large benefit cuts (that lead to surpluses) if reform is delayed until period 6 or 7 partially as a result of the types of policies I consider. I have only allowed the government to make a one-time, permanent change to a single parameter. In reality, the government might pursue a more nuanced policy in which they cut benefits dramatically for a short period of time to pay down some outstanding government debt, and then gradually increase benefits again when the debt is at a sustainable level.

The model with fixed probability of a time-dependent benefit cut (section II) provides a sharp contrast to the model with a fixed probability of the same benefit cut (section II). The two numeric examples presented have the same initial conditions, the same probability of reform next period, and the same constant policy parameters. The only difference is the uniform benefit cut enacts the same

[10] Reform in period 5 requires a benefit cut that creates small surpluses each period. The surpluses are small enough that they never fully offset the debt accumulated in the first four periods. Thus, the steady state still has positive bonds. Reform in period 6 or 7 requires a large enough benefit cut that the surpluses drive bonds to a negative steady state.

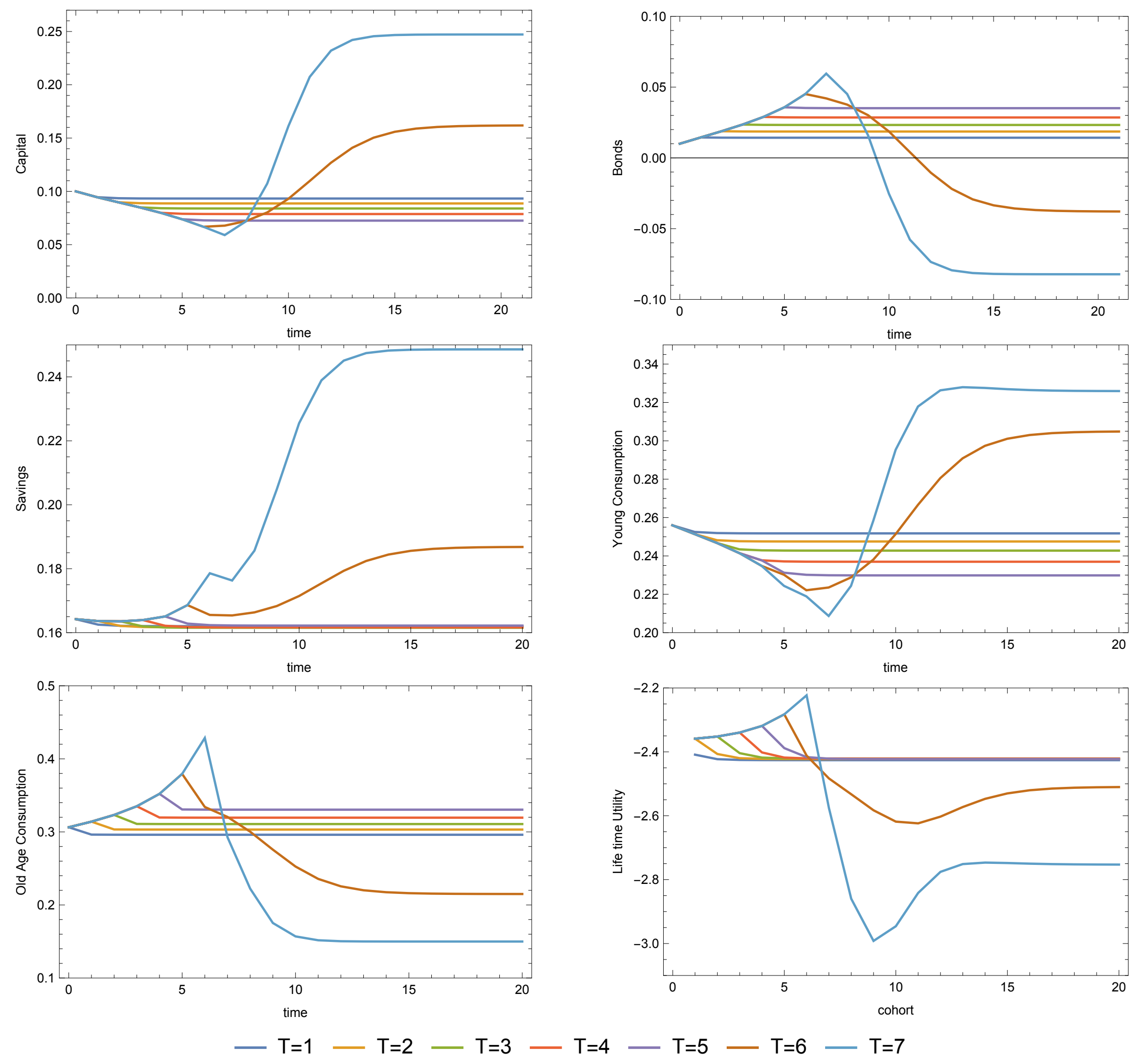


FIGURE 8. Time paths for the economy facing a fixed probability p of history dependent benefit cut $\phi_{new,t}$. The benefit cut required in each period is listed in Table 5. As reform is delayed, larger benefit cuts are required to ensure convergence to a a steady state. Reform that takes place in the first five periods results in qualitatively similar outcomes: capital falls and bonds increase until the reform is enacted. Reform in period 6 or 7 are qualitatively different since the government is required to cut benefits enough that they run a (large) surplus each period. This results in steady state negative bonds, and very high savings which drives up the capital stock. In this parameterization p is 0.9. The initial values for capital and bonds are $(0.1, 0.01)$ for this example. Initial benefits,ϕ_{old}, are set to to 0.17. Taxes,τ, are set at 0.106. All other parameters are equal to their baseline value $(\alpha = 0.22, \beta = 0.86, n = 0.37, g = 0.1)$.

Reform time	1	2	3	4	5	6	7
$\phi_{new,t}$	0.150	0.148	0.144	0.137	0.124	0.086	-0.022
k steady state	0.093	0.089	0.084	0.079	0.073	0.162	0.247
b steady state	0.014	0.019	0.023	0.029	0.035	-0.038	-0.082

TABLE 5. This table lists the rational expectations next period benefit level for each possible period of reform $\phi_{new,t}$. The table also includes the steady state associated with each reform. The benefit cut required for stability increases as the time to reform gets longer. Likewise, the steady state capital after reform gets smaller if reform is delayed (at first). If reform is delayed long enough, the benefit cut will be so large that the government will run a surplus each period, which will increase the steady state capital and result in negative steady state bonds. This is the case if reform doesn't happen until period 6 or 7. Note, that reform in period 6 or 7 is unlikely in this parameterization, since p=0.9. The initial values for capital and bonds are $(0.1, 0.01)$ for this example. Initial benefits are set to ϕ_{old}=0.17. Taxes are τ=0.106. All other parameters are equal to their baseline value ($\alpha = 0.22, \beta = 0.86, n = 0.37, g = 0.1$).

policy change regardless of when the change happens, while the time-dependent model makes the smallest benefit cut possible each period to ensure long-run stability. An obvious difference between the two models are the steady states after each reform. The uniform model results in the same steady state, regardless of when reform happens. In contrast, the time-dependent model converges to a different steady state for each possible reform date.

An additional differences between the models is the evolution of the optimal savings choice. The uniform model results in essentially the same savings decision being made each period, while the history-dependent model has much different savings choices if reform is delayed. In the uniform model, an agent in period $t = 1$ awaiting reform, makes her savings choice on her expected social security benefit which depends on next period's wage, w_2, as well as the benefit rate, ϕ_2. The savings rate is either equal to the old rate ϕ_{old} or the new rate ϕ_{new}. If reform doesn't take place, an agent in period $t = 2$ makes almost exactly the same choice. The young agent in period 2 also saves based on expected benefits, which depend on the next period's wage, w_3, and the benefit rate, ϕ_3. Again, the benefit will either remain unchanged at ϕ_{old} or will fall to ϕ_{new}. The only difference between the savings choice of a young agent awaiting reform in period 1 and a young agent awaiting reform in period 2, is that the next-period wage will be different. The expected benefit rate is the same. This results in a savings choice that looks qualitatively similar regardless of when reform takes place. If reform doesn't happen until period 7, first period consumption and savings falls for each generation until the reform takes place.

This contrasts the optimal savings choice in the time-dependent model, which changes each period that reform is delayed. An agent in period 1 awaiting reform makes a savings decision based on expected benefits which depend on the wage, w_2, and the benefit rate. The benefit rate is either the old rate, ϕ_{old}, or the new rate associated with this reform, $\phi_{new,1}$. If reform doesn't happen, the young agent in period 2, makes a savings choice based on expected prices, w_3, and expected benefit rate which either stays the same at ϕ_{old}, or is cut to the lower rate $\phi_{new,2}$. Thus, the agent awaiting reform in period 2, saves more than the agent awaiting reform in period 1, since the potential benefit cut is larger. If reform doesn't happen until the 7th period, savings increases each time period (with each generation of young), while the youth consumption falls.

The final difference I want to highlight, is the lifetime utility of each generation in both models. Agents in the uniform reform model have qualitatively similar changes in utility. Agents that live during the reform (who experience the benefit cut) have a dramatic reduction in lifetime utility compared to their parents. The generations that come after the reform see their utility increase compared to the reform generation. In contrast, in the model with time-dependent reform, the lifetime utility of agents can be very different depending on when reform occurs. If reform occurs within the first five period, agents alive before the reform have higher utility that those alive after. Agents alive during the change have lower lifetime utility than their parents, but they have higher utility than their children. However, if reform is delayed until period 6 or 7, agents alive during the policy change have much lower utility. Their utility is much lower than their parents, and is also lower than their children's. Lifetime utility by generation does not monotonically decline if reform takes place in period 6 or 7.

It is also possible for the government to enact history dependent tax reform. Future tax rates do not effect the agents behavior, so history dependent tax reform is computationally equivalent to perfect foresight comparative dynamics. Suppose the government increases taxes this period with probability p; if taxes are changed, they are increased to the minimum amount to ensure convergence to a steady state. If taxes are not reformed by period $t = 7$, taxes are changed with certainty. The time paths for a numeric example are presented in Figure 9. The example is similar to the history-dependent benefit cut example above. The initial conditions are the same, and the government takes actions with the same probability $p = 0.9$. The only difference is in this example, taxes are increased rather than benefits being cut. As in the benefit case, the history-dependent tax changes converge to different steady states for each possible reform. Delaying reform until the last period, results in very large tax hike that actually leads to young agents choosing to borrow in the first period. They are able to borrow against their future social security benefits, which are known with certainty. If reform happens earlier, the new tax rate is low enough that agents still choose to save a positive amount in steady state.

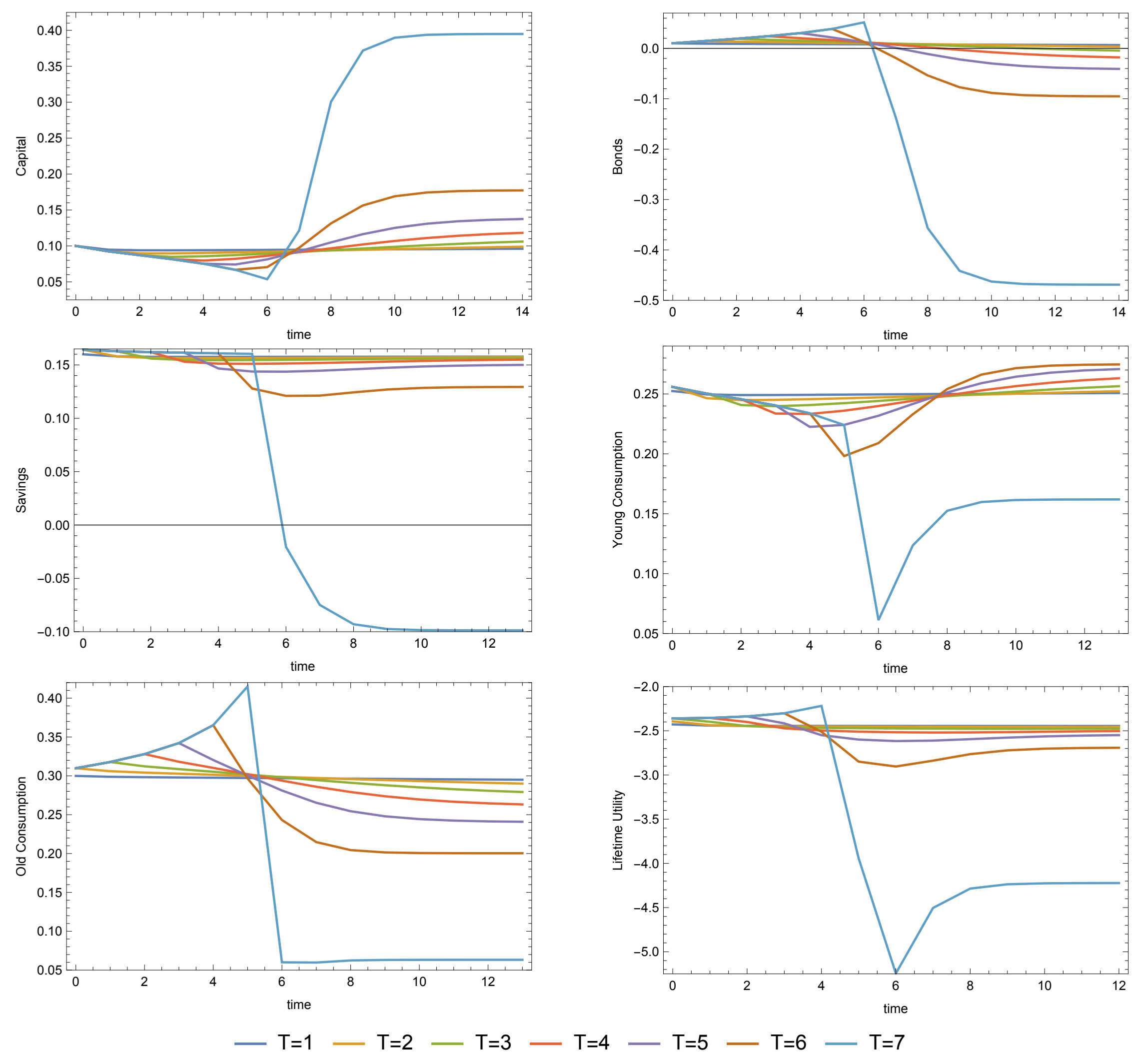


FIGURE 9. Time paths for the economy facing a fixed probability p of history dependent tax increase $\tau_{new,t}$. The tax increase required in each period is listed in Table 6. As reform is delayed, larger tax increases are required to ensure convergence to a a steady state. Reform that takes place in the first six periods results in qualitatively similar outcomes: capital falls and bonds increase until the reform is enacted. Reform in period 7 is different since the tax increase is so large. In this parameterization p is 0.9. The initial values for capital and bonds are $(0.1, 0.01)$ for this example. benefits, ϕ, are set to 0.17. Taxes, τ, are 0.106 before reform. All other parameters are equal to their baseline value $(\alpha = 0.22, \beta = 0.86, n = 0.37, g = 0.1)$.

Reform date	1	2	3	4	5	6	7
τ_{new}	0.123	0.125	0.13	0.14	0.164	0.242	0.901
k stead state	0.099	0.105	0.112	0.122	0.139	0.177	0.395
b steady state	0.004	-0.002	-0.01	-0.021	-0.042	-0.095	-0.469

TABLE 6. This table lists the rational expectations tax increases needed if reform takes place in a given period. Reform takes place during the current period with probability p=0.9. If reform does not take place, there is a p=0.9 probability it takes place next period. If reform has not happened by period 7, it occurs with certainty. If reform is delayed, the benefit cut must be larger. The required tax cuts are large enough that the government runs an endogenous surplus, which leads to steady states with negative bonds. The reforms in this table correspond to Figure 9.

Welfare Analysis

The policy analysis above reveals a trade-off between the size of policy reform (i.e., the magnitude of benefit cuts or tax increases) and the timing of the reform (i.e., how long reform can be postponed). Current agents prefer larger policy changes in the future, and future agents would prefer smaller changes that happen sooner. It is not clear a priori which type of reform would be welfare maximizing for society.

Consider a benevolent social planner than must choose a new social security benefit rate ϕ_{new} and a reform date T to ensure long-run stability for an economy. For simplicity, assume that the tax rate τ is given and the planner cannot change it.[11] The parameters ϕ_{new} and T are chosen jointly to ensure the economy converges to a steady state. The planner's problem can be maximizing a social welfare function:

[11] This assumption could be relaxed to allow the Planner to change either the tax rate, the benefit rate, or both.

$$W = \sum_{t=0}^{T-1} \rho^t (1+n)^t \left[(1+n)c_{1,t}(\phi) + c_{2,t}(\phi)\right] + \sum_{t=T}^{\overline{T}} \rho^t (1+n)^t \left[(1+n)c_{1,t}(\phi_{new}) + c_{2,t}(\phi_{new})\right] \tag{2.27}$$

where $\overline{T} > T$ represents the end of the planning horizon. The discount factor ρ represents both a social discount factor, to weight future generations less, and a discount factor to offset wage growth due to technological innovation. Thus, $\rho \leq \frac{1}{1+g}$. There is some variation in the OLG literature on what should be used as the social discount factor. Feldstein (1985) considers a social discount factor of 1 (which he calls additive welfare) and also $\rho = 1/(1+\eta)$ for $\eta \leq n$. When the discount factor is set equal to $1/(1+n)$, all generations are counted equally, but individuals in larger generations are weighted less (the equivalent discount factor in a model with growth is $\rho = 1/((1+n)(1+g))$) . Sanchez-Marcos and Sanchez-Martin (2006) use a social discount factor of $\rho = \frac{1}{R}$. Olovsson (2010) and Bohn (2009) both use discount factors equivalent to $\rho = u'(c_{2,t})/u'(c_{1,t})$. Bohn (2009) explains that welfare weights of $\rho = 1/E_0\left[u'(c_{1,t})/u'(c_{2,t})\right]$ are the only weights that maximize welfare.

The social welfare function W captures the trade-off between delaying reform and the size of reforms. For example, consider an economy on an explosive path such that last possible date of reform is in $t = 6 = T$, as discussed in section II. The social planner could wait until period $t = 6$ to make policy changes, or the planner could make smaller changes sooner. The welfare associated with the minimum reform at time period $t = 1, 2, \cdots, 6$, is presented in Table 7. If the social planner has a long planning horizon, reforming as soon as possible maximizes welfare. This is unsurprising, since reforming sooner allows the planner to make

Social Welfare of various reforms

	reform date	1	2	3	4	5	6
	ϕ_{new}	0.1497	0.1477	0.1430	0.1333	0.1110	0.0432
$\overline{T}$	100	**2538**	2537	2508	24812	24275	20771
	90	11372	**11389**	11261	11128	10880	9309
	80	5093	**5100**	5068	4995	4875	4172
	70	2278	**2283**	2283	2246	2184	1869
	60	1016	1019	**1020**	1016	977.6	837.0
	50	451.1	452.2	453.1	**453.7**	436.8	374.3
	40	197.6	198.1	198.5	**198.8**	195.2	166.8
	30	84.02	84.24	84.43	84.58	**84.65**	74.32
	20	33.08	33.18	33.27	33.35	33.42	**33.44**
	10	10.25	10.29	10.34	10.38	10.45	**10.55**

TABLE 7. Welfare for various policy reforms, given different planning horizons. In each case, the economy is on an explosive path. The benefit cut considered for each period is the smallest benefit cut that converges to a steady state. Reforming earlier allows the government to make a smaller cut. The bolded values indicate the welfare maximizing reform. For long planning horizons (110 periods) making the small reform in the first period maximizes the welfare function. For a short planning horizon (10 periods), waiting until the last possible minute to reform is the best choice. For planning horizons in between, choosing a policy change date in the middle is the best. The initial benefit rate $\phi = 0.17$. The tax rate $\tau = 0.106$. The initial values for capital and bonds are $(0.1, .0.01)$. The discount factor is $\rho{=}\beta/(1{+}g)$, although the qualitative results are invariant to a variety of discount factors. The other parameters of the model are set to their baseline values.

a smaller benefit cut, which helps future generations, relative to a delayed reform that would require a larger benefit cut. Conversely, if the social planner has a short planning horizon, then delaying reform can maximize welfare, since the older generation(s) who receive large benefits before the reform account for a large fraction of the total welfare. The table uses the discount factor $\rho = \beta/(1+g)$. The results are qualitatively the same if the discount factor is changed to $\rho = 1/(1+g)$ or $\rho = \tilde{\beta}/(1+g)$ for $\tilde{\beta}$ near β (the numeric values change, but welfare is still maximized by the same reform as in the table). Note, that the welfare analysis is different if a generational discount factor is used. This will be discussed below.

If the social discount factor is small enough ($\rho < 1/(1+n)(1+g)$), the planner may use an infinite planning horizon, since the welfare function will converge[12]. After $c_{1,t}$ and $c_{2,t}$ reach their steady state values, the welfare function collapses into a simple geometric series. Suppose the consumption terms reach their steady states in $t = T_{ss}$, and that the steady state value are c_{1ss} and c_{2ss}. Let $\rho = \hat{\rho} < 1/(1+n)(1+g)$. Then, the last terms of the infinite horizon welfare function can be computed as follows:

$$\begin{aligned}
&\sum_{t=T_{ss}}^{\infty} \hat{\rho}\left[(1+n)c_{1,ss} + c_{2,ss}\right] \\
=&\sum_{t=0}^{\infty} \hat{\rho}\left[(1+n)c_{1,ss} + c_{2,ss}\right] - \sum_{t=0}^{T_{ss}} \hat{\rho}\left[(1+n)c_{1,ss} + c_{2,ss}\right] \\
=&\frac{(1+n)c_{1,ss} + c_{2,ss}}{1-\hat{\rho}} - \left[(1+n)c_{1,ss} + c_{2,ss}\right]\frac{1-\hat{\rho}^{T_{ss}+1}}{1-\hat{\rho}} \\
=&\left[(1+n)c_{1,ss} + c_{2,ss}\right]\frac{\hat{\rho}^{T_{ss}+1}}{1-\hat{\rho}}
\end{aligned}$$

The first terms of the infinite horizon welfare function are calculated according to equation (2.27). As an example, the infinite horizon welfare is presented for six different possible reforms in Table 8. The social discount factor used is $\rho = 0.999/(1+n)(1+g)$. The results are qualitatively unchanged if a discount factor of $\beta/(1+n)(1+g)$ is used. Under this specification, using an infinite horizon, welfare is maximized by reforming in the 5th period. Note, that these are the same reforms as those considered in Table 7; the only change is the discount factor.

Table 8 also includes welfare analysis for shorter planning horizons, using the same discount factor of $\rho = 0.999/(1+n)(1+g)$. In contrast to the welfare

[12]Note, that a discount factor of this size is similar to the generational discount factor used by Feldstein. The discount factor accounts for both population growth and technology growth. If $\rho = 1/(1+n)(1+g)$, each generation is treated equally; a discount factor of $\rho < 1/(1+n)(1+g)$, implies that the social planner discounts for population and technology growth and also discounts the future.

Social Welfare of various reforms

	reform date	1	2	3	4	5	6
	ϕ_{new}	0.1497	0.1477	0.1430	0.1333	0.1110	0.0432
$\overline{T}$	∞	6.359	6.387	6.412	6.436	**6.451**	6.423
	100	6.359	6.387	6.412	6.436	**6.451**	6.422
	90	6.358	6.386	6.411	6.435	**6.451**	6.422
	80	6.356	6.384	6.409	6.434	**6.449**	6.420
	70	6.351	6.379	6.405	6.429	**6.444**	6.416
	60	6.339	6.367	6.393	6.417	**6.433**	6.404
	50	6.308	6.335	6.361	6.386	**6.402**	6.373
	40	6.224	6.252	6.277	6.302	**6.323**	6.291
	30	6.006	6.033	6.058	6.083	**6.109**	6.080
	20	5.435	5.461	5.484	5.508	5.534	**5.567**
	10	3.938	3.961	3.981	4.003	4.028	**4.067**

TABLE 8. Welfare for various policy reforms, given different planning horizons. In each case, the economy is on an explosive path. The benefit cut considered for each period is the smallest benefit cut that converges to a steady state. Reforming earlier allows the government to make a smaller cut. The bolded values indicate the welfare maximizing reform. The initial benefit rate ϕ=0.17. The tax rate τ=0.106. The initial values for capital and bonds are $(0.1, .0.01)$. The discount factor is ρ=0.999/(1+n)(1+g). This discount factor was chosen to be smaller than 1/(1+n)(1+g), to ensure that the infinite sum converges when considering a social planner with an infinite planning horizon. The other parameters of the model are set to their baseline values.

analysis in Table 7 above, delaying reform until period 5 is almost always welfare maximizing using a discount factor of $\rho = 0.999/(1+n)(1+g)$. This is because future generations do not count more than current generations, even though they are larger. So, the social planner effectively puts more weight on the current generations who benefit from delaying reform.

The welfare analysis in the non-stochastic model emphasizes the trade-off between reform size and reform timing. Depending on the planner's discount function and planning horizon, any reform timing can be optimal. This illustrates the importance of the social discount factor when there are intergenerational effects. If the planner discounts future generations heavily; then optimal policy will current generation. Similarly, if the planner has a short planning horizon, she will favor current generations. In the numeric examples considered, the planning horizon had to be very long (over 100 periods) *and* the social discount factor could not be too small in order for reforming in the first period to be welfare maximizing. In a two-period model, each period corresponds to roughly 30-40 years; so a 100 period planning horizon is roughly 300 years. The S.S.A. makes 75-year long-run projections, and the C.B.O. publishes a 10-year projection each year. Perhaps a planning horizon of 20 periods or less would better represent government behavior. When a short planning horizon is used, delaying reform is the optimal choice for a policymaker.

Welfare Analysis with Policy Uncertainty

In the previous section, the social welfare function was used to demonstrate the trade-off between making large reforms several periods in the future from making smaller reforms sooner. The social welfare function can also be used to compare the trade-off between facing a given reform at a certain date, with facing uncertain reform.

Suppose the economy is on an explosive path, and the government will reform social security benefits next period with probability p. If the benefit cut takes place, it will be the minimum possible benefit cut. Agents in the model are able to predict the specific benefit cut that would happen in each period, were reform to occur (as in section II). Suppose further that the probability of reform, $p = 0.25$, so that the expected amount of time before reform is four period (expected duration before reform $= p^{-1}$). The minimum benefit cuts needed for periods 1-4 are presented in Table 9.

period	1	2	3	4
$\phi_{new,t}$	0.1497	0.1477	0.1434	0.1345

TABLE 9. Minimum reform needed in each period, given that there is a probability p=0.25 chance of reform in each period. Table includes the benefit cuts for periods 1-4 only, as the expected duration of the old benefit rate is 4 periods. The path of the economy that faces uncertainty and enacts the benefit cut $\phi_{new,4}$ is depicted in Figure 10 as the blue line (circles). The steady state following this reform is $(0.077, 0.030)$. The initial values for capital and bonds are $(0.1, 0.01)$, the tax rate is τ=0.106, and the initial benefit rate is ϕ=0.17. All other parameters are set as in previous examples.

We would like to compare the welfare of the agents in this economy (with policy uncertainty) to the welfare of agents who face a similar policy change with certainty. One natural comparison would be to compare the welfare of agents along the expected time path with uncertainty (that is, the time path when the reform takes place in period 4), with agents who face reform with certainty in the same period (that is, agents who know they will face the policy change in period 4). Since the minimum possible reform in any period is history dependent, the minimum reform for the economy with uncertainty that happens to reform in period 4, will not the be same as the minimum possible reform for an economy that faces reform in period 4 with certainty. This is illustrated in Figure 10. Thus, the welfare comparison is between agents in an economy facing uncertain reform that happens to occur in period 4 (benefits are cut to $\phi_{new,4}$), with the welfare of agents in an economy facing a larger certain reform in period 4 (benefits are cut to $\phi_{new,certain} < \phi_{new,4}$). I compared the the welfare values for these two economies using different discount factors and different planning horizons, and in each case, welfare was higher in the economy with certainty. This result is driven by uncertainty; not by the fact the benefit cut in the certain case is slightly larger than in the uncertain case. Recall, taxes aren't changed in either scenario; the only policy change is a reduction in benefits. Reducing benefits lowers old age consumption and lifetime utility. (Reducing benefits also has general equilibrium impacts by changing the trajectory of the state variables, but these effects appear to be smaller near the policy change). I also verified numerically that welfare is lower in the economy with certainty if benefits are cut further. That is, welfare is lower if benefits are cut to $\phi_{lower} < \phi_{new,certain}$ in period 4, as compared to when benefits are cut to $\phi_{new,certain}$ in period 4.

Figure 10 depicts the time path for capital and bonds for four different scenarios. First, the graph depicts the explosive path of the economy with no reform (capital quickly goes to zero, and bonds increase; this is green line with diamond markers). Second, the graph shows the time-path for the economy that faces uncertainty, and happens to have reform in period 4 (capital and bonds converge to a steady state; this is the blue line depicted with circle points). Benefits are cut to $\phi_{new,4}$ following the fourth period. Third, the graph shows the time path for an economy that cuts benefits to $\phi_{new,4}$ with certainty after 4 periods (yellow line, small square markers). This path traces the explosive path for the first four periods. After the reform, bonds grow slowly and capital still approaches zero. The benefit cut $\phi_{new,4}$ was not enough to make the economy converge since the agents in the first 4 periods did not face any uncertainty and did not engage in an precautionary saving. Finally, the graph depicts the time path for the minimum possible reform that can be enacted in period 4 that will converge (orange line, triangle markers). This benefit cut is larger than the benefit cut under uncertainty (that is, $\phi_{new,certain} < \phi_{new,4}$).

The welfare analysis of this section is only an illustrative example. It is possible (although unlikely) that welfare could be higher for an economy facing

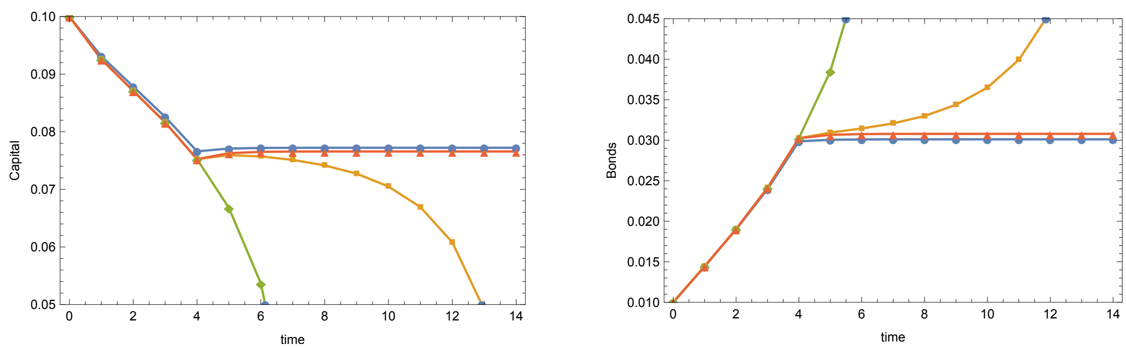


FIGURE 10. Time paths for capital and bond under four different scenarios. The green path (diamonds) shows the economy with no policy change. The blue path (circles), shows the economy with uncertainty that happens to have reform in period 4 (benefits are cut to $\phi_{new,4} = 0.1345$). This path converges to the steady state (0.0772, 0.302). The yellow path (squares), shows an economy with certainty that enacts the same policy change ($\phi_{new,4}$) in period 4, but does not have any uncertainty. This path is still explosive, since the benefit cut is not large enough to ensure convergence. The orange path (triangles) depicts the path for an economy that enacts the minimum possible reform in period 4 with certainty ($\phi_{new,certain} = 0.1332 < \phi_{new,4} = 0.1345$). This path converges to the steady state (0.0765, 0.0307).

uncertain reform than an economy facing a comparable certain reform under some parameterizations. This seems unlikely, however, given the way policy uncertainty impacts the savings decision of the young agents. Agents who face a potential benefit cut over-save compared to those who know their benefit in advance. This extra precautionary saving moves the agents away from their optimal bundle of young vs. old consumption.

Finally, the welfare analysis of this and the previous section both rely on aggregate, intergeneration measures of well-being. I have shown that in aggregate, some reforms are preferable to others (or that reform with certain can be preferable to uncertain reform). This does not mean that every generation prefers the same reform or the same level of uncertainty. Individual generations would prefer their own taxes to be as low as possible and their social security benefits to be as high as possible. Thus, even if benefit cuts in period x are preferable to benefit in in period $y > x$, the agents alive in period x would still have higher utility if the reform were delayed until period y. The agents in this model are only forward-looking one period, so they don't care if benefits are cut in two periods. A richer model is needed to fully capture the potential trade-offs within and between generations.

Possible Extensions

A planar OLG model with endogenous government deficits and debt provides an interesting framework to examine the existence and stability of steady states in relation to government policy. A natural extension of this paper would be to examine a typology of possible steady states in the model. Specially, it would be interesting to examine the stability properties of steady states (determinant vs. explosive) for steady states with positive bonds ($b > 0$), negative government bonds

($b < 0$), dynamic inefficiency (with either positive or negative bonds), and dynamic efficiency (with either positive or negative bonds).

In Appendix A, I introduce a Leeper tax to the model which allows social security taxes to respond endogenously to government debt. A possible extension of this model would be to examine the stability properties of steady states under various Leeper taxes. It may be possible to use a Leeper tax to stabilize a zero-bond steady state that is explosive in the absence of the Leeper tax. The Leeper tax might also be used as an automatic adjustment to avoid explosive dynamics that result from large government debt holdings. A non-linear Leeper tax may also provide interesting results.

Conclusion

In this chapter, I developed an analytical OLG framework to examine endogenous government deficits (surpluses) arising out of a social security system. The model highlights the risk of running large deficits that may result in explosive paths. The model can be used to evaluate different policy reforms for economies on unstable trajectories.

The policy analysis of section II highlights the welfare impacts of delaying reform. If the economy is on an unstable path, it is essentially paying retirees benefits that it (the government) cannot afford. Long-term, benefits need to be cut or taxes need to be raised. Before the reform takes place, retirees are receiving artificially high benefits. Their consumption is higher than the counterfactual where benefits paid are smaller. Because of this, agents who live during or after the reform, have lower lifetime utility than those who live before reform. The artificially high consumption of retirees before reform is financed by borrowing against the consumption of future generations. Moreover, if reform is delayed, the ultimate benefit cut (or tax increase) enacted will be larger in magnitude, as the economy will have moved further out of the basin of attraction of the stable steady state. Thus, the utility loss to agents after a delayed reform is even larger than the utility loss to agents alive after a more quickly implemented reform. This is seen most easily by the utility by generation graphs in Figures 6 and 7. Governments wishing to avoid a large drop in generational utility have an incentive to reform sooner.

The key takeaway from this policy analysis is the trade-off between delaying reform (which benefits current generations) and making a smaller reform now (which benefits future generations). Policy makers that are concerned about the welfare of future generations have an incentive to act sooner rather than later to minimize the negative impact on future generations. However, the reverse is also true. Policy makers who are concerned about current generations should delay reform since it is painful. If policy makers are responding to current generations, perhaps by seeking election, they have strong incentives to delay reform. Yet, reform cannot be delayed forever. Thus, even policy makers who are motivated to maintain the benefits of current generations, need to credibly promise reform in the future.

This model also allows me to consider the macroeconomic impacts of social security policy uncertainty. Young agents who face a potential social security benefit cut, engage in precautionary saving. This raises the capital stock and allows the government to perpetuate what would have been explosive policies for a longer period of time. However, even in the model with uncertainty, eventual policy change is needed to resolve explosive paths. In section II, I discuss the tradeoffs between living in a world with certain policy reform and uncertain reform. For the numeric example considered, (most) agents would be better off in a world with certainty, rather than a world with uncertainty. Agents only prefer uncertainty if the realization of the uncertain policy process means that they don't experience a benefit cut (or tax increase) in their lifetime (when they would have otherwise in the certain model). All other agents are worse off, since they weren't able to perfectly smooth their consumption between periods.

CHAPTER III

POLICY UNCERTAINTY IN A MULTI-PERIOD OLG MODEL WITH RATIONAL EXPECTATIONS

Introduction

In this Chapter, I extend the two-period model to three periods. The three period model is inherently forward looking, which allows for more interesting policy analysis. With three periods, the young worker's saving-consumption decision depends both on the predetermined capital and bonds for the next period, and also on her expectations over capital and bonds in the following period. This forward looking element of the model allows multi-period uncertainty analysis in the fashion of Davig et al. (2011).[1]

I describe the model and its equilibrium in Section III. In Section III, I discuss the number and stability of steady states. The model parameters are described in section III. In section III, I use the model to examine the effects of changes to social security policy. I examine both perfect foresight changes and stochastic policy changes. I consider the long-run implications of social security policy uncertainty in section III, and Section III concludes.

Model

Household problem

Households live for three periods, choose asset allocation (savings) when young a^1, assets in middle age a^2, and consumption in all three stages of life, c^1,

[1]See also Davig and Leeper 2011b, 2011a.

c^2, c^3, to maximize utility, taking price, and government social security policy (tax rates[2] τ, and benefit z) as given. Agents receive the wage w_t for labor provided in period t. The gross real return on savings in period t is given by R_{t+1}. Note that superscripts refer to life-cycle stages (i.e., age), and subscripts refer to time periods. Thus a^2_{t+1} is second period savings in time $t+1$.

The household's problem can be written as follows:

$$\max_{a^1_t,\, a^2_{t+1}} u(c^1_t) + \beta E_t u(c^2_{t+1}) + \beta^2 E_t u(c^3_{t+2}) \tag{3.1}$$

subject to

$$c^1_t + a^1_t \leq (1-\tau_t) w_t l^1_t \tag{3.2}$$

$$c^2_{t+1} + a^2_{t+1} \leq R_{t+1} a^1_t + (1-\tau_{t+1}) w_{t+1} l^2_{t+1} \tag{3.3}$$

$$c^3_{t+2} \leq R_{t+2} a^2_{t+1} + z^3_{t+2} \tag{3.4}$$

Where $E_t(x)$ indicates the time t expectation of x. I will normalize the (exogenous) age-labor endowments to unity for all generations such that $l^1_t = l^2_t = 1$ for all t. I will impose this normalization in all of the equations that follow.

Labor is supplied inelastically and preferences are given as the standard CRRA function:

$$\begin{aligned} u(c) =& \frac{c^{1-\sigma}}{1-\sigma} \quad \text{if } \sigma \neq 1 \\ u(c) =& \ln(c) \quad \text{if } \sigma = 1 \end{aligned} \tag{3.5}$$

The inverse IES for this utility function is simply σ.

The household first order conditions are:

$$u'(c^1_t) = \beta E_t[R_{t+1} u'(c^2_{t+1})] \tag{3.6}$$

$$((1-\tau_t) w_t - a^1_t)^{-\sigma} = \beta E_t \left[R_{t+1} (R_{t+1} a^1_t + (1-\tau_{t+1}) w_{t+1} - a^2_{t+1})^{-\sigma} \right]$$

[2] The model can accommodate time-varying policy parameters where $\tau_t \neq \tau_{t-1}$

$$u'(c^2_{t+1}) = \beta E_{t+1}[R_{t+2}u'(c^3_{t+2})] \tag{3.7}$$

$$(R_{t+1}a^1_t + (1-\tau_{t+1})w_{t+1} - a^2_{t+1})^{-\sigma} = \beta E_{t+1}\left[R_{t+2}(R_{t+2}a^2_{t+1} + z^3_{t+2})^{-\sigma}\right]$$

Equations (3.6) and (3.7) show the intertemporal trade off between consumption today and consumption tomorrow. The agent (trivially) chooses to consume all of her resources in the final period of life, that is $c^3_{t+2} = R_{t+2}a^2_{t+1} + z^3_{t+2}$ and $a^3_{t+2} = 0$.

Note that equations (3.6) and (3.7) have been printed above for an individual who is young in time t and middle aged in time $t+1$. Equation (3.7) can be iterated back one period to show the decision of a middle aged agent in time t.

To start off the economy, I assume that in period zero, there are three households (young, middle, and old) who enter the economy with given asset holdings according to their age. I assume that the initial young enter the economy with zero assets, and the initial middle and old enter with a^1_{-1} and a^2_{-1}, respectively. Successive cohorts enter the model with zero assets when they are young. N_t indicates generation t and is given by number of young at time t. the population (and hence the labor force) grows at rate n such that

$$N_t = (1+n)N_{t-1} \tag{3.8}$$

Production

The consumption good in the economy (Y_t) is produced by single firm (or equivalently many small firms) using a constant elasticity of substitution technology that takes aggregate capital (K_t) and labor hours (H_t) as inputs and produces the consumption good according to:

$$Y_t = F(K_t, H_t) = \left(\alpha K_t^{1-\frac{1}{\varepsilon}} + (1-\alpha) H_t^{1-\frac{1}{\varepsilon}}\right)^{\frac{1}{1-\frac{1}{\varepsilon}}}$$

The parameter α measures the intensity of use of capital in production. The parameter ε measures the elasticity of substitution in production. When this parameter is set to one, this production function collapses to the familiar Cobb-Douglass form: $Y_t = K_t^{\alpha} H_t^{1-\alpha}$. For the analytical expressions in the remainder of this paper, I will assume the Cobb-Douglass functional form. The numerical exercises are qualitatively similar when a different value of ε is selected.[3]

Factor markets are competitive and capital and labor (hours worked) are paid their marginal products. The gross real interest rate R_t is given by:

$$R_t = F_K(K_t, H_t) + 1 - \delta \tag{3.9}$$

where δ is the rate of depreciation. The wage rate w_t is given by

$$w_t = F_H(K_t, H_t) \tag{3.10}$$

Government

The government runs a modified pay-as-you-go social security system. The government pays retirement benefits to the old generation by taxing the working generations and by (possibly) issuing debt.

The pay-roll tax rate is τ_t. The tax has two components, a baseline tax rate τ_t^0, and a Leeper tax τ_t^1 that (potentially) responds to the level of government debt. Although actual social security taxes do not change based on government debt, a Leeper tax can be thought of as a way to capture legislative unease with increasing

[3]We abstract from technological growth, though it is straightforward to incorporate this into the model. If technological process was included in the model, aggregate variables would grow that the rate $(1 + n)(1 + g)$ along the balanced growth path (if g indicates the growth rate of technology). In all other regards, the model is identical.

debt. A Leeper tax is also a useful modeling tool, since it can be used to calibrate the model such that a dynamically efficient steady state with positive bonds is possible. I consider a Leeper tax of the following form:

$$\tau_t = \tau_t^0 + \tau_t^1 (B_t/H_t). \tag{3.11}$$

Here $\tau^0 \in [0, 1]$ is the baseline tax rate when government debt is zero, $\tau^1 \geq 0$ is the incremental tax, B_t/H_t is government debt per labor hours. When $\tau^1 = 0$ the taxes are exogenous and do not respond to government borrowings. When $\tau^1 > 0$, taxes increase when government debt increases. This slows the accumulation of debt and increases the basin of attraction of the stable steady state.[4] Notationally, a tax rate *without* a superscript (τ_t) will refer to the entire pay-roll tax $\tau_t = \tau_t^0 + \tau_t^1(B_t/H_t)$.[5]

Social security benefits z_t are paid according to a benefit earning rule:

$$z_t^s = \phi_t w_t. \tag{3.12}$$

where z_t^s represents the individual benefit paid to an agent of age s in time t. The parameter ϕ_t is the replacement rate and shows how much of a workers' wage-indexed average period earnings are replaced by social security benefits. The current wage (w_t) represents the average wage-indexed income of a retired person in

[4]Leeper taxes are generally modeled as responding to the previous period's debt (see Davig et al. (2011) as an example). However, in this model, government debt (B_t) and capital (K_t) are both predetermined. Thus, a tax rate in time t that responds to bond in time t is reasonable, since agents with perfect foresight know B_t at the beginning of time t.

[5]The Leeper tax given by equation (3.11) leaves open the possibility of a total tax rate greater than one hundred percent if bonds levels are high or if τ^1 is large (i.e., $\tau_t^0 + \tau_t^1(B_t/H_t) > 1$, for large B_t/H_t and/or large τ^1). I avoid this by imposing the restriction τ^1 is either the exogenous parameter value chosen by the economist, or the (smaller) parameter value such that $\tau_t^0 + \tau_t^1(B_t/H_t) = 1$. In practice, the calibration of the model results in numeric values for capital and bonds that are both very small. In the calibrated model, bonds are generally in the range [-0.03, 0.01], and thus it is never necessary to impose the restriction. For example, if $\tau^0 = 0.1$, $\tau^1 = 0.8$ and bonds are 0.01, then the total payroll tax burden for the consumer is $0.1 + 0.8 \times 0.01 = 0.108$. In all of the policy experiments that follow, the majority of the tax burden will come from the baseline tax rate τ^0, and the Leeper tax τ^1 will be used only to create the possibility of a dynamic efficiency with positive bonds.

period t. It shows a retired person's average period income in current wages. The model can accommodate time-varying replacement rates $\phi_t \neq \phi_{t+1}$.

The benefit earning rule (3.12), is similar to the actual benefit earning rule used by the Social Security Administration. There is no within-generation heterogeneity in this model, so there is no need for the benefit earning rule to be a piecewise-linear function of earnings (as is the case in the United States). All individuals within the same generation work the same number of hours (inelastically) for the same wage, thus their average earnings and social security benefits are the same.

The government revenue generated from the social security payroll tax may be greater or less than the government transfer to retirees in any period. If social security taxes are less than social security benefits, the social security program runs a deficit. The time t deficit is defined as the time t social security benefits less the pay-roll tax revenue:

$$\text{Deficit}_t = N_{t-2}\phi_t w_t - (N_{t-1} + N_t)(\tau_t^0 + \tau_t^1(B_t/H_t))w_t$$

The number of tax payers is $N_t + N_{t-1}$ and represents the young and middle aged. The number of old is N_{t-2} and they receive benefits according to (3.12).

The government issuance of bonds is equal to the gross interest on outstanding debt plus the social security deficit from the previous period.

$$B_{t+1} = R_t B_t + N_{t-2}\phi_t w_t - (N_{t-1} + N_t)(\tau_t^0 + \tau_t^1(B_t/H_t))w_t \tag{3.13}$$

Market Clearing

There are four markets: labor, capital, bonds, and goods. Prices adjust in equilibrium to ensure all markets clear.

Labor market clearing requires the total number of hours worked by the young and middle aged to equal the labor input of the representative firm:

$$H_t = N_t + N_{t-1} \tag{3.14}$$

Asset market clearing requires that aggregate capital and bonds next period are equal to total savings of the young and middle aged today:

$$A_{t+1} = K_{t+1} + B_{t+1} = a_{1,t}N_t + a_{2,t}N_{t-1} \tag{3.15}$$

Goods market clearing requires that aggregate consumption and next period capital are equal to aggregate production and undepreciated capital:

$$N_t c_t^1 + N_{t-1}c_t^2 + N_{t-2}c_t^3 + K_{t+1} = K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t \tag{3.16}$$

The goods market clears by Walras law, and so it is not necessary to track this equation as part of the equilibrium.

Bond market clearing is ensured by the government's flow budget constraint, reprinted below:

$$B_{t+1} = R_t B_t + N_{t-2}\phi_t w_t - (N_{t-1} + N_t)(\tau_t^0 + \tau_t^1(B_t/H_t))w_t \tag{3.13}$$

<u>Market Clearing Equations in Per-Labor-Hour Terms</u>

Along the balanced growth path, cohort size (N_t), labor hours worked (H_t), output (Y_t), capital (K_t), and bonds (B_t) all grow at rate n. Therefore, it will be convenient to rewrite the market clearing equations (3.13), (3.15), and (3.16) in per-hours terms by defining $b_t = B_t/H_t$, and $k_t = K_t/H_t$. I will refer to variables in per-hours terms as "efficient."

The government equation can be written in efficient terms by dividing both sides by H_t and multiplying the left hand side by the fraction H_{t+1}/H_{t+1}.[6]

$$\frac{B_{t+1}}{H_t}\frac{H_{t+1}}{H_{t+1}} = \frac{R_t B_t}{H_t} + \frac{N_{t-2}\phi_t w_t}{H_t} - \frac{(N_{t-1}+N_t)(\tau_t^0 + \tau_t^1(B_t/H_t - b^*))w_t}{H_t}$$

$$b_{t+1}(1+n) = R_t b_t + \frac{1}{(2+n)(1+n)}\phi_t w_t - (\tau_t^0 + \tau_t^1(b_t - b^*))w_t \tag{3.17}$$

The capital market clearing equation can be written in efficient terms by dividing both sides by H_t and multiplying the left hand side by the fraction H_{t+1}/H_{t+1}.

$$\frac{K_{t+1}+B_{t+1}}{H_t}\frac{H_{t+1}}{H_{t+1}} = a_t^1\frac{N_t}{H_t} + a_t^2\frac{N_{t-1}}{H_t}$$

$$(k_{t+1}+b_{t+1})(1+n) = a_t^1\frac{1+n}{2+n} + a_t^2\frac{1}{2+n} \tag{3.18}$$

The goods market clearing can be written in efficient terms by dividing both sides by H_t

$$\frac{N_t}{H_t}c_t^1 + \frac{N_{t-1}}{H_t}c_t^2 + \frac{N_{t-2}}{H_t}c_t^3 + \frac{K_{t+1}}{H_t} = \frac{K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t}{H_t}$$

$$\frac{1+n}{2+n}c_t^1 + \frac{1}{2+n}c_t^2 + \frac{1}{(1+n)(2+n)}c_t^3 + (1+n)k_{t+1} = Ak_t^\alpha + (1-\delta)k_t \tag{3.19}$$

Equilibrium

Given initial conditions k_0, b_0, a_{-1}^1 and a_{-1}^2, an initial cohort of young N_0, middle $N_{-1} = \frac{N_0}{1+n}$, and old $N_{-2} = \frac{N_0}{(1+n)^2}$, and a population growth rate of n such that $N_t = (1+n)N_{t-1}$, a competitive equilibrium is a sequences of functions for the households $\{a_t^1, a_t^2\}_{t=0}^\infty$, production plans for the firm, $\{k_t\}_{t=1}^\infty$, government bonds

[6]Note, H_t can be written as $H_t = \frac{2+n}{1+n}N_t$, which implies $H_t = (2+n)N_{t-1}$ and $H_t = (1+n)(2+n)N_{t-2}$. These identities are used to simplify 3.17, and 3.18.

$\{b_t\}_{t=1}^{\infty}$ factor prices $\{R_t, w_t\}_{t=0}^{\infty}$, government policy variables $\{\tau_t^0, \tau_t^1, \phi_t\}_{t=0}^{\infty}$, that satisfy the following conditions:

1. Given factor prices and government policy variables, individuals' decisions solve the household optimization problem (3.2), (3.3), and (3.4)

2. Factor prices are derived competitively according to (3.9) and (3.10)

3. All markets clear according to (3.14), (3.18), and (3.17)

The equilibrium equations hold for $t = 0, 1, ...\infty$ are reprinted below (substituting in the Leeper tax $\tau_t = \tau_t^0 + \tau_t^1 b_t$).

$$\begin{aligned} &\left((1-\tau_t^0-\tau_t^1 b_t)w(k_t)-a_t^1\right)^{-\sigma} \\ &= \beta E_t[R(k_{t+1})\left(R(k_{t+1})a_t^1+(1-\tau_{t+1}^0-\tau_{t+1}^1(b_{t+1}))w(k_{t+1})-a_{t+1}^2\right)^{-\sigma}] \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\left(R(k_t)a_{t-1}^1+(1-\tau_t^0-\tau_t^1 b_t)w(k_t)-a_t^2\right)^{-\sigma} \\ &= \beta E_{t+1}[R(k_{t+1})\left(R(k_{t+1})a_t^2+\phi_{t+1}w(k_{t+1})\right)^{-\sigma}] \end{aligned} \tag{3.21}$$

$$b_{t+1}(1+n) = R(k_t)b_t + \frac{1}{(2+n)(1+n)}\phi_t w(k_t) - (\tau_t^0+\tau_t^1 b_t)w(k_t) \tag{3.22}$$

$$(k_t+b_t)(1+n) = a_{t-1}^1\frac{1+n}{2+n} + a_{t-1}^2\frac{1}{2+n} \tag{3.23}$$

Note that equation (3.21) has been iterated back to show the decision of a middle aged person in period t, that happens simultaneously with the decision of a young person in period t characterized by (3.20). Equation (3.23) has also been iterated back one period so that it holds for $t = 0, , 1, ...\infty$. Note finally that $k_{-1} = 0$.

A stationary equilibrium is a competitive equilibrium in which per capita variables and functions, as well as prices and policies, are constant, and aggregate variables grow at the constant growth rate of the population n.

The steady state is a collection $\{k, b, a^1, a^2\}$ that solves:

$$
\begin{aligned}
&((1-\tau^0-\tau^1 b)w(k)-a^1)^{-\sigma} \\
&=\beta R(k)(R(k)a^1+(1-\tau^0-\tau^1 b)w(k)-a^2)^{-\sigma}
\end{aligned}
\tag{3.24}
$$

$$
\left(R(k)a^1+(1-\tau^0-\tau^1 b)w(k)-a^2\right)^{-\sigma}=\beta R(k)\left(R(k)a^2+\phi w(k)\right)^{-\sigma} \tag{3.25}
$$

$$
b(1+n)=R(k)b+\frac{1}{(2+n)(1+n)}\phi w(k)-(\tau^0+\tau^1 b)w(k) \tag{3.26}
$$

$$
(k+b)(1+n)=a^1\frac{1+n}{2+n}+a^2\frac{1}{2+n} \tag{3.27}
$$

where $R(k)$ and $w(k)$ are given by (3.9) and (3.10).

Steady State Analysis

The number and stability of steady states in this model depends crucially on the government's social security policy. Government deficits and debt are both endogenous and depend on the underlying policy parameters. When the government runs relatively small deficits or surpluses, the model has two steady states, one of which is stable, the other of which is explosive. As government deficits (or surpluses) increase, the steady states move closer together, until at a critical policy value, only one steady state exists. If deficits (surpluses) increase beyond this critical size, no steady states exist. This change in the number of steady states is called a saddle-node bifurcation, and is discussed in great detail in Chalk (2000) and Azariadis (1993) (Chapter 8 and Chapter 20). Specific examples of steady states, along with stability analysis, will be presented below in section III. Before examining examples, however, I will present the general framework for stability analysis in this model.

Linearization and Eigenvalue Analysis

The stability of steady states can be studied using eigenvalue analysis of the linearized system, following Laitner (1990).

Define $X_t = (a^1_{t-1}, a^2_t, k_t, b_t)'$. Given initial conditions $(a^1_{-1}, a^2_{-1}, k_0, b_0)$,the non-linear system governing the economy can be described as

$$G(X_t, E_t J(X_{t+1})) = 0 \tag{3.28}$$

where the notation $E_t J(X_{t+1})$ indicates the time t expectation of a function J of X_{t+1}.

The system $G(X_t, E_t J(X_{t+1})) = 0$ is given by:

$$((1-\tau_t)w(k_t) - a^1_t)^{-\sigma} - \beta E_t[R(k_{t+1})(R(k_{t+1})a^1_t + (1-\tau_{t+1})w(k_{t+1}) - a^2_{t+1})^{-\sigma}] = 0$$

$$(R(k_t)a^1_{t-1} + (1-\tau_t)w(k_t) - a^2_t)^{-\sigma} - \beta E_t[R(k_{t+1})(R(k_{t+1})a^2_t + \phi_{t+1}w(k_{t+1}))^{-\sigma}] = 0$$

$$b_{t+1}(1+n) - R(k_t)b_t - \frac{1}{(2+n)(1+n)}\phi_t w(k_t) + \tau_t w(k_t) = 0$$

$$(k_{t+1} + b_{t+1})(1+n) - a^1_t \frac{1+n}{2+n} - a^2_t \frac{1}{2+n} = 0$$

Equation (3.28) implicitly defines $E_t X_{t+1}$ as a function of X_t, using the chain rule.[7] Define $F(X_t)$ as the implicit function such that $E_t X_{t+1} = F(X_t)$. Non-linear solution techniques, outlined in Appendix B are used to approximate $F(X_t)$. Alternatively, The system $F(X_t)$ can be approximated linearly. The linearized system is useful in assessing the stability of steady states.

Define $Z_t = (\tau^0_t, \tau^0_{t+1}, \tau^1_t, \tau^1_{t+1}, \phi_t, \phi_{t+1})'$. Total differentiation of equation (3.28) using the chain-rule, yields a system that is linear in $dE_t X_{t+1}$, dX_t, and dZ_t:

$$H_1 dX_t + H_2 dE_t X_{t+1} + H_3 dZ_t = 0 \tag{3.29}$$

[7]Recall that the tax rate without a subscript refers to the entire tax $\tau_t = \tau^0_t + \tau^1_t b_t$. The tax rate is written the suppressed form τ_t for this section to save space.

where H_1, H_2, and H_3 are matrices of partial derivatives.[8] Multiplying through by the negative inverse of H_2 yields:

$$dE_t X_{t+1} = M_1 dX_t + M_2 dZ_t \tag{3.30}$$

where $M_1 = -H_2^{-1} H_1$ and $M_2 = -H_2^{-1} H_3$.

The system $E_t X_{t+1} = F(X_t)$ can be approximated at a steady state X^* using a first order Taylor approximation. Using the matrix notation from the total differentiation above,

$$F(X) \approx F(X^*) + M_1 dX^*(X - X^*) \tag{3.31}$$

Following Laitner (1990), the stability of a given steady state depends on the eigenvalues of matrix M_1 evaluated at the steady state. Three cases are possible.

Case 1 stability: the matrix M_1 has three distinct, nonzero eigenvalues λ_i, with modulus less than one for $i = 1, 2, 3$; and one distinct, nonzero eigenvalues λ_4, with modulus greater than one.

Case 2 instability (explosive): the matrix M_1 has distinct, nonzero eigenvalues λ_i with modulus greater than one for more than one eigenvalues.

Case 3 indeterminacy: the matrix M_1 has distinct, nonzero eigenvalues λ_i with modulus less than one for all four eigenvalues ($i = 1, 2, 3, 4$).

Stability (Case 1) requires three eigenvalues lie inside the unit circle because the model has three predetermined variables. The predetermined variables are capital, bonds, and asset level of the middle aged (a_t^2). Capital and bonds are backward-looking variables. Period $t + 1$ capital and bonds (k_{t+1}, b_{t+1}) are defined as functions of time t variables only (k_t, b_t, a_t^1, and a_t^2). Thus, period $t + 1$ capital and bonds are known in period t. The asset holdings chosen by the middle aged

[8] H_1 and H_2 are size 4×4, and H_3 has size 4×6.

in period t (a_t^2) is also a predetermined variable. The middle aged savings choice depends on last period's young saving choice (a_{t-1}^1), current capital (k_t), and future capital (k_{t+1}). Since period $t+1$ capital is known in time t, everything in the a_t^2 equation is known in time t. The remaining variable, asset choice of the young (a_t^1), is a free variable that is not known in period t. The savings decision of the young depends on their forecast of their savings decision in the next period, which in turn, depends on their forecast of the capital stock two periods ahead.

Examples of Steady States

In the previous chapter, I illustrate the existence and stability of steady states in the two-period model using a phase digram. In a three-period model, a simple phase diagram is not possible, but the steady state equations can still be displayed in two dimensions in a special case.

In general, the optimal savings choices of the young and middle-aged in the model cannot be solved analytically. However, in the special case of Log Utility ($\sigma = 1$) and Cobb-Douglass Production ($\varepsilon = 1$), the optimal savings of both generations have closed form solutions that are functions of capital and bonds only. Substituting the steady state versions of these expressions into the steady state bond equation (3.27) and asset market clearing equation (3.26) yields a two-dimensional system in (k, b):

$$b(1+n) = R(k)b + \frac{1}{(2+n)(1+n)}\phi w(k) - (\tau^0 + \tau^1 b)w(k) \tag{3.32}$$

$$(k+b)(1+n) = a^{1*}(k,b)\frac{1+n}{2+n} + a^{2*}(k,b)\frac{1}{2+n} \tag{3.33}$$

where $a^{1*}(k,b)$ and $a^{2*}(k,b)$ are the analytical solution to (3.24) and (3.25) when $\sigma = 1$, and $R(k)$ and $w(k)$ indicate competitive factor prices.

Contour plots of the steady state bond equation (3.32) and the steady state asset market clearing equation (3.33) each plot the pairs (k, b) for which the specific equation holds. The intersection of the two equations show the steady states of the system.

Government policy and dynamic efficiency both play a key role in the determining the number and stability of steady states. The model has at most two steady states; however, one or zero steady states is also possible. The number of steady states results from a saddle-node bifurcation. The model has two steady states when government deficits are not "too large" (that is, if the benefit replacement rate ϕ is not too high relative to the tax rate τ and the population growth rate n). The steady state with higher capital is stable (case 1) and the lower capital steady state is explosive (case 2). As the size of the government deficit increases (due to changes in ϕ, τ, or n), the steady states more closer together. When the deficit is just large enough, a single steady state exists.[9] When deficits are too large, no steady states exist.

I illustrate the saddle-node bifurcation with two dynamically inefficient steady states and increasing deficits in Figure 11. This case is comparable to the set-up of Chalk 2000 and my own two-period model analysis. Note the saddle node bifurcation can occur at other parameterizations; the graph is presented was chosen because it is qualitatively similar to Chalk (2000), and Figure 4 of the previous chapter. The steady state bond equation (3.32) is a hyperbola with an asymptote

[9]This steady state is not unique–there are many possible parameter combinations that yield a single steady state.

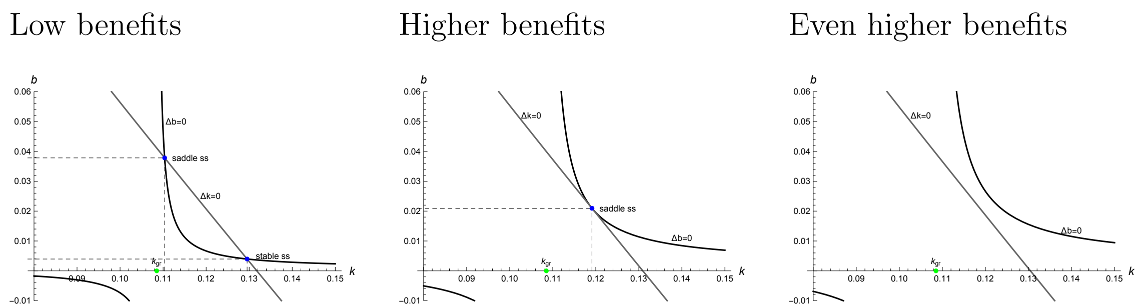


FIGURE 11. Steady state contour graphs of equations (3.32) and (3.33) in (k, b) space (capital is plotted on the horizontal axis, bonds on the vertical). These contour plots illustrate the steady states for an economy that is dynamically inefficient and running a social security program with a deficit. As the size of the deficit is increased (because the tax rate is decreased), the constant bond loci shifts out (up and to the left). When taxes are small enough, the bond locus shifts so that it no longer crosses the asset locus and no steady states exist. This is an illustration of the saddle-node bifurcation. The parameters for this example have been selected to generate graphs that look qualitatively similar to Chalk 2000.

as the golden rule level of capital.[10] The steady state asset equation (3.33) is an inverted parabola. In the figures the top of the parabola is not visible.

The steady state contour plots for an economy with positive bonds and a Leeper tax are presented in Figure 12. The graph looks qualitatively similar to the contour plots for dynamic inefficiency with a deficit in Figure 11. The key similarities are that the bond locus intersects the asset locus above the zero-bond axis, and the steady states both occur on the same "arm" of the bond locus. The bond locus shifts out (up and to the right) as the social security benefits increase (or taxes decrease). If policy changes are large enough, the saddle node bifurcation occurs and the steady states disappear. The key feature of a positive Leeper tax ($\tau^1 > 0$) is that a positive bond steady state is possible at a dynamically efficient level of capital. I find this to be a compelling parameterization, since the

[10]The asymptote is at the golden rule level of capital when the Leeper tax $\tau^1 = 0$; the asymptote moves to the left if a positive Leeper tax is introduced.

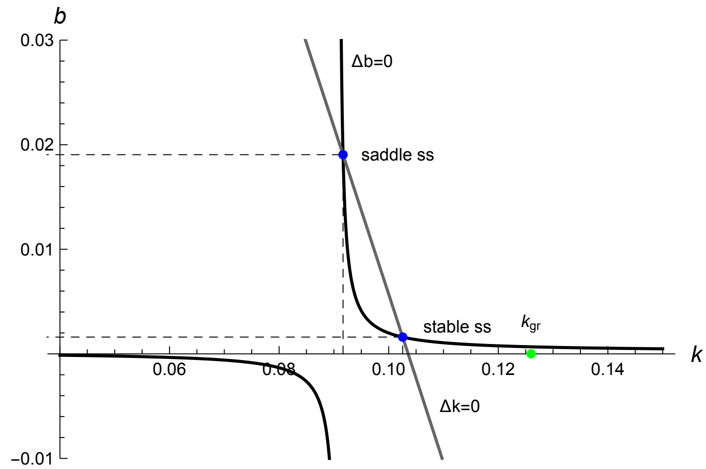


FIGURE 12. Steady state contour graphs of model with Leeper tax in (k, b) space (capital is plotted on the horizontal axis, bonds on the vertical). These contour plots illustrate the steady states for an economy that is dynamically efficient and running a social security program with a small surplus.

United States holds debt and the economy is (likely) dynamically efficient. If the Leeper tax is not included (or if τ^1 is too small), then dynamically efficient steady states only occur with negative bonds (government asset holdings), which is less compelling as a model of the U.S. I will use the parameterization of Figure 12 as the backdrop for policy experiments.

The baseline parameterization presented above in Figure 12 has two steady states. The high capital steady state is determinant (or stable); three of the eigenvalues of the linearized system evaluated at this steady state fall inside the unit circle and one falls outside. The low capital steady state is explosive (or unstable); two of the eigenvalues of the linearized system evaluated at the steady state fall outside the unit circle. The Appendix section B includes additional examples of steady states. The steady states and the eigenvalues of the linearized system evaluated at the steady state are presented in a summary Table B.1.

Parameterization

Agents are economically active as soon as they enter the model and live for a total of three periods. Each period can be thought of as twenty years. I assume that agents enter the model at age twenty-five. Thus, young agents enters the model at time t when they are twenty-five years old, become middled aged at time $t+1$ when they are forty-five years old, retire at time $t+2$ when they are sixty-five years old, and die at the end of period $t+1$ when they are eighty-five years old. The remaining model parameters correspond the baseline model presented in Auerbach and Kotlikoff (1987) with two exceptions.

In this paper, the utility preference parameter, σ, is set to one in the baseline. This corresponds to log utility. This choice is for analytical simplicity, which allows the four steady state equations to be consolidated into two equations that can then be graphed in (k, b) space. It is common for this parameter to be higher in discrete time models. For example, Auerbach and Kotlikoff (1987) set this parameter to 4, and Krueger and Kubler (2006) target an IES of 0.5 (which would correspond to $\sigma = 2$). However, log utility is not uncommon in the literature and is within the range of empirical estimates (see Gourinchas and Parker (2002),Bullard and Feigenbaum (2007), and Feigenbaum (2008)). The policy experiments are qualitatively similar if a higher value for σ is chosen.

Second, I have assumed full deprecation for the baseline in this paper. This assumption is not crucial to the results, although it does increase convergence speed of the solution algorithm.[11] Qualitatively similar results occur with lower rates of depreciation (including the extreme case of no deprecation $\delta = 0$). The baseline

[11]Full deprecation is a reasonable assumption for an OLG model with 20 year periods. Assuming an annual deprecation rate of 5% corresponds to $\delta = 1 - 0.95^20 = 0.64$ in this model. Annual deprecation of 10% corresponds to $\delta = 1 - 0.9^{20} = 0.88$.

	Parameter	Value	Notes
α	capital share of output	0.25	AK $\gamma = 0.25$
β	discount factor	$\left(\frac{1}{1+0.015}\right)^{20}$	AK $\rho = 0.015$
n	population growth rate	$1-(1-0.01)^{20}$	AK $n = 0.01$
δ	depreciation	1	chosen for computational speed
σ	Inverse Elasticity of Intertemporal Substitution	1	Log utility chosen for simplicity of graphs

TABLE 10. Baseline parameters. "AK" in the right column refers to the baseline parameters of Auerbach and Kotlikoff (1987). In the three period model, and single period corresponds to roughly 20 years; thus annual parameters from AK have been converted to 20-year parameters.

parameters are presented in Table 10. Social security parameters (taxes τ^0, τ^1 and benefit replacement rate ϕ) are generally set to correspond to the US system which has a payroll tax rate of 10.6% to fund the Old Age Survivor portion of social security, and an average benefit replacement rate of 38% (see Olovsson (2010) for an example for calibrating social security in an OLG model with few periods).

Note, in the absence of social security (when $\tau^0 = \tau^1 = \phi = 0$), the steady states of the model are dynamically inefficient. As social security is introduced, this crowds out capital, and results in steady state that are dynamically efficient. This parameterization loosely follows Imrohoroglu, Imrohoroglu, and Joines (1995), which also assumes dynamic inefficiency in the absence of social security and dynamic efficiency with the program. The assumption of dynamic efficiency with social security is also consistent with many computational life-cycle models, including Krueger and Kubler (2002), (2004), and (2006).

Policy Experiments: Pension Reforms

Perfect Foresight Expected Policy Changes

Before examining policy uncertainty, it is useful to describe how the model responds to expected, perfect foresight changes in policy. As a simple example, consider an economy in a steady state without public pensions. Social security is introduced in period $t = 6$, this decreases the savings of the young and middle aged and drives down the capital stock. Under the specific parameterization considered, the government owns debt at the final steady state, and the social security system runs a small surplus to pay the interest on the debt. The transition paths are presented in Figure 13.

The policy changes are anticipated with perfect foresight, so we see changes in the savings decisions *prior* to the introduction of social security. We observe the largest changes in the two periods immediately preceding the policy change (since agents experience the policy change in their lifetimes). Very small changes also occur in the periods before that as a result of the general equilibrium feedback.

For example, consider the generation who retires at time $t = 6$. This generation is young at time $t = 4$, middle aged at time $t = 5$, and retired with social security at time $t = 6$. This generation experiences a windfall gain relative to other generations, since they do not pay taxes, yet they receive social security benefits. Thus, they save less in both working periods of their life than previous generations ($a_4^1 < a_3^1$ and $a_5^2 < a_4^2$).

The generation that retires in $t = 7$ also experiences a windfall gain: they only pay taxes in middle age, and still receives a social security benefit. This generation increases savings while they are young, to help smooth the drop in

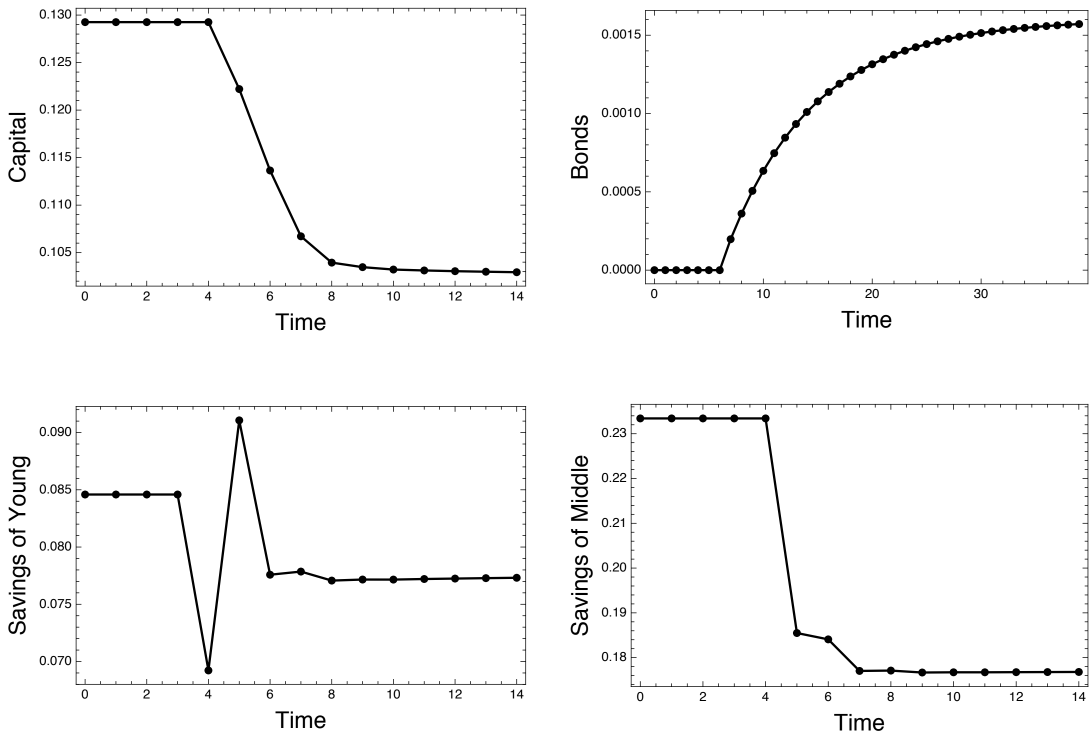


FIGURE 13. Time paths for an economy that starts in a steady state without public pensions (τ^0=τ^1=ϕ=0). Social security is introduced in period t=6; the social security system has a Leeper tax and runs a small surplus to pay interest on the outstanding government bonds (τ^0=0.077, τ^1=0.8, ϕ=0.2). All other parameters set at baseline values.

consumption they will experience by paying taxes in middle age ($a_5^1 > a_4^1$). Finally, generations that retire at or after $t = 8$ are taxed when young and middle aged and receive a social security benefit. These generations save less compared to their predecessors.

Next, consider an economy that is on a (temporarily) explosive trajectory because the social security system is running an unsustainable deficit, such that no steady states exist for the economy.[12] Suppose also that the government has

[12]Perhaps the government was running a sustainable social security system (with taxes that were high enough to cover benefits), but then the growth rate of the population slowed, causing the total taxes collected each period to be lower than the total benefits paid.

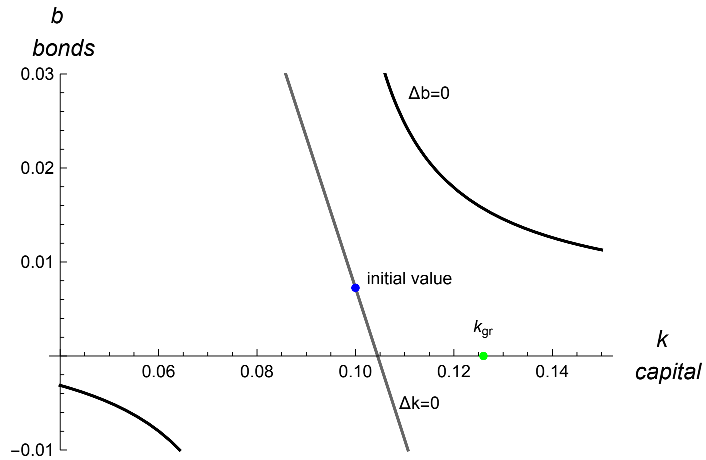


FIGURE 14. Steady state contour graphs of model with Leeper tax in (k, b) space (capital is plotted on the horizontal axis, bonds on the vertical). No steady states exist for this parameterization. The taxes raised are insufficient to cover benefits paid each period. $\tau^0 = 0.065$, $\tau^1 = 0.8$, and $\phi = 0.2$. The social security system has the same benefit level as in Figure 12, but the taxes are lower. All other parameters are at the baseline values. The initial values of capital and bonds for the policy experiment presented in Figure 15 is also included in the graph. Note, that the initial value is not a steady state.

existing government debt ($b > 0$). It is possible for the government to remedy the social security system, and ensure convergence to a steady state, by simply raising taxes or lowering benefits as in Figure 15. The size of the benefit cut was chosen such that the steady state capital is the same after the benefit cut or the tax increase.[13] In both examples, initial government policy is unsustainable with taxes insufficient to cover benefits. If policy were left unchecked, government bonds would increase infinitely. The contour plots of the steady state equations for the unsustainable policy are presented in Figure 14. Note that no steady states exist. Increasing taxes or decreasing benefits shifts the bond locus down and to the left, which allows for the possibility of steady states.

[13]The steady states, initial values, and policy parameters for these experiments are presented in Table 11.

Fig	Initial policy			New policy			Initial values				Final steady state(s)			
	τ^0	τ^1	ϕ	τ^0	τ^1	ϕ	k	b	a^1	a^2	k	b	a^1	a^2
13	0	0	0	0.077	0.8	0.2	0.129	0	0.085	0.233	0.103	0.002	0.078	0.178
15	0.065	0.8	0.2	0.077	0.8	0.2	0.103	0.002	0.078	0.177	0.103	0.002	0.078	0.178
15	0.065	0.8	0.2	0.065	0.8	0.173	0.103	0.002	0.078	0.177	0.103	0.006	0.081	0.185

TABLE 11. Summary of perfect foresight policy experiments. Experiment 13: Economy in steady state, social security introduced in t=6. Experiment 15 (circles): Economy on explosive path, taxes increased in period t=4. Experiment 15 (triangles): Economy on explosive path, benefits cut in period t=4.

In Figure 15, I start the economy off with positive bonds and unsustainable social security policy. The government reforms social security in period $t = 4$ by raising taxes (indicated on the graph with circles) or cutting benefits (presented in graph as triangles). This tax increase or benefit cut results in a small social security surplus each period that is sufficient to pay the interest on the government's debt. During the first three periods, before the reform, the capital stock is falling which drives down wages. Agents respond by increasing their savings relative to their parents to smooth their consumption across their lifecycle. Following either policy reform, capital, bonds, and the savings of both generations converge to the new steady state.

Consider first the response to an announced tax increase in period $t = 4$. Young agents in period $t = 3$ increase their savings (a_3^1) to offset the loss in consumption in their middle age due to higher payroll taxes. Middle-aged savings a^2, also increases leading up to the tax cut, but this seems to be driven mainly by the falling capital stock. In contrast, if social security benefits are cut in period $t = 4$, we see a large response in the middle-aged savings of agents in period $t = 3$. These agents see that their social security benefits will be lower than previous generations, and so they save more. The savings of the young also increases before a benefit cut. The young in period $t = 2$ see that their benefits will be cut in $t = 4$, and so they save more.

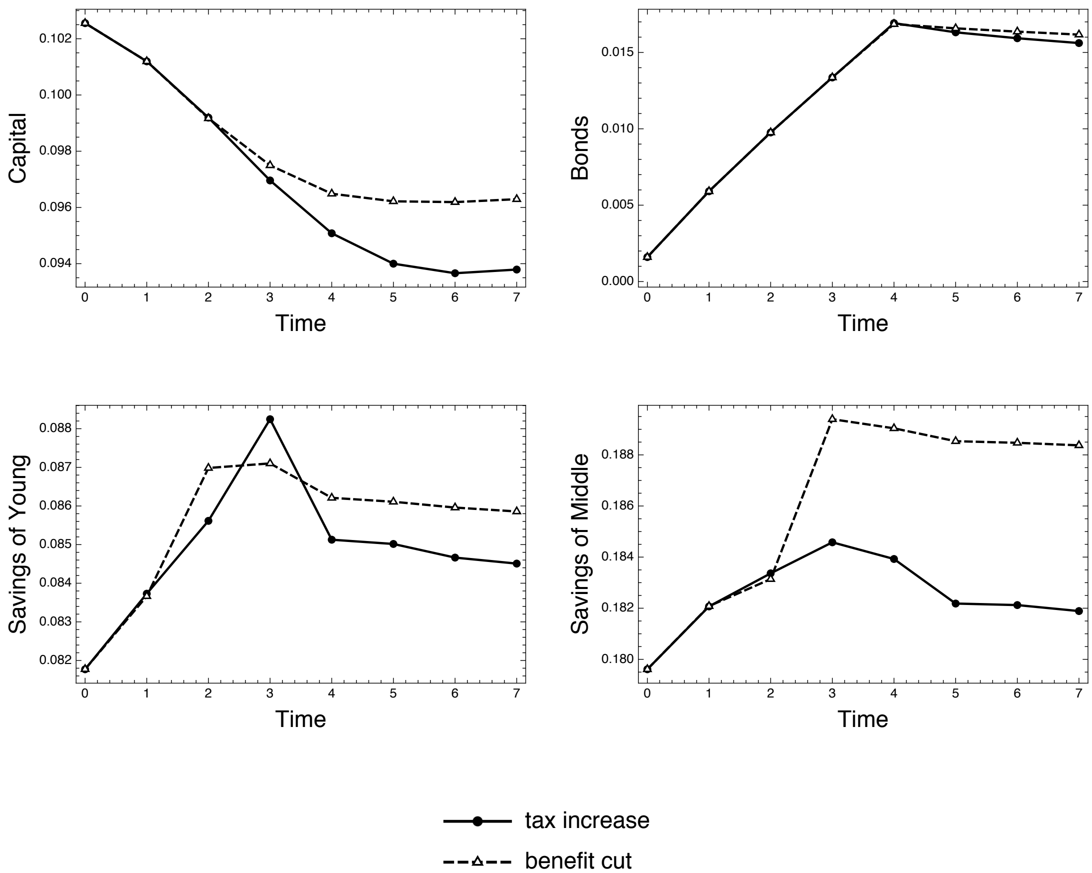


FIGURE 15. Time path of endogenous variables for an economy that starts on an explosive trajectory (bonds increasing infinitely and capital falling). Taxes are increased (circles) or benefits are cut (triangles) in period t=4 and the economy converges to the steady state. Agents respond *prior* to the change in policy, since they fully anticipate the reform. The social security parameters before the reform are τ^0=0.065, τ^1=0.8, and ϕ=0.2. The tax reform increases τ^0 to 0.077. The benefit reform cuts ϕ to 0.173. All other parameters are set to the baseline values. Initial values and steady states are listed in Table 11.

Let $\omega_t = (\tau_t^0, \tau_t^1, \phi_t)'$ describe the government policy parameters at time t. Suppose that the social security program will be reformed (permanently) at some unknown future date, such that $\omega_{new} \neq \omega_{old}$. Suppose that the realization of ω_{t+1} depends on ω_t, and is contained in the finite set Ω. Suppose also that each possible reform converges to a steady state (the path is non-explosive). Suppose also, that if reform is not enacted by period $\hat{T}$, that policy $\hat{\omega}$ will be enacted with certainty in period $\hat{T}+1$. (This final assumption is to guarantee that debt does not become explosive in the long-run).

Let the probability of realizing a particular value of ω_{t+1} given ω_t be described by $\pi(\omega_{t+1}|\omega_t)$. Using this notation, the expected value in the household first order equations (3.6) and (3.7) can be written as:

$$u'(c_t^1) = \beta \sum_{\omega_{t+1} \in \, \Omega} \pi(\omega_{t+1}|\omega_t) R_{t+1} u'(c_{t+1}^2) \tag{3.34}$$

$$u'(c_t^2) = \beta \sum_{\omega_{t+1} \in \, \Omega} \pi(\omega_{t+1}|\omega_t) R_{t+1} u'(c_{t+1}^3) \tag{3.35}$$

These first order conditions, combined with the asset market clearing equation (3.18) and the bond transition equation (3.17), define the equilibrium path of the economy with policy uncertainty.

Introducing aggregate uncertainty to the model raises some concerns. Krueger and Kubler (2004) explain that in a OLG model with aggregate uncertainty, a steady state wealth distribution will not exist in general. Additionally, even when "exogenous aggregate shocks can take only a finitely many values the equilibrium allocations in general do not have finite support (p.2).” I overcome this problem by modeling aggregate policy uncertainty in a simple, stylized way. Policy uncertainty only exists for a small number of periods, $\hat{T}$, and policy reform can only be one of a small number of policy options. These simplifications, combined with the fact agents only live three periods, allow me to calculate the equilibrium paths for the economy with policy uncertainty. To begin, I consider policy uncertainty that spans two-periods only (section III). Next, I consider simultaneous timing and policy uncertainty (section III). In the appendix, I present examples with timing uncertainty that spans three periods (section B).

Simple Two-Period Timing Uncertainty

As the simplest possible type of uncertainty, suppose government policy $\omega = (\tau_t^0, \tau_t^1, \phi_t)'$ will either be changed in period S with probability p, or if it is not changed in period S, it is changed with certainty in period $S+1$. Let ω_{old} indicate the policy parameters before the reform, and ω_{new} indicate policy after. Under this simple type of policy uncertainty, there are only two possible paths for the economy: either reform occurs in period S, or it occurs in period $S+1$. For notational ease, I will distinguish the two paths using tildes and hats: $\tilde{x}$ for the path of the economy when reform occurs in period S, and $\hat{x}$ for the path of the economy when reform happens in $S+1$.

This simple uncertainty only impacts two generations: the generation who is young at time $S-2$ (and thus isn't sure what the value of ϕ_S will be when they retire), and the generation who is young at time $S-1$ (who isn't sure about the value of τ_S^0 and τ_S^1 for their middle age). Decisions made by all other generations are determined by the non-stochastic Euler equations (3.6) and (3.7).

The generation who is young at time $S-2$ chooses a^1_{S-2} using the non-stochastic Euler equation (3.36) and a^2_{S-1} using the stochastic Euler equation (3.37) or (3.38).

$$\begin{aligned}&((1-\tau_{S-2})w(k_{S-2})-a^1_{S-2})^{-\sigma}\\&=\beta(R(k_{S-1})(R(k_{S-1})a^1_{S-2}+(1-\tau_{S-1})w(k_{S-1})-a^2_{S-1})^{-\sigma})\end{aligned} \tag{3.36}$$

For path $\tilde{x}$:

$$\begin{aligned}&(R(\tilde{k}_{S-1})\tilde{a}^1_{S-2}+(1-\tilde{\tau}_{S-1})w(\tilde{k}_{S-1})-\tilde{a}^2_{S-1})^{-\sigma}\\&=\beta[p(R(\tilde{k}_S)(R(k_S)\tilde{a}^2_{S-1}+\tilde{\phi}_{new}w(\tilde{k}_S))^{-\sigma})\\&+(1-p)(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^2_{S-1}+\hat{\phi}_{old}w(\hat{k}_S))^{-\sigma})]\end{aligned} \tag{3.37}$$

For path $\hat{x}$:

$$\begin{aligned}&(R(\hat{k}_{S-1})\hat{a}^1_{S-2}+(1-\hat{\tau}_{S-1})w(\hat{k}_{S-1})-\hat{a}^2_{S-1})^{-\sigma}\\&=\beta[p(R(\tilde{k}_S)(R(k_S)\tilde{a}^2_{S-1}+\tilde{\phi}_{new}w(\tilde{k}_S))^{-\sigma})\\&+(1-p)(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^2_{S-1}+\hat{\phi}_{old}w(\hat{k}_S))^{-\sigma})]\end{aligned} \tag{3.38}$$

The agents in either state of the world have the same information set and form the same expectations thus:

$$\tilde{a}^1_{S-2}=\hat{a}^1_{S-2}$$
$$\tilde{a}^2_{S-1}=\hat{a}^2_{S-1}$$

The generation who is young at time $S-1$ choices a^1_{S-1} using the stochastic Euler equation (3.39) or (3.41). Uncertainty is resolved in period S (either reform occurs, or it occurs in period $S+1$ with certainty). The agents choose a^2_S using the equation (3.40) or (3.42).

Path $\tilde{x}$:

$$\begin{aligned}
&((1-\tilde{\tau}_{S-1})w(\tilde{k}_{S-1}) - \tilde{a}^1_{S-1})^{-\sigma} \\
&= \beta[p(R(k_{\tilde{S}})(R(\tilde{k}_S)\tilde{a}^1_{S-1} + (1-\tilde{\tau}_S)w(\tilde{k}_S) - \tilde{a}^2_S)^{-\sigma}) \\
&+ (1-p)(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^1_{S-1} + (1-\hat{\tau}_S)w(\hat{k}_S) - \hat{a}^2_S)^{-\sigma})]
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
&p((R(\tilde{k}_S)\tilde{a}^1_{S-1} + (1-\tilde{\tau}_S)w(\tilde{k}_S) - \tilde{a}^2_S)^{-\sigma}) \\
&+ (1-p)((R(\hat{k}_S)\hat{a}^1_{S-1} + (1-\hat{\tau}_S)w(\hat{k}_S) - \hat{a}^2_S)^{-\sigma}) \\
&= \beta(R(\tilde{k}_{S+1})(R(\tilde{k}_{S+1})\tilde{a}^2_S + \tilde{\phi}_{S+1}w(\tilde{k}_{S+1}))^{-\sigma})
\end{aligned} \tag{3.40}$$

Path $\hat{x}$:

$$\begin{aligned}
&((1-\hat{\tau}_{S-1})w(\hat{k}_{S-1}) - \hat{a}^1_{S-1})^{-\sigma} \\
&= \beta[p(R(k_{\tilde{S}})(R(\tilde{k}_S)\tilde{a}^1_{S-1} + (1-\tilde{\tau}_S)w(\tilde{k}_S) - \tilde{a}^2_S)^{-\sigma}) \\
&+ (1-p)(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^1_{S-1} + (1-\hat{\tau}_S)w(\hat{k}_S) - \hat{a}^2_S)^{-\sigma})]
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
&p((R(\tilde{k}_S)\tilde{a}^1_{S-1} + (1-\tilde{\tau}_S)w(\tilde{k}_S) - \tilde{a}^2_S)^{-\sigma}) \\
&+ (1-p)((R(\hat{k}_S)\hat{a}^1_{S-1} + (1-\hat{\tau}_S)w(\hat{k}_S) - \hat{a}^2_S)^{-\sigma}) \\
&= \beta(R(\hat{k}_{S+1})(R(\hat{k}_{S+1})\hat{a}^2_S + \hat{\phi}_{S+1}w(\hat{k}_{S+1}))^{-\sigma})
\end{aligned} \tag{3.42}$$

Prior to reform, agents along either path of the economy have the same information set and form the same expectations; however, in period S, the agents will make different choices depending on the realization of the stochastic policy process. Thus:

$$\tilde{a}^1_{S-1} = \hat{a}^1_{S-1}$$

$$\tilde{a}^2_S \neq \hat{a}^2_S$$

Following the resolution of uncertainty in period S, all subsequent generations make decision under perfect foresight using (3.6) and (3.7). The choices of the agents differ between the two paths.

$$\tilde{a}^1_{S+j} \neq \hat{a}^1_{S+j}$$

$$\tilde{a}^2_{S+j} \neq \hat{a}^2_{S+j} \quad \text{for} \quad j \in [1, T]$$

Note that decisions made by agents prior to period $S-1$ will be the same on either path of the economy.

$$\tilde{a}^1_{S-i} = \hat{a}^1_{S-i}$$

$$\tilde{a}^2_{S-i} = \hat{a}^2_{S-i} \quad \text{for} \quad i \geq 1$$

Note finally that the path of the state variables will the same for the two paths until the reform is enacted.

I solve for the paths $\{\tilde{a}^1_{t-1}, \tilde{a}^2_{t-1}, \tilde{k}_t, \tilde{b}_t, \hat{a}^1_{t-1}, \hat{a}^2_{t-1}, \hat{k}_t, \hat{b}_t\}^T_{t=1}$ simultaneously given initial conditions a^1_{-1}, a^2_{-1}, k_0 and b_0.

Although this simple policy uncertainty is a small deviation from the perfect foresight model, it is enough to produce discernible changes to equilibrium dynamics. I will present four examples below to highlight the impact of this simple policy timing uncertainty. Before presenting examples, I will discuss the expectation formation process for the individuals in the economy.

The majority of agents in the model do not form expectations, as they do not face any uncertainty. They simply solve the household maximization problem under perfect foresight and choose asset holdings a^1 and a^2 to maximize their utility. Only two cohorts of agents face uncertainty in this simple set up. These agents choose asset allocations according to the equations listed above. The agents choose assets to maximize the mathematical expected value of their lifetime utility. The agents expectations are consistent with the underlying stochastic process that governs policy. The agents are fully rational and act according to expected utility theory.

This is an important difference between the policy experiments in Bütler (1999) and the policy experiments presented in this paper. Bütler examines how agents' expectations about social security policy influence macroeconomic variables. However, as Phelan (1999), points out, the policy expectation of the agents in Bütler's paper are not model consistent. Bütler's agents are not rational and do not form expectations based on an underlying stochastic process that governs policy. The agents expectations are effectively different "perceptions" or beliefs regarding how and when policy will change. The beliefs are not tied to actual policy process. The analysis presented in this paper can be viewed a response to Phelan's comment.

Using the simple two-period timing uncertainty presented in this section, there are only two possible time paths for the economy. Either policy is changed in period S, or it is changed in period $S + 1$. I solve the equilibrium equations for both possible time paths (including the agent level expectations) and present them graphically. In more complex models with greater aggregate uncertainty (like Davig et al. (2011), Davig and Leeper (2011b), or Krueger and Kubler (2004)), it is common to present a time path for the economy that is constructed by taking the average of several, possibly thousands, draws from the stochastic process. Since the number of possible stochastic paths is so small in this model, I simply present the evolution of the economy for all paths. I also present the time paths for an alternate economy that does not have any policy uncertainty, as a benchmark to isolate the impact of the uncertainty.

First, I consider an economy that is in a steady state that experiences a large social security tax increase (τ^0 increases, τ^1 remains the same) in either period $t = 3$ or period $t = 4$ (Figure 16). Agents alive prior to period $t = 3$ know there is a probability $p = 0.5$ that taxes will be increased in $t = 3$, and that if taxes are unchanged in $t = 3$, they will be increased the following period.[14] Young agents who expect that they may have to pay higher taxes when they are middled aged respond by increasing their savings. Thus, regardless of when the policy change is implemented, we see the young increase their savings in period $t = 2$ in anticipate of possibly higher taxes in period $t = 3$. This is particularly evident when comparing the savings of the young who experience the tax increase in period $t = 4$ under uncertainty (that is, they thought the tax increase was possible in either period 3 or 4, indicated in the graph with diamonds) compared to the savings of the young who expect the tax increase with perfect foresight in $t = 4$ (indicated in the graph with squares). The agents in the model with uncertainty save more in periods 2 and 3 leading up to the tax increase. After the tax increase is implemented, savings fall for both the young and middle aged, capital increases, and bonds decrease as the government runs large surpluses.

It is also interesting to compare the middle aged savings of an agent who knows taxes will be increased in her old age to the middle aged savings of an agent who faces a probability p chance that taxes are increased in her old age and a probability $1 - p$ chance that taxes are increased the following period. Although agents in the model do not pay taxes in retirement, they anticipate the general equilibrium feedback effects of younger generations decreasing savings in response

[14] The tax increase creates a large social security surplus and leads the government to accumulate assets. This does not represent a policy that is likely to occurs in the U.S., but it provides a useful experiment to examine the responses to agents to tax rate uncertainty.

to a tax increase (reduced savings lowers the capital stock, driving down the wage and social security benefit). Thus, middle aged agents who see taxes increasing the following period with certainty save more than agents who have the possibility that taxes won't be increased until after their death. This is evident in the graph by comparing the middle aged savings a_2^2 of agents facing certain reform in period $t = 3$, with the middle aged savings of agents who expected reform was possible in $t = 3$ or $t = 4$. The initial values, policy parameters, and final steady state for this experiment are listed in Table 12.

I consider a similar example in Figure 17. The economy starts in a steady state with social security and the government greatly reduces social security benefits in either period $t = 3$ or period $t = 4$. Young agents who think that their benefits might be cut relative to their parents, increase their savings. This is particularly evident comparing young savings in period $t = 1$ (a_1^1) for reform uncertainty with young savings in period $t = 1$ for an economy that doesn't cut benefits until period 4 with certainty. The young agents in the world with certainty know their benefits will not be cut, so they do not increase a_1^1 compared to the initial generation of young. However, the young agents who think benefits will be cut in period $t = 3$ with probability $p = 0.5$, save more to offset the possible loss in retirement income. We also see large effects in the savings choices of the middle aged. Middle aged agents in period $t = 2$ who face a probability $p = 0.5$ that their social security benefit will be cut (indicated with circles and diamonds) save more than the prior generation and more than agents who know benefits will not be reduced until period $t = 4$ (indicated with squares). However, these agents save less than agents in the counterfactual world who know their benefits will be cut with

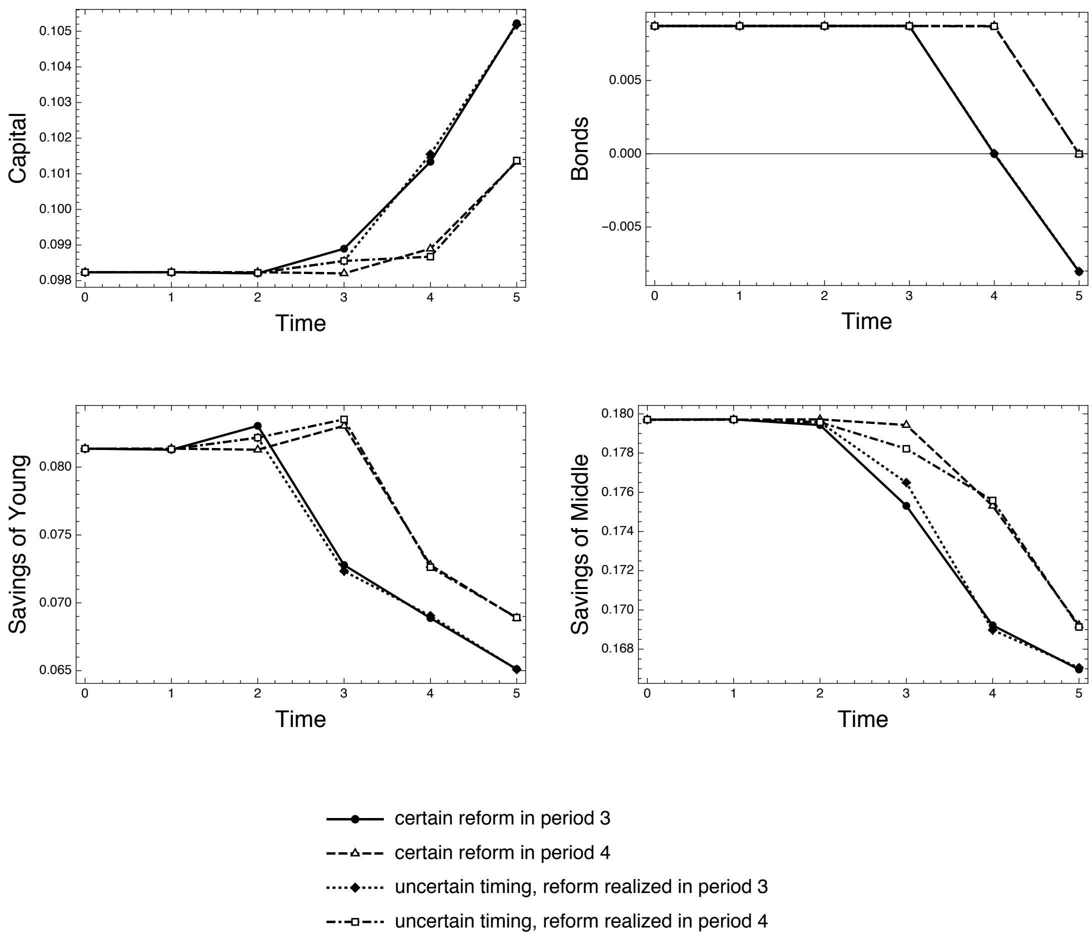


FIGURE 16. Time paths for an economy that is in a steady state and then experiences a large social security **tax increase** either in period 3 or 4. Time paths are plotted for an economy that faces probability p=0.5 of reform in period 3. If reform is not enacted in period 3, it happens with certainty in period 4. There are two possible paths for the economy under this type of stochastic policy, either the reform takes place in period 3 (these paths are labeled uncertain reform in 3), or it happens in period 4 (these paths are labeled uncertain reform in 4). Time paths are plotted in addition for reform that happens with certainty in either period. The top left panel shows capital, top right shows bonds, bottom left shows the savings (or asset allocations) of the young (a^1), and the bottom right shows the savings of the middle aged (a^2).

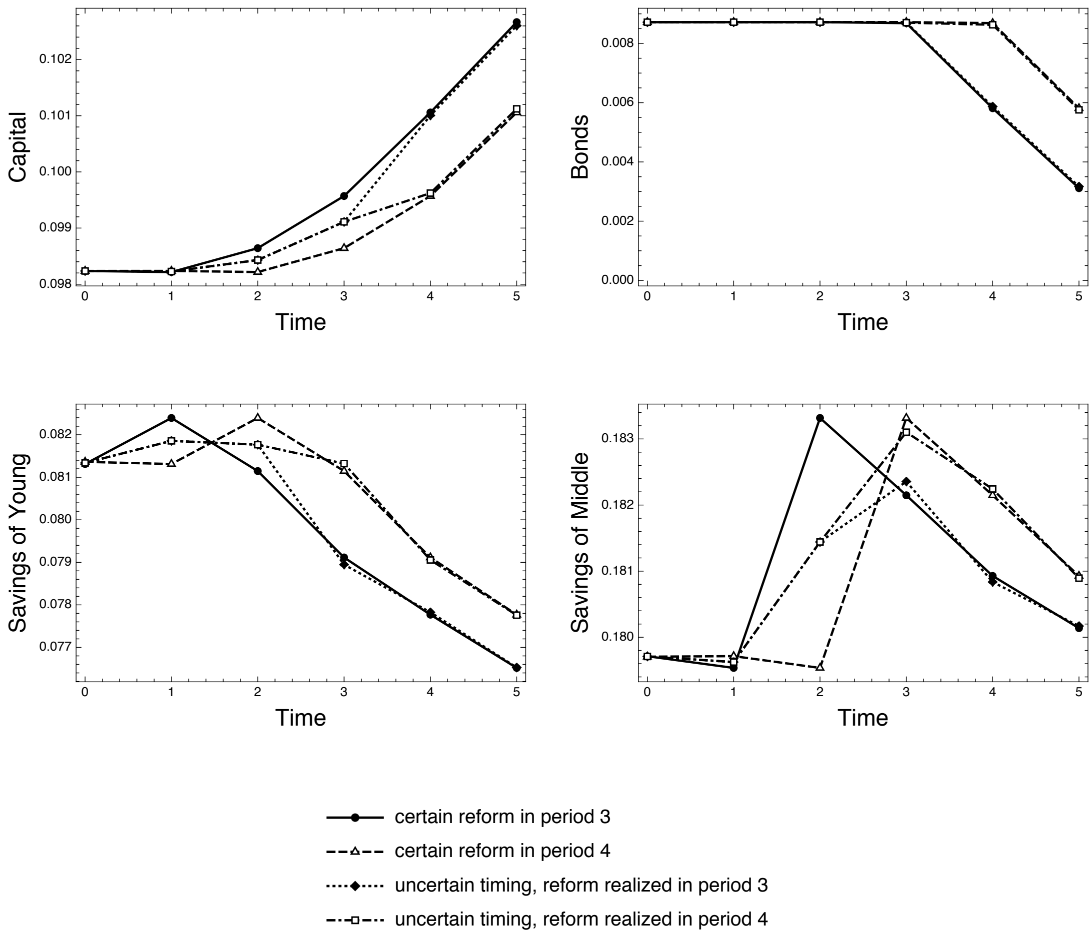


FIGURE 17. Time paths for an economy that is in a steady state and then experiences a large social security **benefit reduction** either in period 3 or 4. Time paths are plotted for an economy that faces probability p=0.5 of reform in period 3. If reform is not enacted in period 3, it happens with certainty in period 4. There are two possible paths for the economy under this type of stochastic policy, either the reform takes place in period 3 (these paths are labeled uncertain reform in 3), or it happens in period 4 (these paths are labeled uncertain reform in 4). Time paths are plotted in addition for reform that happens with certainty in either period. The top left panel shows capital, top right shows bonds, bottom left shows the savings (or asset allocations) of the young (a^1), and the bottom right shows the savings of the middle aged (a^2).

certainty in period $t = 3$ (indicated in the graph with triangles). The initial values, policy parameters, and final steady state for this experiment are listed in Table 12.

Next I consider an economy that is on an unsustainable path (deficits are too large and no steady states exist), that remedies the situation by increasing the social security tax rate in either period $t = 3$ or period $t = 4$ (Figure 18).[15] Reform is expected in period $t = 3$ with probability $p = 0.5$, and if reform does not happen in period $t = 3$, it happens in period $t = 4$. The tax change in this example is relatively small compared to the tax change in examined in Figure 16, thus the time paths for uncertain reform are closer to the time paths with no uncertainty. There is still a discernible increase in young savings (a_2^1) for agents who think they might be taxed higher next period, compared to those who know then tax increase won't happen until after they retire. The most visible difference between uncertain and certain reform in this example is evident in middle aged savings in period 3 (a_3^2). If reform takes place in period $t = 4$ with certainty, we observe middle aged savings increase dramatically in $t = 3$. If reform is possible in period $t = 3$, but doesn't actually take place until period $t = 4$, the middle aged savings increases more slowly. Specifically a_3^2 is lower in the model with uncertainty. The key difference is that when they were young (in period $t = 1$), the agents in the model with uncertainty saved more because they though the tax increase was possible in their middle age. They entered period $t = 2$ with more assets than the agents in the counterfactual world where the reform occurs in period $t = 4$ with certainty. Thus, they did not need to save as much as the agents in the model with certainty. Although it is not visible in the graph, all four time paths converge to the same steady state that is determined by the policy parameters after the tax increase. The initial values, policy parameters, and steady state are listed in Table 12.

[15] The initial parameter values are identical to the perfect foresight example presented in Figures 14 and 15. The magnitude of the tax increase is also identical to the example in Figure 15.

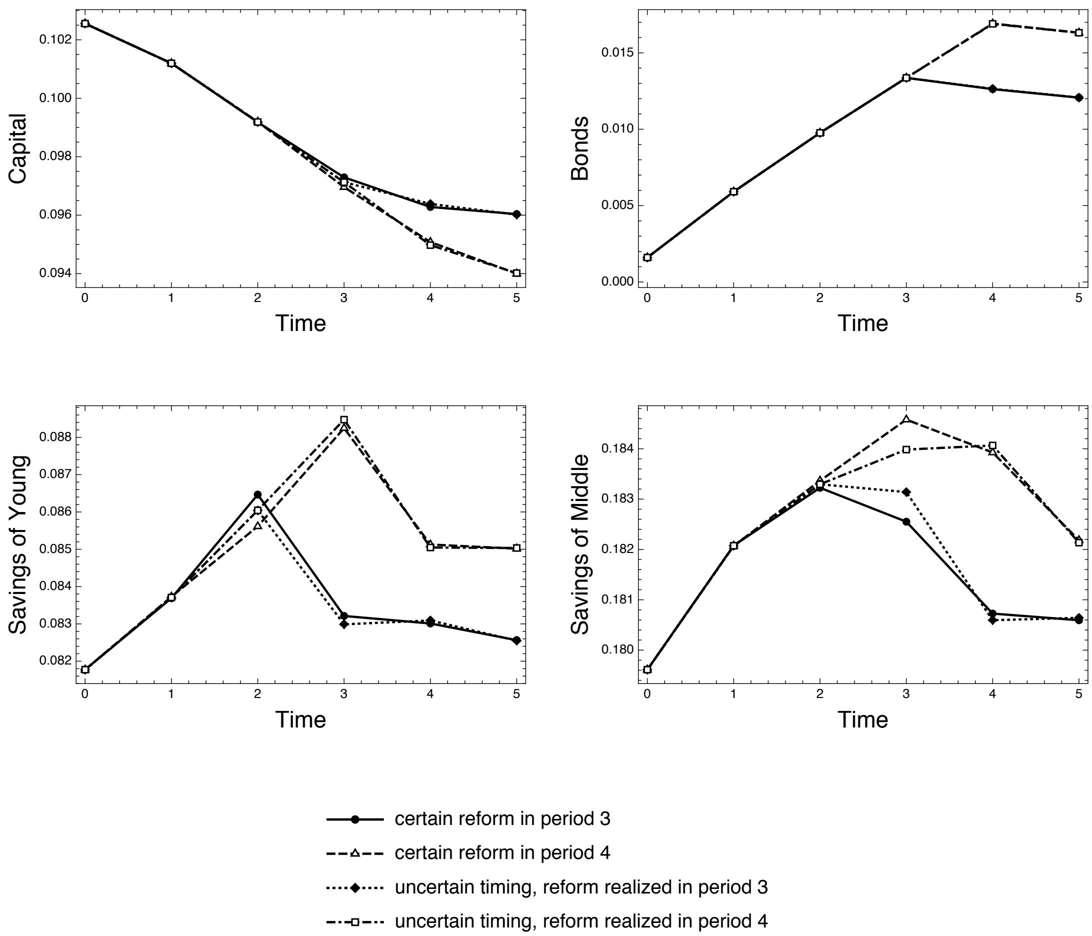


FIGURE 18. Time paths for an economy that has unsustainable policy that is reformed via **tax increase** either in period 3 or 4. Time paths are plotted for an economy that faces probability p=0.5 of reform in period 3. If reform is not enacted in period 3, it happens with certainty in period 4. There are two possible paths for the economy under this type of stochastic policy, either the reform takes place in period 3 (these paths are labeled uncertain reform in 3), or it happens in period 4 (these paths are labeled uncertain reform in 4). Time paths are plotted in addition for reform that happens with certainty in either period. The top left panel shows capital, top right shows bonds, bottom left shows the savings (or asset allocations) of the young (a^1), and the bottom right shows the savings of the middle aged (a^2).

Fig	Initial policy			New policy			Initial values				Final steady state(s)			
	τ^0	τ^1	ϕ	τ^0	τ^1	ϕ	k	b	a^1	a^2	k	b	a^1	a^2
16	0.076	0.8	0.2	0.1	0.8	0.2	0.098	0.009	0.081	0.18	0.121	-0.031	0.058	0.163
17	0.076	0.8	0.2	0.077	0.8	0.18	0.098	0.009	0.081	0.18	0.113	-0.012	0.071	0.176
18	0.065	0.8	0.2	0.077	0.8	0.2	0.103	0.002	0.078	0.177	0.103	0.002	0.078	0.178
19	0.065	0.8	0.2	0.065	0.8	0.173	0.103	0.002	0.078	0.177	0.103	0.006	0.081	0.185

TABLE 12. Policy Experiments with simple two-period timing uncertainty. Experiment 16: Economy in steady state with social security, taxes raised in either period t=3 or t=4. Experiment 17: Economy in steady state with social security, benefits cut in either period t=3 or t=4. Experiment 18: Economy on explosive path, taxes increased in in either period t=3 or t=4. Experiment 19: Economy on explosive path, benefits cut in either period t=3 or t=4. Two-period timing uncertainty.

The final example I consider with simple two-period timing uncertainty is an economy that is on an explosive path that reforms social security via benefit reduction in either period $t = 3$ or $t = 4$ (Figure 19). The initial values and the size of the benefit cut are identical to the perfect foresight example in Figure 15. Agents who are young in period $t = 1$ and face the possibility of a benefit cut during their retirement in $t = 3$ save more than agents who know benefits will not be cut until $t = 4$. Conversely, young agents who aren't sure if benefits will be cut in their lifetime save less than agents who know benefits will be cut in $t = 3$ with certainty. We see a similar pattern looking at the savings of the middle age in period $t = 2$. Agents who might have their benefits cut in $t = 3$ save more than the agents who know benefits won't be cut until $t = 4$, and less than the agents who know benefits will be cut in $t = 3$. The initial values, policy parameters, and steady state are listed in Table 12.

Timing and Policy Uncertainty

The analysis of the previous section only includes policy timing uncertainty. In Appendix B, I also consider simple policy type uncertainty (with a known date of reform). In reality, Americans face uncertainty about the future of social security

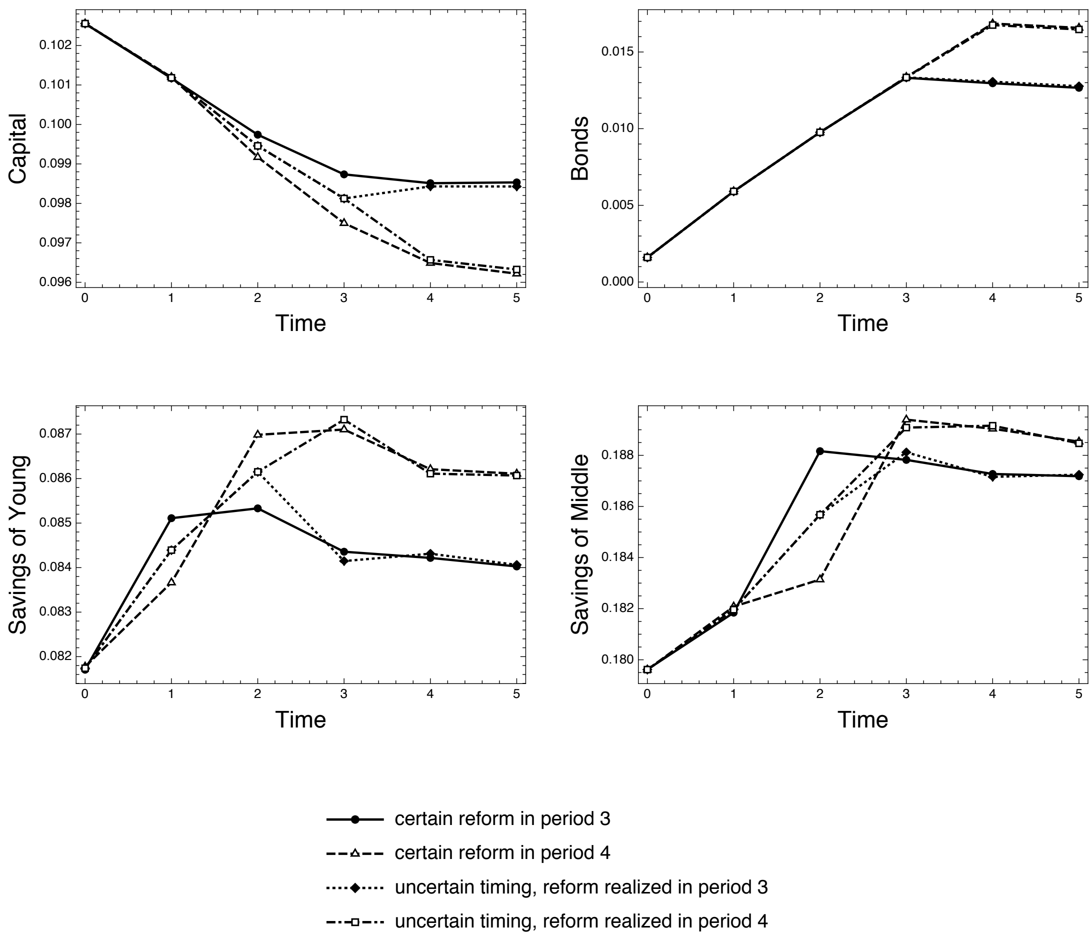


FIGURE 19. Time paths for an economy that has unsustainable policy that is reformed via **benefit reduction** either in period 3 or 4. Time paths are plotted for reform that happens with certainty in either period. Time paths are plotted in addition for an economy that faces probability p=0.5 of reform in period 3. If reform is not enacted in period 3, it happens with certainty in period 4. There are two possible paths for the economy under this type of stochastic policy, either the reform takes place in period 3 (these paths are labeled uncertain reform in 3), or it happens in period 4 (these paths are labeled uncertain reform in 4).

policy that span time and policy options. Social security could be reformed by cutting taxes, raising benefits, or both. Additionally, that reform could take place today, or tomorrow, or ten years from now. In this final section, I explore simple examples of timing *and* policy uncertainty. For example, consider a government that will implement either reform A or reform B, and the reform will either take place in period S, or in period $S+1$. Under this regime, four policy paths are possible. Suppose the probability of each path is as follows: probability p of reform A in period S, probability q of reform A in $S+1$, probability r of reform B in S, and probability s of reform B in $S+1$.[16]

The generation who is young at time $S-2$ chooses a^1_{S-2} using the non-stochastic Euler equation (as in previous sections, see 3.36) and a^2_{S-1} using a stochastic Euler equation such as (3.43). There will be four distinct Euler equations (one for each path of the economy). Only one path is presented below.[17]

For path $\tilde{x}$:

$$\begin{aligned}
&(R(\tilde{k}_{S-1})\tilde{a}^1_{S-2} + (1-\tilde{\tau}_{S-1})w(\tilde{k}_{S-1}) - \tilde{a}^2_{S-1})^{-\sigma} \\
&= \beta[p(R(\tilde{k}_S)(R(k_S)\tilde{a}^2_{S-1} + \tilde{\phi}_S w(\tilde{k}_S))^{-\sigma}) \\
&+ q(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^2_{S-1} + \hat{\phi}_S w(\hat{k}_S))^{-\sigma}) \\
&+ r(R(\bar{k}_S)(R(\bar{k}_S)\bar{a}^2_{S-1} + \bar{\phi}_S w(\bar{k}_S))^{-\sigma}) \\
&+ s(R(\breve{k}_S)(R(\breve{k}_S)\breve{a}^2_{S-1} + \breve{\phi}_S w(\breve{k}_S))^{-\sigma})]
\end{aligned} \tag{3.43}$$

The generation who is young at time $S-1$ choices a^1_{S-1} using a stochastic Euler equation such as (3.44). Uncertainty is resolved in period S only if reform

[16] $p+q+r+s=1$, and $p<1$, $q<1$, $r<1$, and $s<1$

[17] The distinct path are denoted using the following notation: reform A in S as $\tilde{x}$, reform A in $S+1$ as $\hat{x}$, reform B in S as $\bar{x}$, and reform B in $S+1$ as $\breve{x}$, for $x = k, b, a^1, a^2, \tau, \phi$.

occurs in that period; if not there is still uncertainty about which reform will take place in period $S+1$. The agents choose a_S^2 using the equation (3.45).

Path $\tilde{x}$:

$$\begin{aligned}
&((1-\tilde{\tau}_{S-1})w(\tilde{k}_{S-1})-\tilde{a}_{S-1}^1)^{-\sigma}\\
&=\beta[p(R(\tilde{k}_S)(R(\tilde{k}_S)\tilde{a}_{S-1}^1+(1-\tilde{\tau}_S)w(\tilde{k}_S)-\tilde{a}_S^2)^{-\sigma})\\
&+q(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}_{S-1}^1+(1-\hat{\tau}_S)w(\hat{k}_S)-\hat{a}_S^2)^{-\sigma})\\
&+r(R(\bar{k}_S)(R(\bar{k}_S)\bar{a}_{S-1}^1+(1-\bar{\tau}_S)w(\bar{k}_S)-\bar{a}_S^2)^{-\sigma})\\
&+s(R(\breve{k}_S)(R(\breve{k}_S)\breve{a}_{S-1}^1+(1-\breve{\tau}_S)w(\breve{k}_S)-\breve{a}_S^2)^{-\sigma})]
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
&p(R(\tilde{k}_S)\tilde{a}_{S-1}^1+(1-\tilde{\tau}_S)w(\tilde{k}_S)-\tilde{a}_S^2)^{-\sigma})\\
&+q(R(\hat{k}_S)\hat{a}_{S-1}^1+(1-\hat{\tau}_S)w(\hat{k}_S)-\hat{a}_S^2)^{-\sigma})\\
&+r(R(\bar{k}_S)\bar{a}_{S-1}^1+(1-\bar{\tau}_S)w(\bar{k}_S)-\bar{a}_S^2)^{-\sigma})\\
&+s(R(\breve{k}_S)\breve{a}_{S-1}^1+(1-\breve{\tau}_S)w(\breve{k}_S)-\breve{a}_S^2)^{-\sigma})\\
&=\beta(R(\tilde{k}_{S+1})(R(\tilde{k}_{S+1})\tilde{a}_S^2+\tilde{\phi}_{S+1}w(\tilde{k}_{S+1}))^{-\sigma})
\end{aligned} \tag{3.45}$$

Generations born in period S face uncertainty if reform did not take place in period S. They know reform will occur in period $S+1$, but they don't know form it will take. Along path $\tilde{x}$ (reform A in period S) and path $\bar{x}$ (reform B in period S), young agents in period S choose savings using non-stochastic Euler Equations.

Young agents in period S using the following first order equations for path $\hat{x}$ (reform A in period $S+1$, there is a similar set of FOC for path $\breve{x}$):

$$\begin{aligned}
&((1-\hat{\tau}_S)w(\hat{k}_S)-\hat{a}_S^1)^{-\sigma}\\
&=\beta[q/(q+s)(R(\hat{k}_{S+1})(R(\hat{k}_{S+1})\hat{a}_S^1+(1-\hat{\tau}_{S+1})w(\hat{k}_{S+1})-\hat{a}_{S+1}^2)^{-\sigma}\\
&+s/(q+s)(R(\breve{k}_{S+1})(R(\breve{k}_{S+1})\breve{a}_S^1+(1-\breve{\tau}_{S+1})w(\breve{k}_{S+1})-\breve{a}_{S+1}^2)^{-\sigma}]
\end{aligned} \tag{3.46}$$

$$
\begin{aligned}
&q/(q+s)(R(\hat{k}_{S+1})\hat{a}_S^1 + (1-\hat{\tau}_{S+1})w(\hat{k}_{S+1}) - \hat{a}_{S+1}^2)^{-\sigma} \\
&+ s/(q+s)(R(\breve{k}_{S+1})\breve{a}_S^1 + (1-\breve{\tau}_{S+1})w(\breve{k}_{S+1}) - \breve{a}_{S+1}^2)^{-\sigma} \\
&= \beta(R(\hat{k}_{S+2})(R(\hat{k}_{S+2})\hat{a}_{S+1}^2 + \hat{\phi}_{S+2}w(\hat{k}_{S+2}))^{-\sigma}
\end{aligned} \tag{3.47}
$$

As an example, consider an economy that is in a steady state, and then experiences a change of policy in either period $t = 3$ or $t = 4$. Suppose there is probability p that taxes are raised in period $t = 3$, probability q that taxes are raised in period $t = 4$, probability r that benefits are cut in $t = 3$, and a probability s that benefits are cut in period $t = 4$ (where $p = q = r = s = 0.25$). The time paths for the four possible outcomes are plotted in Figure 20. The four time paths for the same policy changes occurring with certainty have been excluded from the graph for simplicity. This exercise combines the timing uncertainty of Figures 16 and 17 with the simple policy option uncertainty.

Young agents in period $t = 1$ who aren't sure if reform will happen in their lifecycle and also aren't sure if benefits will be cut or if taxes will be raised engage in a small amount of precautionary savings. Only one of the four possible policy paths increases the young agents need to save (a benefit reduction in period 3). Thus, the young agent doesn't respond very much. In period $t = 2$, the young agents increases savings to offset the possible tax increase during their middle age and the possible benefit cut. If reform occurs in period $t = 3$, agents respond accordingly. If reform does not take place in period $t = 3$, the young agent still faced some uncertainty; benefits might be cut in period 4 or taxes might be increased. The young agent in period $t = 3$ facing this uncertainty increases savings to offset the possibility of higher taxes next period.

This example has important policy implications. Survey data suggests that many American households expect social security to be reformed and only expect

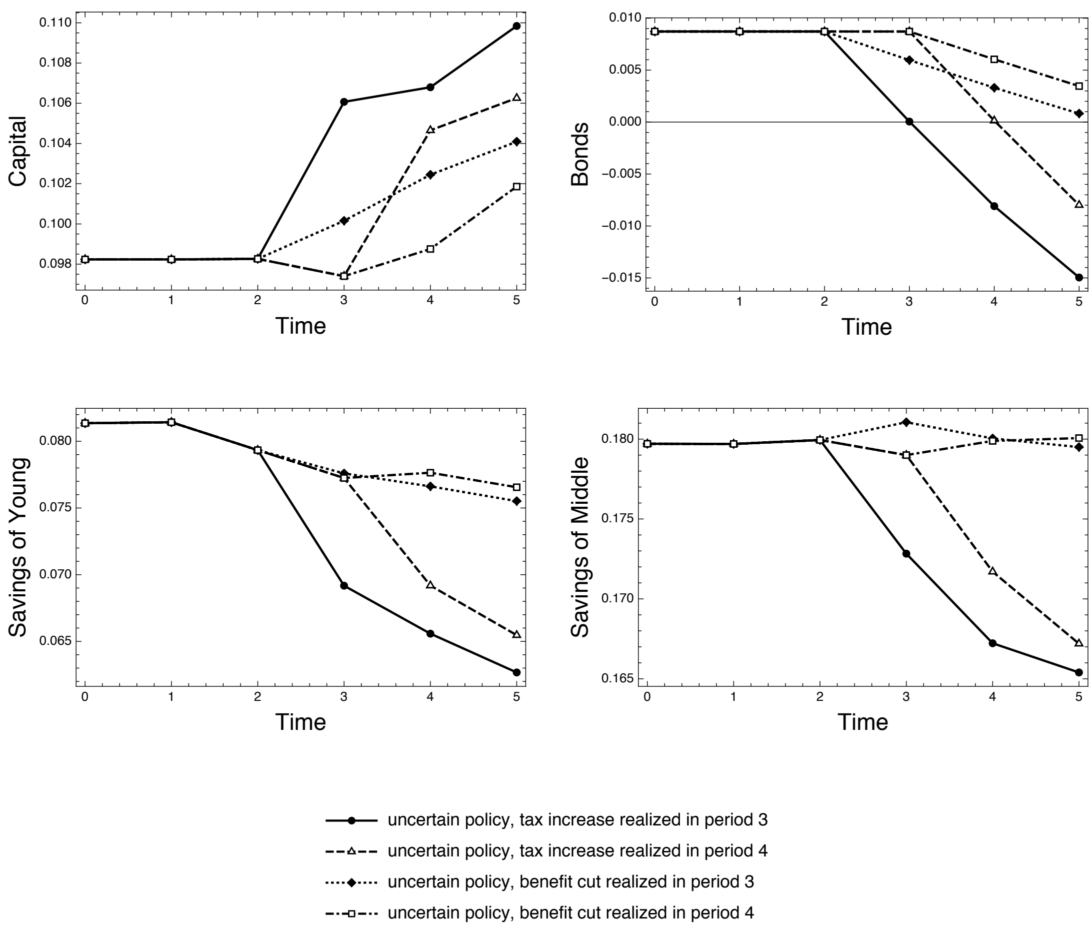


FIGURE 20. Time paths for an economy that begins in a steady state and then either experiences i) a tax increase in t=3, ii) a tax increase in t=4, iii) a benefit cut in t=3, or iv) a benefit cut in t=4. There are four possible time paths possible given this type of uncertainty.

Figure	Initial policy			New policy			Initial values				Final steady state(s)			
	τ^0	τ^1	ϕ	τ^0	τ^1	ϕ	k	b	a^1	a^2	k	b	a^1	a^2
20	0.0756	0.8	0.2	0.1	0.8	0.2	0.098	0.009	0.081	0.18	0.121	-0.031	0.058	0.163
				0.0756	0.8	0.18					0.113	-0.012	0.071	0.176

TABLE 13. Policy experiment with timing and policy uncertainty. Experiment 20: Economy in a steady state. Possible reforms (each with probability 0.25): taxes raised in t=3, benefits cut in t=3, taxes raised in t=4, or benefits cut in t=4. Policy option and two-period timing uncertainty.

to receive a fraction of promised benefits.[18] Yet, in contrast to standard economic theory, these households are *not* increasing their private savings.

Standard lifecycle models suggest that households who expect to receive smaller social security benefits should save more. In the context of this model, the relationship between savings and social security benefits is negative, that is $\partial a_t^1/\partial\phi_{t+2} < 0$ and $\partial a_t^2/\partial\phi_{t+1} < 0$. Similarly, young agents who expect to pay higher taxes next period, save more today; $\partial a_t^1/\partial\phi_{t+1} > 0$.

The introduction of policy and timing uncertainty can partially explain the paradox that households expect social security reform, yet they do not increase their savings. The time $t = 1$ young agents in Figure 20 do not increase their savings very much, even though they believe there is a 50% chance of reform in their lifetime. The incentive to increase savings to offset a potential benefit cut is offset by the incentive to consume now and hope for policy that doesn't directly impact the lifetime resources of the agent.[19]

I intend to explore this relationship further in my future work.

[18]See Dominitz and Manski (2006) for an overview of the social security expectations literature

[19] Young savings is only a function of the benefit rate two period ahead; $\partial a_t^1/\partial\phi_{t+i} = 0$ for all $i \neq 2$ and only responds to current and one period ahead taxes: $\partial a_t^1/\partial\tau_{t+i} = 0$ for all $i < 0$ and $i \geq 2$. Middle aged savings responds to benefit rate the next period, but no other periods $\partial a_t^2/\partial\phi_{t+i} = 0$ for all $i \neq 1$. Middle aged savings does not depend on future taxes $\partial a_t^2/\partial\tau_{t+i} = 0$ for all $i \geq 2$.

Long-run Consequences of Policy Uncertainty

The analysis of the previous sections focused on the short-run, intergenerational consequences of policy uncertainty. The savings decisions of agents who face uncertainty also have interesting long-run, macroeconomic consequences. The possibility of unstable government policy (low taxes and/or high benefits that lead to explosive government debt) can destabilize the economy. Unsurprisingly, the possibility of low taxes (or high benefits) decreases the agents' incentives to save and decreases the capital stock, even if the unstable policy is not realized.

Interestingly, the reverse is also possible. The potential for tax increases (or benefit cuts) in the future, may be enough to stabilize the path of the economy even if the tax cut (or benefit increase) is not realized. The precautionary savings of the agents facing uncertainty can be enough to increase the capital stock to lie within the basin of attraction of the stable steady state. That is, policy uncertainty can generate a rational expectations equilibrium that converge to a steady state for given initial conditions that would *not* converge to a steady state in the absence of policy uncertainty.

As an example, consider an economy that is on a (temporarily) explosive path. The government reforms social security in period $t = 4$ by implementing a benefit cut. In the absence of policy uncertainty, benefit cut A is *insufficient* to stabilize the economy; that is, benefit cut A is not an equilibrium path. Benefit cut B, on the other hand, is larger, and is sufficient to cause the economy to converge to a steady state.

If there is a probability p that the government implements policy A, and a probability $(1 - p)$ of policy B, the realization of either policy can be a stable equilibrium path, as long as p is small enough. This is depicted graphically in Figure 21 and Figure 22. In this example, the initial benefit replacement rate is $\phi = 0.2$. Policy A (which is unstable on it's own) reduces the benefit replacement rate to $\phi = 0.1743$. Policy B cuts the benefit replacement rate to $\phi = 0.17$. The probability of benefit cut A is quite small at $p = 0.05$.

Figure 21 focuses on the first few periods of the economy. Note that the implementation of policy B with certainty, and realization of policy B under uncertainty are nearly identical. This is because the probability of policy B was quite high in the model with uncertainty. Figure 22 shows the long-run converge of the model. Benefit cut B converges to a higher level of capital and a lower level of bonds, since it is a larger benefit cut. The realization of benefit cut A in the model with uncertainty converges to a steady state with lower capital and higher bonds. The realization of benefit cut A also takes much longer to converge than policy B. The capital stock is depressed for generations following the realization of policy A.

Conclusion

Uncertainty about social security policy impacts agents over their lifecycle and impacts the economy for generations following a reform. I develop a three-period overlapping generations model with endogenous government deficits and debt to examine the interaction between social security policy and the long-run stability of the economy. The general equilibrium model is forward looking and agents care about future policy. The model is simple enough to accommodate aggregate uncertainty regarding the government's social security policy and can be solved computationally. Agents who face uncertain social security policy in their lifetime behave different than agents in a perfect foresight model who know

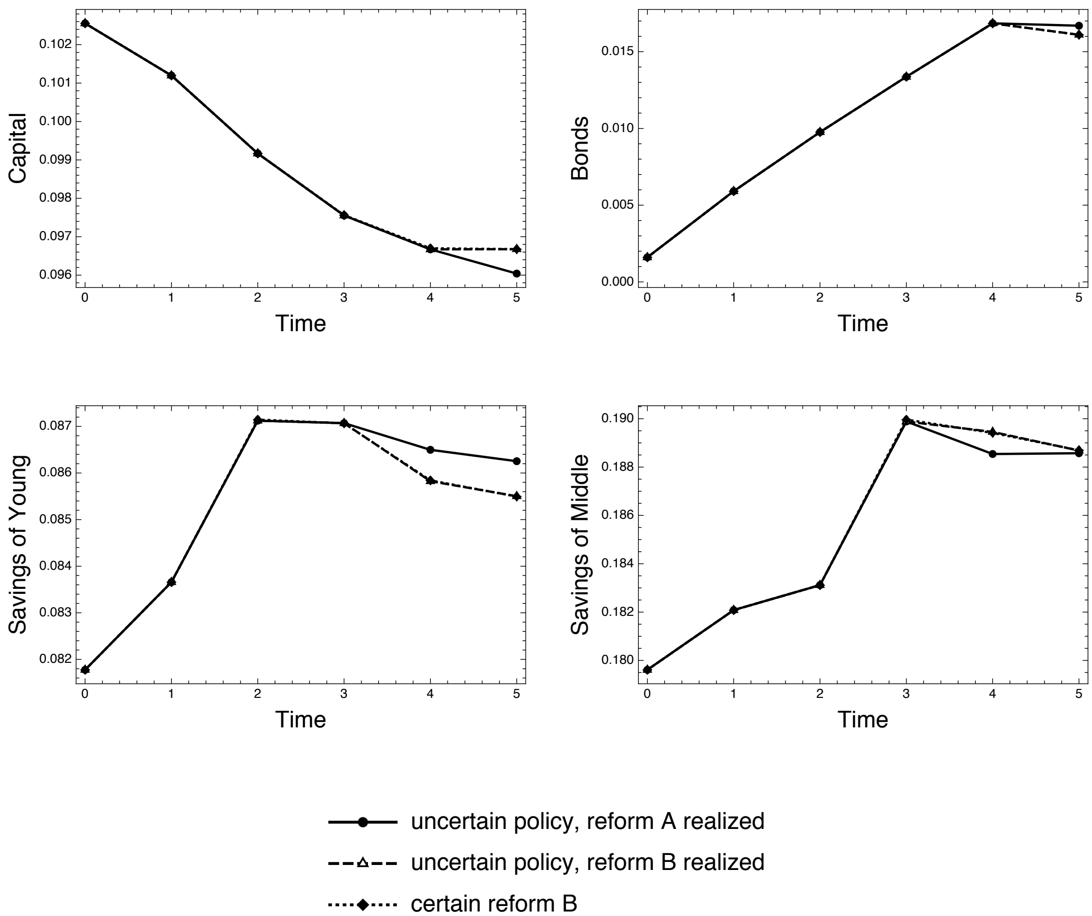


FIGURE 21. Time paths for an economy that is on an explosive path until social security benefits are cut in period t=4. There is a probability p=0.05 of policy A where benefits are cut to 0.1743 and a probability (1−p)=0.95 policy B where benefits are cut to 0.17. Policy A does not converge to a steady state if it is enacted with certainty. However, where there is a large enough probability of benefit but B, the precautionary savings of agents leading up to the reform increases the capital stock enough that the realization of policy A does, in fact, converge to a steady state. The graph also includes the equilibrium path for an economy that implements policy B with certainty. The equilibrium paths for realization of policy B in the uncertain model are nearly identical to the equilibrium paths for policy B implemented with certainty.

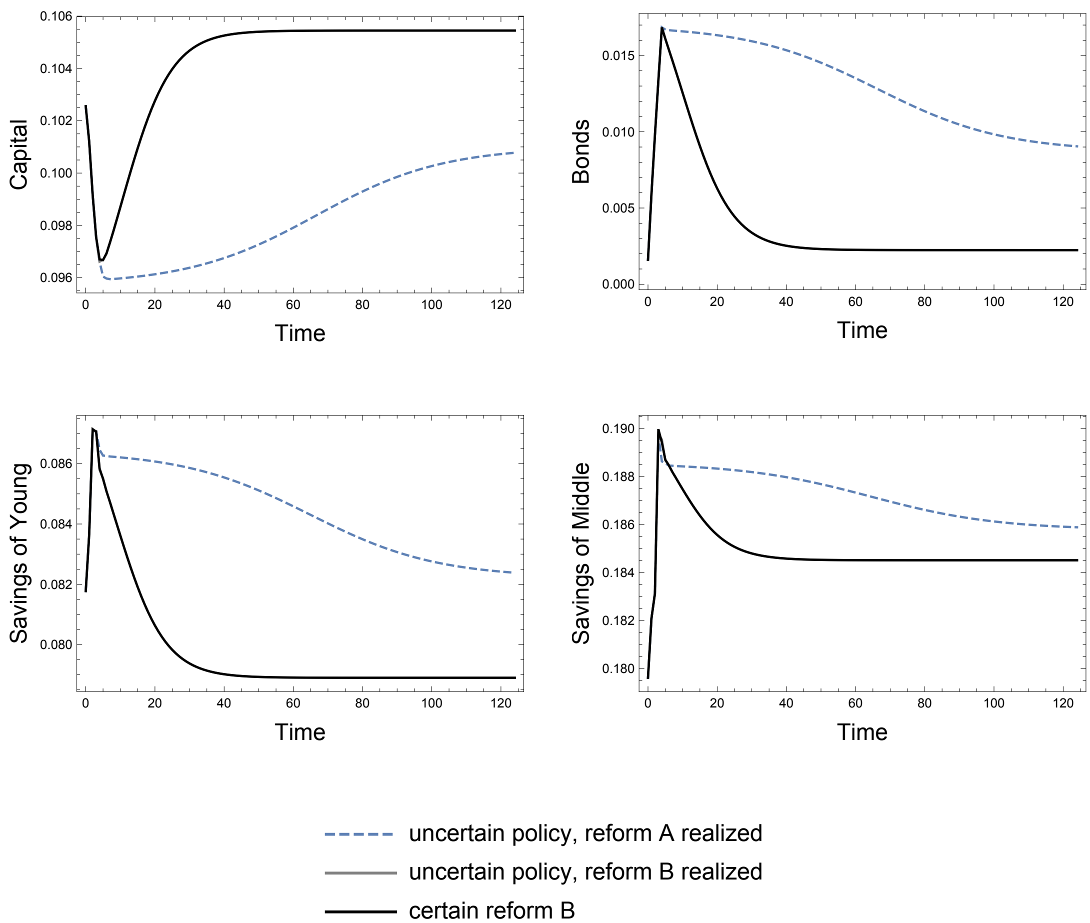


FIGURE 22. This is the same policy experiment as Figure 21. These graphs show the entire equilibrium paths for the realization of policy A and the realization of policy B in a model with uncertainty (where there is a 0.05 probability that policy A is enacted). The time paths for an economy where policy B is enacted with certainty is also depicted. The time paths for the realization of policy B under uncertainty are so similar to the time paths for the economy that enacts B with certainty, that the graphs lie directly on top of each other.

the future path of all policy parameters. Agents increase their savings if there is a possibility that their taxes will be increased or if their social security benefits will be decreased. Agents save less if there is a possibility that reform doesn't happen until after their lifetime, or the possibility that reform is small. The introduction of uncertainty regarding future policy partially explains the empirical finding that many American households simultaneously predict social security will be reformed and yet do not engage in precautionary saving.

The choices of agents who face policy uncertainty have lasting consequences even after the policy uncertainty has been resolved. The savings choices of agents determine the capital stock the following period, which in turn influences the savings and consumption decisions of agents. Even a small amount of uncertainty, for example a reform that takes place in either period S or in period $S+1$, can change the capital stock and influence the transition path of the economy for generations.

Policy uncertainty can destabilize an otherwise stable transition to a steady state, and policy uncertainty can make an otherwise unsustainable path converge to a steady state. As illustrated in section III, the possibility of a large reform can make the realization of a small reform stable in the long-run. This result is similar to the findings of Davig et al. (2011) who show in a New-Keynesian model with unsustainably large government transfers, that dire policy outcomes like hyperinflation can be avoided if there is a small probability that actual transfers will be less than promised transfers. In the New-Keynesian model the possibility of the government reneging on promised transfers increases the incentive for agents to save. In the OLG framework of this paper, the potential for a larger (or sooner) reform also causes agents to save more, which increases the basin of attraction for a given steady state and opens the possibility that more paths will converge in the long-run.

The short-run impacts of policy uncertainty, specifically the changes in savings decisions of agents prior to a reform, are similar to the results in Bütler (1999); expectations leading up to a reform have large impacts on aggregate variables and on consumption across generations. The analysis of the paper solidifies the results of Bütler, by providing a baseline model will forward-looking, fully rational agents. The agents in this paper do not suffer from "misperception" nor are they surprised by unannounced changes to policy. The expectations of agents are consistent with the (simple) stochastic process that governs the policy parameters.

Caliendo et al. (2015) calculate the welfare cost to a worker of facing uncertainty about the future of social security. They find that the welfare cost of uncertainty to be up to one percent of total lifetime consumption for agents who do not save (agents who save optimally experience a smaller welfare loss). The model includes many realistic details of the lifecycle (like stochastic survival probability) and can accommodate rich policy uncertainty that spans a distribution of potential reforms that take place in many possible periods. The partial equilibrium model focuses on a single agent, and does not consider the transition of the economy to a steady state following a reform. I have shown that policy uncertainty can have aggregate impacts for generations following a policy reform. The general equilibrium feedback of changes in saving can cause the capital stock to be lower (or higher) for fifty of more periods as the economy transitions to a steady state. Combining the insights from my three-period model with the results of Caliendo et al. (2015) suggest that welfare of the economy could be impacted for many generations if there is any uncertainty regarding the future path of social security.

CHAPTER IV

LIFE-CYCLE HORIZON LEARNING AND POLICY UNCERTIANTY

Introduction

In this chapter, I continue my analysis of policy uncertainty in a richer, multi-period lifecycle model. The way in which agents form expectations has a large impact on both the short-run responses to changes in policy (or possible changes in policy) and the long-run level of capital and output in the economy. I consider rational expectations and adaptive expectations. My paper contributes to a growing literature that examines the response to anticipated fiscal policy in models with adaptive expectations (see G. W. Evans et al. (2009), Mitra et al. (2013), Gasteiger and Zhang (2014), and Caprioli (2015)).

The rational expectations hypothesis is standard in macroeconomics and I develop a rational expectations model as the baseline for my analysis. I also relax the assumption of rational expectations and explore the consequences of agents using adaptive expectations in a model with an aging society and social security reform. One criticism of the rational expectations hypothesis is that it requires agents to possess more knowledge about the structure of the economy than any econometrician possesses about the actual economy. Models of adaptive learning relax that requirement, and allow for boundedly rational agents (Sargent (1993), G. W. Evans and Honkapohja (2001)).

Adaptive learning has several benefits. Models of adaptive learning produce more realistic dynamics that better fit observed data. Eusepi and Preston (2011) demonstrate that infinite-horizon learning amplifies technology shocks in a real business cycle model and creates more realistic dynamics. W. A. Branch and Mcgough (2011) show that a calibrated model with heterogenous beliefs (some agents are fully rational, others use N-step ahead adaptive learning) provides a closer fit to business cycle data than the rational expectations baseline. Models of adaptive learning are also better able to match survey data on expectations (W. A. Branch (2004)), asset price volatility (Bullard and Duffy (2001), Adam, Marcet, and Nicolini (2016)), and experimental data on expectation formation (Adam (2007) and Pfajfar and Santoro (2010)). Additionally, adaptive learning can be viewed as a robustness check to examine the stability properties of a rational expectations equilibrium (see G. W. Evans and Honkapohja (2001) and (2009)).

In this paper, I introduce a two new frameworks for modeling bounded rationality that I call life-cycle horizon learning, and finite horizon life-cycle learning. I simulate the underlying demographic changes and the two policy changes that the SSA suggests would be sufficient to fund future benefits. I find the welfare cost of implementing a reform is much larger for some cohorts in the learning models, particularly in the life-cycle horizon learning model, compared to a with model fully rational agents. Cohorts of agents alive before and after reforms can be negatively impacted by the cyclical dynamics adaptive expectations introduce into the model.

I also explore the consequences of policy uncertainty by modeling reform that can take place in one of two different dates and can take the form of a tax increase or a benefit cut. I find that the welfare cost of this type of policy uncertainty is small in the rational expectations model. The consumption equivalent variation that makes an agent indifferent between an announced policy reform or policy uncertainty (between four different options, tax increases or benefit cuts possible in two different periods) is at most 0.3% of period consumption. I find that the welfare cost to be much larger for agents in a life-cycle horizon learning model. The most harmed agents who experience an announced policy change in a rational expectations framework would have to give up 1.9% of their period consumption to be indifferent between announced policy in a life-cycle horizon model and policy uncertainty in a life-cycle horizon learning model.

I present the model in section IV and introduce life-cycle horizon learning and finite horizon life-cycle learning in sections IV and IV. The model is calibrated in section IV. Policy applications, including social security reform uncertainty are explored in section IV. Welfare analysis is in sections IV and IV. Section IV includes alternative modeling assumptions, and section IV concludes.

Model

Households

Households live for J periods, choose asset allocation (a^j for $j = 1, \cdots J-1$) in the first $J-1$ periods of life, and consumption (c^j for $j = 1, \cdots, J$) in all J stages of life, to maximize utility, taking price, and government social security policy (tax rates τ, and benefit z) as given. Agents receive wage w_t for labor provided in period t, and retire at date $T \leq J$. The gross real return on savings in period t is given by R_{t+1}. Superscripts on variable indicate life-cycle stage (i.e., age), and subscripts indicate time period.[1]

Household's maximize discounted expected lifetime utility (4.1) subject to period budget constraints (4.2). Here $E_t^*(x)$ indicates the time t expectation of x.

[1]As an example, a_{t+1}^2 is age 2 savings (saving in the second stage of life), in time period $t+1$.

The star indicates that the expectations need not be rational.

$$\max_{a^j_{t+j-1}} E^*_t \sum_{j=1}^{J} \beta^{j-1} u(c^j_{t+j-1}) \tag{4.1}$$

$$c^j_{t+j-1} + a^j_{t+j-1} \leq R_{t+j-1} a^{j-1}_{t+j-2} + y^j_{t+j-1} \qquad \text{for } j = 1, \cdots, J \tag{4.2}$$

Here y^j indicates period labor income during working life and social security income after retirement:

$$y^j_{t+j-1} = (1 - \tau_{t+j-1}) w_{t+j-1} \quad \text{for } j < T$$
$$y^j_{t+j-1} = z^j_{t+j-1} \quad \text{for } j \geq T$$

The lifespan is certain and agents do not have a bequest motive, so they exhaust all of their resources in the final period of life.

$$a^J_{t+J-1} = 0 \tag{4.3}$$

The household first order conditions are:

$$u'(c^j_{t+j-1}) = \beta E^*_t [R_{t+j} u'(c^{j+1}_{t+j})] \quad \text{for } j = 1, \cdots, J-1 \tag{4.4}$$

The agent (trivially) chooses to consume all of her resources in the final period of life, that is $c^J_{t+J-1} = R_{t+J-1} a^{J-1}_{t+J-2} + z^J_{t+J-1}$.

Labor is supplied inelastically and preferences are given as the standard constant elasticity of substitution function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad \text{if } \sigma \neq 1$$
$$u(c) = \ln(c) \quad \text{if } \sigma = 1$$

Demographics

To start off the economy, I assume that in period zero, there are J cohorts who enter the economy with given asset holdings according to their age. I assume that the initial young enter the economy with zero assets, and all other cohorts enter with a_{-1}^{j} for $j = 1, \cdots, J-1$. Successive cohorts enter the model with zero assets when they are young.

N_t indicates generation t and is given by number of young (i.e., the generation born) at time t. The population grows at rate n_t such that

$$N_t = (1 + n_t) N_{t-1} \tag{4.5}$$

The population at any time t is the sum of all living cohorts. If the growth rate n is constant, the population can be expressed relative to the initial young or initial old:

$$\sum_{j=1}^{J} N_{t+1-j} = \sum_{j=1}^{J} (1+n)^{1-j} N_t = \sum_{j=1}^{J} (1+n)^{J-j} N_{t+1-J}.$$

Production

The consumption good in the economy (Y_t) is produced by single firm (or equivalently many small firms) using a constant elasticity of substitution technology that takes aggregate capital (K_t) and labor (H_t) as inputs and produces the consumption good according to:

$$Y_t = F(K_t, H_t) = K_t^{\alpha} H_t^{1-\alpha}$$

The parameter α measures the intensity of use of capital in production.[2] Factor markets are competitive and capital and labor (hours worked) are paid their marginal products. The gross real interest rate R_t is given by:

$$R_t = F_K(K_t, H_t) + 1 - \delta \tag{4.6}$$

[2]I abstract from technological growth, although it is straightforward to incorporate this into the model. If technological process was included in the model, aggregate variables would grow that the rate $(1 + n)(1 + g)$ along the balanced growth path (if g indicates the growth rate of technology). In all other regards, the model is identical.

where δ is the rate of depreciation. The wage rate w_t is given by

$$w_t = F_H(K_t, H_t) \tag{4.7}$$

Government

The government runs a modified pay-as-you-go social security system. The government pays retirement benefits to the retired generations by taxing the working generations and by (possibly) issuing debt.

The pay-roll tax rate is τ_t. The tax has two components, a baseline tax rate τ_t^0, and a Leeper tax τ_t^1 that responds to the level of government debt. Although actual social security taxes do not change based on government debt, a Leeper tax can be thought of as a way to capture legislative unease with increasing debt. I consider a Leeper tax of the following form:

$$\tau_t = \tau_t^0 + \tau_t^1(B_t/H_t). \tag{4.8}$$

where $\tau^0 \in [0,1]$ is the baseline tax rate when government debt is zero, $\tau^1 \geq 0$ is the incremental tax, and B_t/H_t is government debt per labor hours. Notationally, a tax rate *without* a superscript (τ_t) will refer to the entire pay-roll tax $\tau_t = \tau_t^0 + \tau_t^1(B_t/H_t)$.[3]

Social security benefits z_t^j are paid according to a benefit earning rule:

$$z_t^j = \phi_t w_{t+T-j}. \tag{4.9}$$

[3]The Leeper tax given by equation (4.8) leaves open the possibility of a total tax rate greater than one hundred percent if bonds levels are high or if τ^1 is large (i.e., $\tau_t^0 + \tau_t^1(B_t/H_t) > 1$, for large B_t/H_t and/or large τ^1). I avoid this by imposing the restriction τ^1 is either the exogenous parameter value chosen by the economist, or the (smaller) parameter value such that $\tau_t^0 + \tau_t^1(B_t/H_t) = 1$. In the policy experiments that follow, the majority of the tax burden will come from the baseline tax rate τ^0, and the Leeper tax τ^1 will be used only to calibrate a dynamically efficient steady state with positive bonds.

where z_t^j represents the benefit paid to a retiree of age j in time t. The parameter ϕ_t is the replacement rate and shows how much of a workers' wage-indexed average period earnings (w_{t+T-j}) are replaced by social security benefits.[4]

The government is not required to balance the social security budget. If social security taxes are less than social security benefits, the social security program runs a deficit. The time t deficit is defined as the time t social security benefits less the pay-roll tax revenue:

$$\text{Deficit}_t = \sum_{j=T}^{J} N_{t+1-j}\phi_t w_{t+T-j} - H_t(\tau_t^0 + \tau_t^1(B_t/H_t))w_t$$

Here H_t indicating the working age population.

The government issuance of bonds is equal to the gross interest on outstanding debt plus the social security deficit from the previous period.

$$B_{t+1} = R_t B_t + \sum_{j=T}^{J} N_{t+1-j}\phi_t w_{t+T-j} - H_t(\tau_t^0 + \tau_t^1(B_t/H_t))w_t \tag{4.10}$$

Markets

There are four markets: labor, capital, bonds, and goods. Prices adjust in equilibrium to ensure all markets clear.

Labor market clearing requires the total number of hours worked equal the labor input of the representative firm:

$$H_t = \sum_{j=1}^{T-1} N_{t+1-j} \tag{4.11}$$

[4]There is no technological growth in the model, so this corresponds to the US system.

Asset market clearing requires that aggregate capital and bonds next period are equal to the savings of each cohort:

$$K_{t+1} + B_{t+1} = \sum_{j=1}^{J} N_{t+1-j} a_t^j \tag{4.12}$$

Bond market clearing is ensured by the government's flow budget constraint, reprinted below:

$$B_{t+1} = R_t B_t + \sum_{j=T}^{J} N_{t+1-j} \phi_t w_{t+T-j} - H_t(\tau_t^0 + \tau_t^1 (B_t/H_t)) w_t \tag{4.10}$$

Goods market clearing requires that aggregate output is equal to aggregate consumption and aggregate investment. The goods market clears by Walras law. The goods market clearing equation is printed below for reference:

$$F(K_t, H_t) = \sum_{j=1}^{T} N_{t+1-j} c_t^j + K_{t+1} - (1-\delta) K_t$$

Along the balanced growth path, cohort size (N_t), labor hours worked (H_t), output (Y_t), capital (K_t), and bonds (B_t) all grow at rate n. Therefore, it will be convenient to rewrite the market clearing equations (4.10) and (4.12) in per-hours terms by defining $b_t = B_t/H_t$, and $k_t = K_t/H_t$. I will refer to variables in per-hours terms as "efficient."

The capital and bond market clearing equations can be written in efficient terms as:

$$(k_{t+1} + b_{t+1})(1 + n_t) = \frac{\sum_{j=1}^{J} N_{t+1-j} a_t^j}{H_t} \tag{4.13}$$

$$(1 + n_t) b_{t+1} = R_t b_t + \frac{\sum_{j=T}^{J} N_{t+1-j} \phi_t w_{t+T-j}}{H_t} - (\tau_t^0 + \tau_t^1 (B_t/H_t)) w_t \tag{4.14}$$

Rational Expectations Equilibrium

Given initial conditions k_0, b_0, $a_{-1}^1, \cdots a_{-1}^{J-1}$, and an initial population $\sum_{j=1}^{J} (1+n)^{1-j} N_0$ (where N_0 is the the initial cohort of young and n is the population growth rate), a competitive equilibrium is a sequences of functions for the household savings $\{a_t^1, a_t^2, \cdots, a_t^J\}_{t=0}^{\infty}$, production plans for the firm, $\{k_t\}_{t=1}^{\infty}$, government bonds $\{b_t\}_{t=1}^{\infty}$, factor prices $\{R_t, w_t\}_{t=0}^{\infty}$, and government policy variables $\{\tau_t^0, \tau_t^1, \phi_t\}_{t=0}^{\infty}$, that satisfy the following conditions:

1. Given factor prices and government policy variables, individuals' decisions solve the household optimization problem (4.1) and (4.2)

2. Factor prices are derived competitively according to (4.6) and (4.7)

3. All markets clear according to (4.11), (4.13), and (4.14)

There are $J+1$ equilibrium equations, which hold for time periods $t = 0, \cdots \infty$. The equilibrium equations include the capital clearing equation (4.13), the bond market clearing equation (4.14), and $J-1$ household first order conditions (4.15). These are reprinted below:

$$(k_{t+1} + b_{t+1})(1+n_t) = \frac{\sum_{j=1}^{J} N_{t+1-j} a_t^j}{H_t} \tag{4.13}$$

$$(1+n_t)b_{t+1} = R_t b_t + \frac{\sum_{j=T}^{J} N_{t+1-j} \phi_t w_{t+T-j}}{H_t} - (\tau_t^0 + \tau_t^1 (B_t/H_t)) w_t \tag{4.14}$$

$$(R_t a_{t-1}^{j-1} + y_t^j - a_t^j)^{-\sigma} = \beta E_t[R_{t+1}(R_{t+1} a_t^j + y_{t+1}^{j+1} - a_{t+1}^{j+1})^{-\sigma}] \tag{4.15}$$

$$\text{for } j = 1, \cdots, J-1$$

Here y^j indicates period labor income during working life and social security income after retirement. Note that $a^0 = 0$, and factor prices are given by (4.6) and (4.7).

The equilibrium definition above accommodates a time-varying population growth rate n_t. In a steady-state, the growth rate of the population is constant. When the growth rate is constant, several terms can be written concisely by defining $\nu = N_t/H_t = (\sum_{j=1}^{T-1}(1+n)^{1-j})^{-1}$. The steady state is a collection

$\{k, b, a^1, \cdots, a^J\}$ that solves:

$$(k+b)(1+n) = \nu \sum_{j=1}^{J} (1+n)^{1-j} a^j \tag{4.16}$$

$$(1+n)b = R(k)b + \nu \sum_{j=T}^{J} (1+n)^{1-j} \phi_t w(k) - (\tau^0 + \tau^1 b) w(k) \tag{4.17}$$

$$(R(k)a^{j-1} + y^j - a^j)^{-\sigma} = \beta \left[R(k)(R(k)a^j + y^{j+1} - a^{j+1})^{-\sigma} \right] \tag{4.18}$$

$$\text{for } j = 1, \cdots, J-1$$

with $y^j = (1 - (\tau^0 + \tau^1 b))w(k)$ for $j < T$, $y^j = \phi w(k)$ for $j \geq T$, $a^J = 0$, and factor prices $R(k)$ and $w(k)$ given by (4.6) and (4.7).

The equilibrium of the model is not unique; for many parameter combinations there are two steady states. The number of steady states depends on the model parameters and can be characterized by a saddle-node bifurcation. Chalk (2000) discusses the saddle-node bifurcation in a similar OLG model with government debt and physical capital.[5]

The intuition of the bifurcation in this model is that as a given parameter changes (like the payroll tax rate τ^0 or the population growth rate n), it impacts the social security deficit or surplus. Suppose the model is calibrated such that two steady states exist. As the deficit increases (endogenously in response to a parameter change), this increases government debt and crowds out capital, pushing the two steady states closer together. At a critical value of the parameter, only one steady state exists; beyond that, deficits are too large, government debt is explosive, and no steady states (and no equilibria) exist.

In the exercises that follow, the model will be calibrated such that two steady states exist. I analyze the determinacy of equilibria by linearizing the model

[5]See Azariadis (1993) (Chapter 8 and Chapter 20) for a discussion of saddle-node bifurcations in planar OLG models.

around a steady state and computing the eigenvalues of the linearized system (see Laitner (1990)). There are three predetermined variables in the model (k, b, and a^{J-1}) and J-2 free variables (a^1,...,a^{J-2}). There are three possible cases. If three eigenvalues of the linearized system are less than one in modulus and the remaining $J-2$ have modulus greater than one, the system is determinate. If more than three eigenvalues lie inside the unit circle, the system is indeterminate. If more than $J-2$ eigenvalues lie outside the unit circle, the system is explosive. I have confirmed numerically that when two steady states exist, the steady state with higher capital is stable (determinate), and the lower capital steady state is explosive. I will focus only on the determinate solutions in this paper.[6]

Life-cycle Horizon Learning

Under the rational expectations hypothesis (RE), households make optimal savings and consumption choices given their (rational) forecasts of future prices and policy. Households make decisions at age zero looking forward over their entire life cycle. These rational agents are fully forward looking, and consider every stage of the life-cycle when making decisions. Under RE agents know the underlying equations that govern the economy and form expectations of future variables using the mathematical expected value.

This paper backs away from RE and proposes a learning model in which agents combine limited knowledge about the structure of the economy with adaptive forecasts for future macroeconomic aggregates. As a baseline, I consider agents who do not have perfect foresight over factor prices. Thus, agents do not know the future path of wages or interest rates over their life-cycle. Agents use

[6]Chalk (2000) also finds that the high capital steady state is determinate and the low capital steady state is explosive in a many-period OLG model with debt and capital.

adaptive expectations to forecast future prices, but behave optimally in all other ways. Agents update their forecasts as more information becomes available. I call this behavioral assumption life-cycle horizon learning (LCH learning).

Under LCH learning, households are forward looking and make optimal savings and consumption choices given their (adaptive) forecasts of future macroeconomic aggregates. Adaptive expectations may be viewed as a special case of adaptive learning suitable for non-stochastic models. The adaptive expectations assumption in this paper is analogous to constant gain least squares learning in a model with random shocks (like a productivity shock). LCH learning is similar to infinite-horizon learning, developed in Marcet and Sargent (1989) and emphasized by Preston (2005 and 2006). The key difference is that infinite-horizon learning models are based on a representative agent who lives for an infinite number of periods. LCH learning applies the same forward looking behavior to finitely lived agents in an overlapping generations life-cycle model.[7] Throughout this paper, I assume homogenous expectations across agents.

Agents in the LCH learning model forecast wages and interest rates using adaptive expectations of the following form:

$$w^e_{t+1} = \gamma w_t + (1-\gamma) w^e_t \tag{4.19}$$

$$R^e_{t+1} = \gamma R_t + (1-\gamma) R^e_t \tag{4.20}$$

with $\gamma \in (0,1)$. Here w^e indicates expected wage, and R^e_t indicates expected interest rate. Agents also form expectations at time t of prices in period $t+j$ for

[7]LCH learning is similar to the model developed in Bullard and Duffy (2001), in which agents forecast inflation over a finite life-cycle and make optimal choices based on those forecasts.

$j > 1$:

$$w^e_{t+j} = w^e_{t,t+j} = w^e_{t+1} \tag{4.21}$$

$$R^e_{t+j} = R^e_{t,t+j} = R^e_{t+1} \tag{4.22}$$

When the Leeper tax is non-zero ($\tau^1 > 0$), then agents need to know future bond levels in order to estimate their after-tax pay. For simplicity, I assume that agents forecast bonds using the same adaptive learning rule they use to forecast prices.

$$b^e_{t+1} = \gamma b_t + (1-\gamma) b^e_t \tag{4.23}$$

$$b^e_{t,t+j} = b^e_{t,t+1} \quad \text{for } j > 1 \tag{4.24}$$

Agents enter the model with knowledge of the previous period's prices, bonds, and expectations. At any moment in time, all agents in the model have the same expectation for future prices and bonds. This assumption is equivalent to assuming agents inherit expectations from the previous generation. I consider alternative way for agents to forecast their income net of taxes in section IV.

LCH learning can be viewed as a small deviation from rational expectations in the sense that agents are still forward looking and only use a different rule to forecast future aggregates. Adaptive expectations are plausible in a world in which the true data generating process is complex. Adaptive expectations are optimal if agents think that capital follows an IMA(1,1) process. That is, expectations of the form given in Equation (4.19) are *rational* if the change in capital has the following form $\Delta k_t = \epsilon_t + \theta \epsilon_{t-1}$ for shocks ϵ and parameter $\theta \in (0, 1)$. The use of adaptive expectations is equivalent to agents believing that capital has a mixture of permanent and transitory shocks (See Muth (1960)). The gain parameter, γ, has a natural interpretation in this context. If all shocks are transitory, the optimal forecasting rule is constant, i.e., γ=0. In contrast, if all shocks are permanent, then the best forecasting rule is a random walk, i.e., $\gamma = 1$. In this particular model, all shocks are permanent, but agents are not endowed with this knowledge. I choose the gain parameter used in simulations to minimize the welfare cost of agents of inaccurately forecasting along the transition path. I explore alternative gain parameters as robustness checks.[8]

In the baseline LCH learning model, agents have full knowledge of government policy. Agents know the future path of social taxes and the benefit replacement rate if there is no policy uncertainty. If future policy is stochastic, agents form expectations of future policy parameters using *rational* expectations. Agents know the finite set of policy parameters (and/or reform dates) and the relative probability of each. I relax this assumption and have agents learn about future government policy adaptively in section IV.

A young agent in the LCH learning model chooses first period savings and consumption and plans future savings and consumption to satisfy her $J-1$ first order equations and her lifetime budget constraint. The young agent's time t plan of second period savings is denoted $a^2_{t,t+1}$. In the following period, her time $t+1$ actual choice of second period savings is given by a^2_{t+1}. Her time $t+1$ plan does not have to be consistent with her time t choice. She can update her savings decision based on the new information she receives in period $t+1$. Abstracting from policy uncertainty (assuming the path of taxes and social security benefit rates are

[8]Evans and Ramey show that by appropriately tuning the free parameters of the forecast rule agents can obtain the best forecast rule within a given class of underparameterized learning rules (G. Evans and Ramey (2006)).

predetermined), the young agent in time t solves:

$$(R_{t+j-1}a^{j-1}_{t+j-2} + y^{j}_{t+j-1} - a^{j}_{t+j-1})^{-\sigma} = \beta R^{e}_{t+j}(R^{e}_{t+j}a^{j}_{t+j-1} + y^{e,j+1}_{t+j} - a^{j+1}_{t,t+j})^{-\sigma} \quad (4.25)$$

$$\text{for } j = 1, \cdots, J-1$$

with $y^{e,j+1}$ indicating expected period labor income or social security income

$$y^{e,j+1}_{t+j} = (1-\tau_{t+j})w^{e}_{t+j} \quad \text{for } j < T$$

$$y^{e,j+1}_{t+j} = \phi_{t+j}w^{e}_{t+j+T-j} \quad \text{for } j \geq T$$

Similarly, an agent of age two solve $J-2$ first order equations, for the remaining $J-2$ periods of her life-cycle. In total, the J cohorts alive in any period solve $\sum_{j=1}^{J-1} j = \frac{(J-1)J}{2}$ first order conditions. Together, the decisions of households of all ages, asset market clearing (4.13), and government bonds (4.14), and the expectation equations (4.19), (4.20), and (4.23), create a recursive system that governs the dynamics of the economy.

Finite Horizon Life-cycle Learning

This paper proposes an additional new learning model for finitely lived agents who form expectations over a short horizon (less than or equal to the length of their lifecycle) called finite horizon life-cycle learning (FHL). This learning model is similar to N-step ahead optimal learning, developed by W. Branch, Evans, and McGough (2013). In both cases, agents make optimal choices based on their forecast of future prices (and potentially other macro variables) and their budget constraint over a finite horizon. Agents using FHL learning look forward for a fixed number of periods and make decisions based on this short-planning horizon. They update their choices each period as their planning horizon moves forward and they receive new information.

In order to solve the FHL model model, it is necessary to specify a terminal condition for asset holdings at the end of the planning horizon. When the planning horizon equals the end of the life-cycle, the terminal condition is simply that the agent dies without debt (and thus chooses optimally to die with zero assets). When the planning horizon is shorter than the life-cycle, it is less obvious what the terminal condition should be.

One simple alternative is to impose that assets are non-negative at the end of the planning horizon. Under this assumption, agents will choose to hold zero assets at the end of the planning horizon. This assumption drives the main results in the short-planning horizon literature and leads to time-inconsistent behavior (see Caliendo and Aadland (2007) and Park and Feigenbaum (2017)). This assumption is somewhat unsatisfactory, because agents treat periods beyond the planning horizon as if they did not exist. An alternative assumption is to have agents hold wealth at the end of the planning horizon. As a baseline, I assume that agents plan to hold the same amount of wealth at the end of the planning horizon that older cohorts held at the same age. Specifically, agents adaptively forecast the steady-state value of age-specific wealth at the end of their planning horizon.

As in the LCH learning model, agents using FHL learning forecast future factor prices and bonds according to (4.19) - (4.23). FHL learners also need to forecast a terminal wealth holding for the end of the planning horizon. I assume that agents forecast

$$a_{t,terminal}^{j,e} = \gamma a_{t-1}^{j} + (1-\gamma) a_{t-1,terminal}^{j,e} \tag{4.26}$$

$$\text{for } j = 1, \cdots, J-1$$

Here, $a^{j,e}_{t,terminal}$ is amount of asset an age $j-1$ agent expects to hold at the end of the period when they are age j. This asset amount is based on the observed asset holding of age j agents from last period, and the forecast from last period.

As an example, suppose the planning horizon length is three. That is, agents looks forward three periods when making decisions. A young agent chooses first period savings and consumption, and plans saving and consumption for period two and three, using three first order equations and a terminal condition:

$$(y^1_t - a^1_t)^{-\sigma} = \beta R^e_{t+1}(R^e_{t+1}a^1_t + y^{e,2}_{t+1} - a^2_{t,t+1})^{-\sigma} \tag{4.27}$$

$$(y^{e,2}_{t+1} - a^2_{t,t+1})^{-\sigma} = \beta R^e_{t+2}(R^e_{t+2}a^2_{t,t+1} + y^{e,3}_{t+2} - a^3_{t,t+2})^{-\sigma} \tag{4.28}$$

$$(y^{e,3}_{t+2} - a^3_{t,t+2})^{-\sigma} = \beta R^e_{t+3}(R^e_{t+3}a^3_{t,t+2} + y^{e,4}_{t+2} - a^4_{t,terminal})^{-\sigma} \tag{4.29}$$

Older agents follow a similar pattern. Agents who have three or fewer periods of life remaining using the standard terminal condition $a^J = 0$.

The FHL learning model can be written as a recursive system. The system includes the first order equations for the households, asset market clearing (4.13), and government bonds (4.14), and the expectation equations (4.19) - (4.23) and (4.26). For a planning horizon of length H, there will be $J-H$ terminal conditions and $H(J-H)+\frac{H(H-1)}{2}$ household first order equations. When the planning horizon is equal to the life-cycle, the system is identical to the life-cycle horizon learning model.

The shortest possible planning horizon is one. Agents who only look forward one period make their savings consumption decision based on their Euler equation, their budget constraint, and the assets they believe they need to hold at the end of the next period (the terminal condition). W. Branch et al. (2013) call this type of learning one-step-ahead optimal learning and compare and contrast it to Euler-equation-learning, a behavioral primitive in which agents do not explicitly consider

Parameter		Value
α	Capital share of income	$\frac{1}{3}$
β	Discount factor	$(\frac{1}{1+0.005})^{10}$
σ	Inverse elasticity of substitution	1
δ	Depreciation	$1-(1-0.10)^{10}$
n	Population growth rate	0.01
A	TFP factor	10

TABLE 14. Baseline parameters for 6 period model

their transversality condition. I plan to explore Euler equation learning in an OLG framework in future work.

Calibration

Agents enter the model at age 25 and live for six periods ($J = 6$). Each period is 10 years, and agents die with certainty at age 85. The capital share of income, discount factor, inverse elasticity of substitution, and depreciation rate are all set to standard parameter values[9]. The total factor productivity parameter A is arbitrarily set to 10 (results do not depend on this assumption). For the majority of the exercises in this paper, the population growth rate is 0.01. The parameter values are listed in Table 14.

The SSA estimates the current ratio of social security beneficiaries to workers to be 0.35 (as of the 2017 Trustee Report). This ratio is expected to increase to 0.46 by 2035, and to 0.5 by 2095 under the medium cost assumptions. The increase in the ratio of beneficiaries to workers is driven by both increasing lifespans for retirees, and declining birthrates for working generations. I abstract from these details in the model, and capture all population changes using the

[9]See, for example, W. Branch et al. (2013). My discount factor β is slightly closer to one and the depreciation rate is slightly higher. I choose the higher discount factor and depreciation rate to better match the size of tax increase necessary to ensure long-run solvency of social security

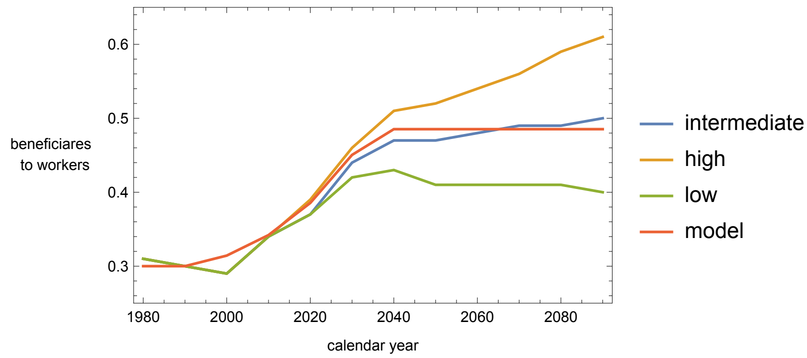


FIGURE 23. Ratio of beneficiaries to retirees. The red link in this graph illustrates the ratio of beneficiaries to retirees in the calibrated, six period model. The demographic change is modeled as a one-time reduction in the population growth rate n from 0.1802 to 0.01. The yellow, blue, and green lines show the high, intermediate, and low cost projections of the ratio from the 2017 SSA Trustees' Report. The model calibration closely tracks the intermediate assumptions from the SSA.

growth rate n. For the majority of the exercises in section IV, I will choose n such that the ratio of retirees to workers is 0.3 and then increases gradually to 0.485, as illustrated in Figure 23.

In the learning models, I set the gain parameter $\gamma = 0.93$. This gain parameter minimizes the maximum welfare cost to an agent of using adaptive forecasts in the LCH model along the transition path that includes a demographic shock and a change to social security. I construct a consumption equivalent variation that compares the utility of consumption of each cohort of agents in the LCH model to the utility of consumption for "infinitesimally rational" cohorts of agents. An infinitesimally rational agent is able to predict future prices with perfect foresight in the LCH world, but is such a small part of the market that she does not change prices. I compute the consumption equivalent variation that makes an infinitesimally rational agent indifferent between her own consumption and the consumption of the life-cycle horizon learners. I chose the gain parameter γ to minimize the maximum welfare cost. It is unsurprising that the gain parameter is close to $\gamma = 1$, since all shocks in the model are permanent. The simulation results are qualitatively sensitive to the gain parameter, and I explore alternative parameterizations as robustness checks.

Applications

Announced Social Security Reform

The aging of society put pressure on the social security system. In the absence of reform, the government would be required to fund social security benefits by issuing debt. There is a limit to how much the government can borrow, even in a dynamically inefficient economy. I illustrate these dynamics in the following example.[10] [11]

Suppose that the government runs a social security program that generates a small surplus (government assets); however, as the population ages, the operating surplus changes to a deficit. The government continues to operate the same social security system which depletes the government assets and causes the government to accrue debt. After a number of periods, the government reforms social security by cutting benefits or raising taxes.

I calibrate this example to correspond to the US system. The initial population growth rate is $n = 0.1802$ and falls to $n = 0.01$ in 1980. This generates a smooth transition in the ratio of retirees to workers from 0.3 to 0.485 after six

[10] The government could also fund social security benefits using tax revenue from some other existing tax. For the sake of this paper, I will argue that diverting other federal tax dollars into the pension system is isomorphic to directly increasing social security taxes, and is thus reform.

[11] See Chalk (2000) for a discussion of maximum government deficits in a dynamically inefficient economy.

periods (which is near SSA projections). The initial policy mimics the current US system: the payroll tax rate is set to $\tau^0 = 0.124$ and the benefit replacement rate is $\phi = 0.4$. Reform is calibrated as either a tax increase to $\tau^0 = 0.1516$, or a benefit cut to $\phi = 0.332$. The SSA estimates these reforms would eliminate the funding shortfall in the social security system. In order to have a dynamically efficient economy with positive government debt, the Leeper tax rate is set to $\tau^1 = 0.045$ before and after reform.

Figure 24 plots the path of capital and bonds for this tax increase reform. The plot includes paths for an economy with Rational Expectations (in green), and for an economy with LCH Learning. Government debt increases before the reform takes place and then converges to the new higher steady state value. Debt is increasing since the government is running a deficit in the social security system. Debt would become explosive, were it not for the reform in the year 2030. The reform is announced, so agents adjust their behavior before the reform date. Capital increases initially as saving goes up, and then falls to the steady state.

Agents in the learning model overestimate the interest rate and underestimate the wage in the first few periods following the change in the population growth rate (which moves the economy away from the initial steady state). This is depicted in Figure 25. The expectations adjust quickly because the gain parameter γ is so large. In this example, the gain is set to 0.93.

The agents in the learning model save more relative to the fully rational agents in the initial periods, since their estimate of prices and bonds is inaccurate. The savings choices of agents are depicted in Figure 26. 1980 is the first period with the lower growth rate. Agents with rational expectations do not make (large) changes in their savings behavior if they know they will not be alive for the reform.

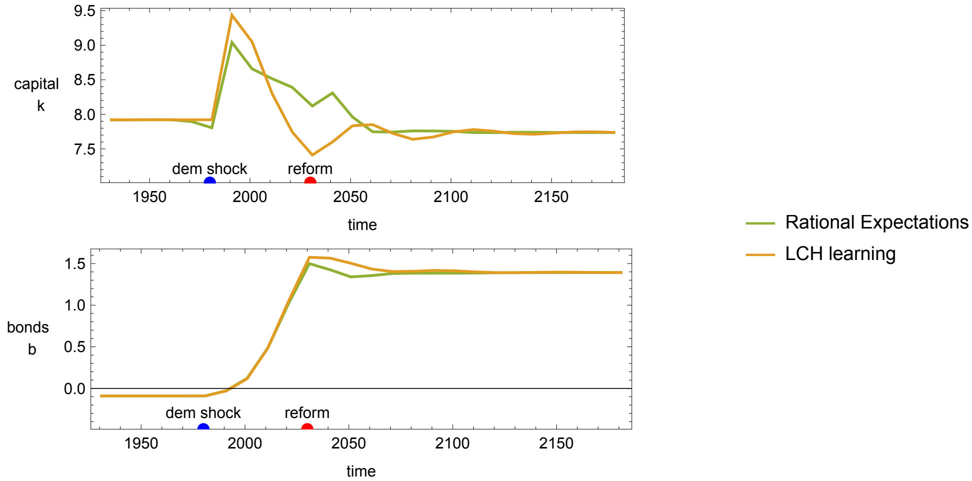


FIGURE 24. Time paths for capital and bonds for RE (green) and LCH (yellow) for an announced tax increase. The initial population growth rate n is 0.1802 and falls to 0.01 in 1980. Demographic changes drive increasing debt until tax increase in 2030. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. The gain parameter $\gamma = 0.93$ for this example.

This explains the spikes in savings along the rational expectations paths leading up to reform. The initial decline in savings along the RE paths are due to the changes in the capital and bonds.

The agents in the model with learning over-save relative to the rational model, since they do not know future prices. Because the agents initially over-save, they are able to save less later in life relative to their plan from the first period. This is evident in Figure 27 by comparing the young (age 1) agent's planned asset holdings for future periods against the actual savings choice she makes when she reaches those periods. For the first several time periods, the agent's planned savings are above the actual savings. The planed savings and actual savings of older agents follows a similar pattern.

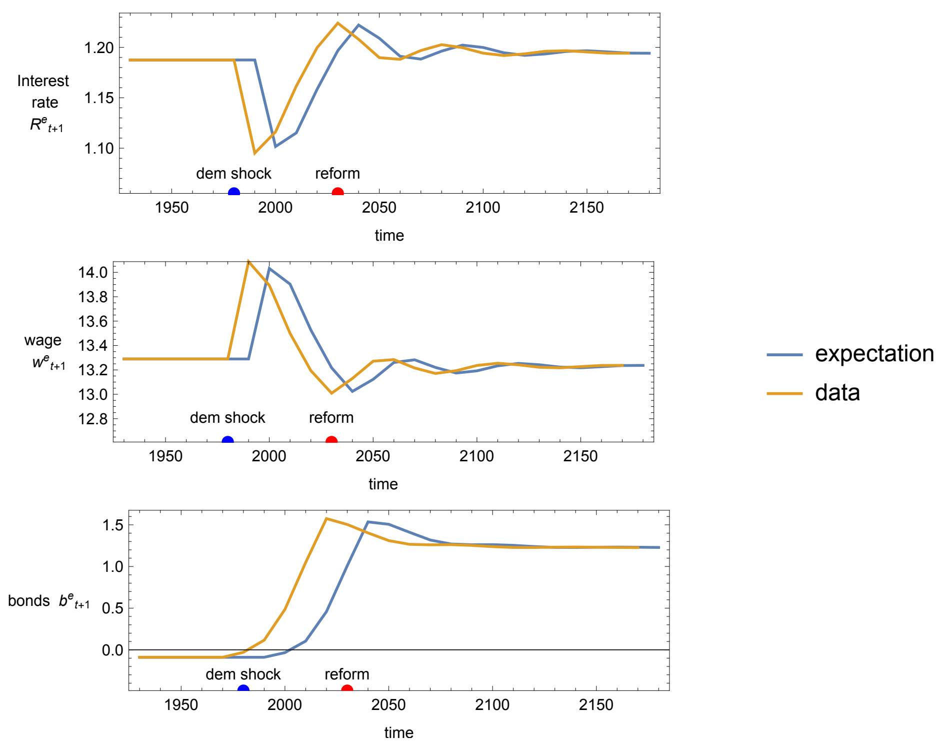


FIGURE 25. Time paths for expected interest rates, wages, and government debt levels (blue lines), and realized interest rates, wages, and debt (yellow lines) in the LCH model with an announced tax increase. Agents overestimate the interest rate and underestimate the wage in the initial periods as they learn that capital and debt are increasing. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Demographic changes drive increasing debt until tax increase in 2030. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. The gain parameter $\gamma = 0.93$.

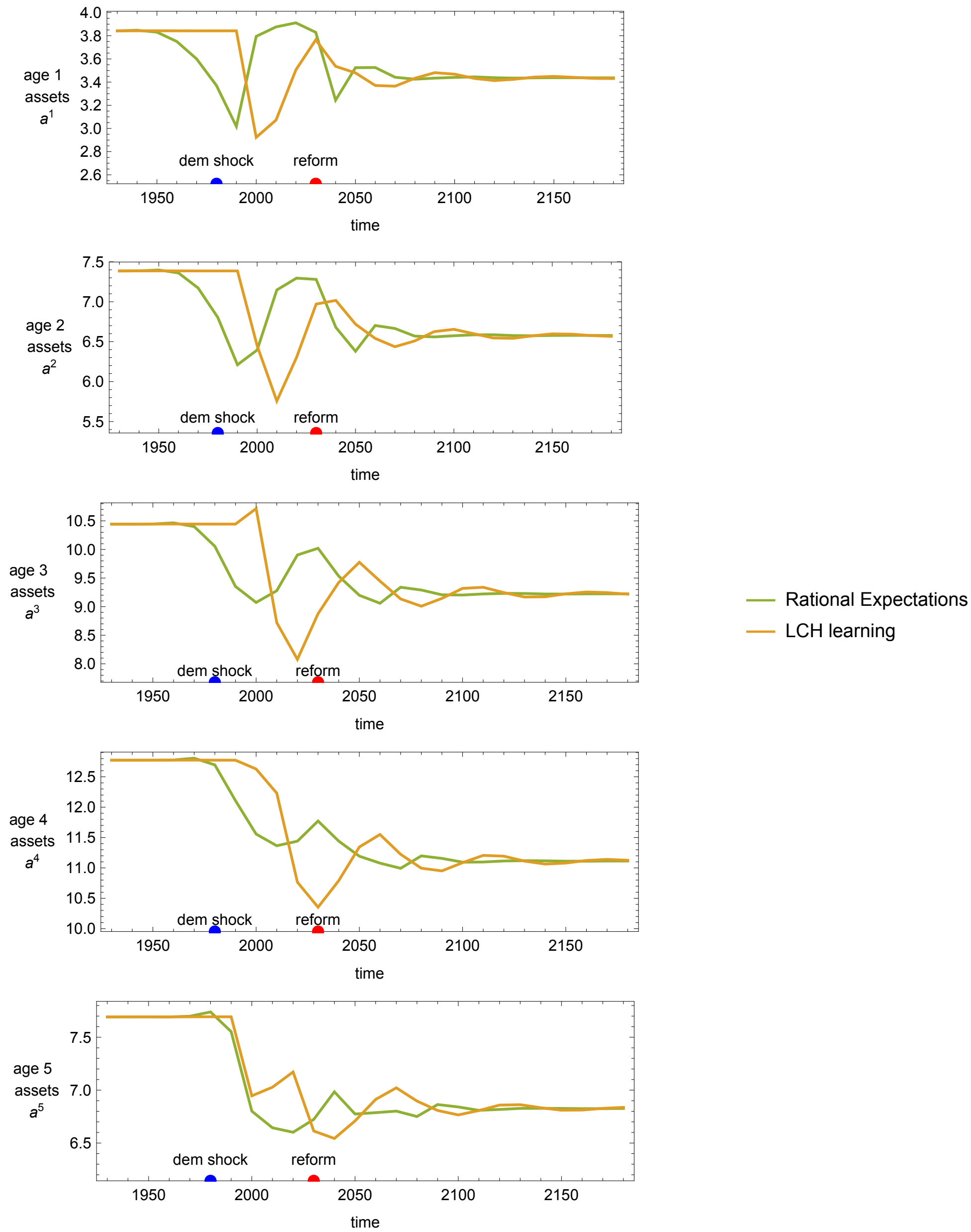


FIGURE 26. Time paths for the savings accounts for RE (green) and LCH (yellow) models with an announced tax increase. Demographic changes drive increasing debt until tax increase in 2030. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. The gain parameter $\gamma = 0.93$.

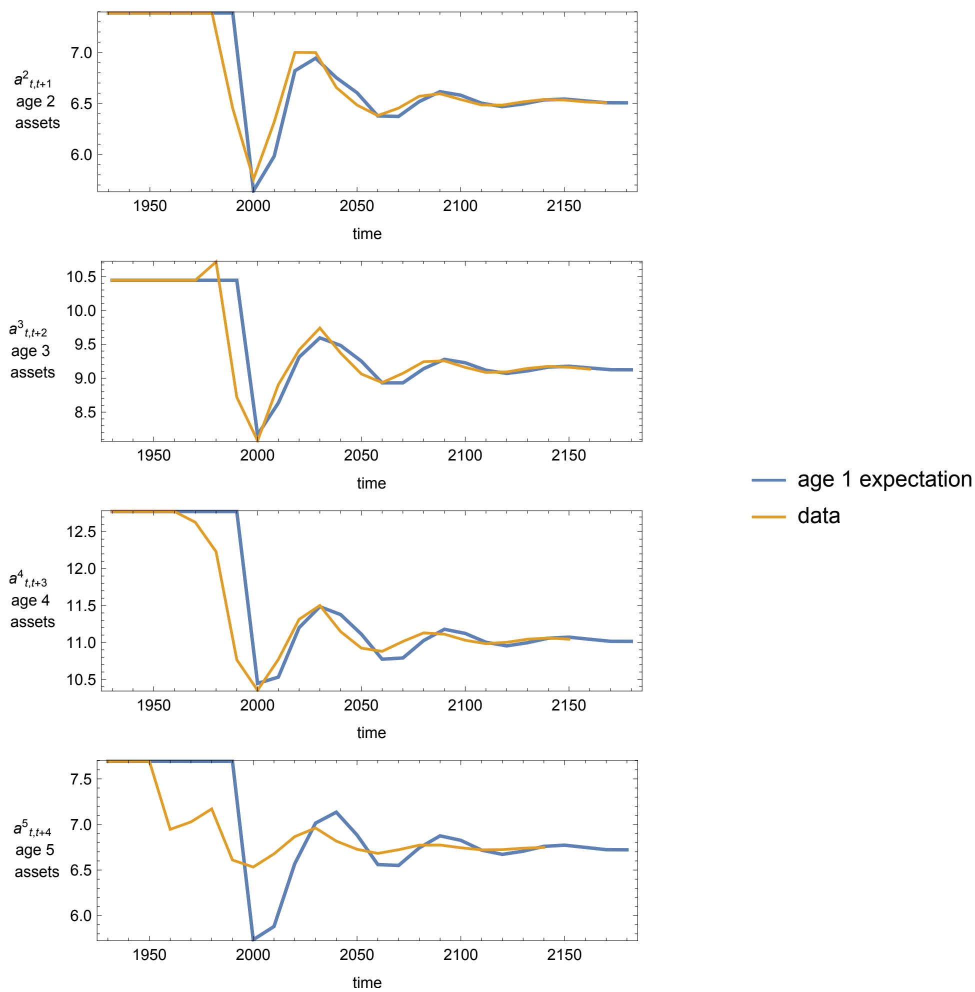


FIGURE 27. This graph shows planned future savings of the period in blue, and actual future savings in yellow. The expectations (or plans) are made when the agent in young (age 1) in period t. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Demographic changes drive increasing debt until tax increase in 2030. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. The gain parameter $\gamma = 0.93$.

Announced Change, Finite Horizon Life-cycle Learning

Agents using FHL learning only make decisions based on their forecasts over their short planning horizon. Agents forecast prices, bonds, and the assets the expect to hold at the end of the planning horizon. Over time, agents learn the steady state values of each of these quantities. If agents' estimates for the assets they'll need at the end of the planning horizon are close to the rational expectations (steady state) values, they make relatively better decisions using a shorter planning horizon than using a longer horizon. This is because agents using a longer horizon forecast over more periods and can make larger mistakes. This is evident by comparing the transition dynamics of different planning horizon lengths in the FLH model for the a social security tax increase in response to a demographic shock. The fluctuations are larger for longer planning horizons and smaller for shorter planning horizons, as depicted in Figure 28. The figure depicts the time path for capital and bonds for a tax increase. The time paths for asset holdings display cyclical behavior, with the largest cycles associated with the longest planning horizon. As in the previous section, this policy experiment replicates the tax increase suggested by the SSA. The initial population growth rate is $n = 0.1802$ and rate falls to $n = 0.01$ in the year 1980. Initial government policy is $\tau^0 = 0.124$, $\tau^1 = 0.045$, and $\phi = 0.4$. The policy change increases τ^0 to 0.1516 in the year 2030.

Agents in the FHL learning model only respond to policy changes when the announced policy enters their planning horizon. Agents making one step ahead forecasts will not respond to announced policy until the period right before the change. Despite this fact, FHL leads to smaller aggregate fluctuations, since agents

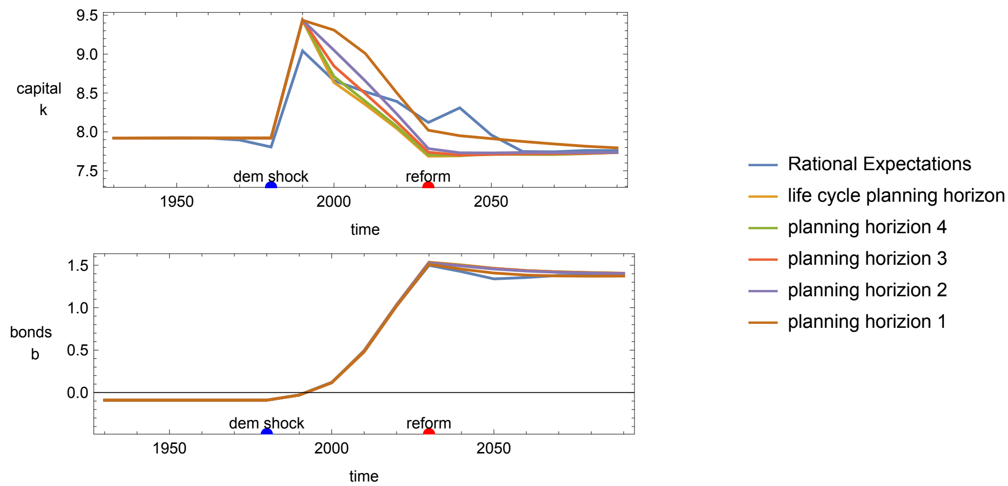


FIGURE 28. Time paths for capital and bonds for RE (blue), LCH (yellow), and FHL (planning horizon 4: green, 3: red, 2: purple, 1: brown) for an announced tax increase. Demographic changes drive increasing debt until tax increase in the year 2030. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. The policy change increases τ^0 to 0.1516. γ=0.93 for all learning rules. The red dashed lines indicate the steady state following the policy change.

using a short planning horizon do not make inaccurate forecasts many periods into the future.

Welfare

Welfare Cost of Social Security Reform

The demographic changes and social security reforms proposed in the previous sections harm some generations and help others. I illustrate these effects by using a Consumption Equivalent Variation (CEV) measure. This CEV measure shows the percent of period consumption that would have to be added to an agent in the initial steady state in order to be indifferent between the steady state and being born at a particular time along a particular transition path. A negative CEV indicates that the agent would have higher utility in the initial steady state. The

CEV is calculated as:

$$\sum_{j=1}^{J} \beta^{j-1} u(c^{j,ss}(1+\Delta)) = \sum_{j=1}^{J} \beta^{j-1} u(c^{j,}_{t+j-1}) \tag{4.30}$$

where $c^{j,ss}$ indicates age j consumption in the initial steady state, $c^{j,}_{t+j-1}$ is consumption of an agent with age j in period $t+j-1$ along a given transition path, $u(c)$ is the period utility function as defined by (4.5), and Δ is the Consumption Equivalent Variation.

The initial steady state is used as a baseline for comparison to capture both the effect of the demographic changes and the social security reform. Note that the initial steady state consumption is *not* possible after the initial period because of demographic changes. It is used only as a reference point to compare the relative harm of the different learning rules. The welfare cost of an announced social security benefit cut is presented in Figure 29 and the welfare cost of an announced tax increase is presented in Figure 30. In both examples the economy begins in a steady state with a ratio of retirees to workers of 0.3. In 1980, the population growth rate falls to 0.01, which leads the ratio of retirees to workers to increase to 0.485 over six periods to correspond with the aging of the US population. The baseline payroll tax rate is set to $\tau^0 = 0.124$ and the benefit replacement rate is $\phi = 0.4$. The benefit cut reform lowers the replacement rate to $\phi = 0.332$, while the tax increase reform increases the payroll tax to $\tau^0 = 0.1516$. Throughout both examples the Leeper tax rate is set to $\tau^1 = 0.045$ before and after reform, and the gain parameter $\gamma = 0.93$.

The underlying demographic changes in these examples harm agents. The model is calibrated to be dynamically efficient with a social security system about the size of the US system, as the norm in this literature. Slowing population

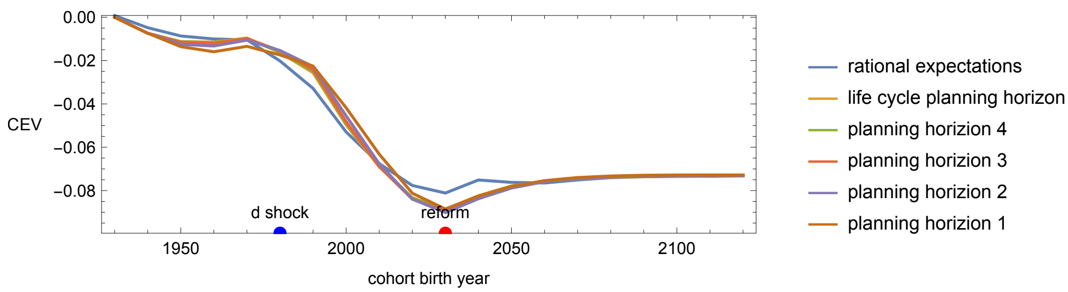


FIGURE 29. Consumption equivalent variation measure of the welfare cost of an announced social security benefit cut. he initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. This drives the need for social security reform. In 2030, the benefit replacement rate falls to ϕ=0.332. The gain parameter γ=0.93. The cost of the reform is highest (lowest CEV) for agents in the LCH learning model born in 2030; agents in the initial steady state would have to give up 8.88% of their period consumption to be indifferent between living in the initial steady and being born in 2030 in the LCH model.

growth makes the economy even more dynamically efficient, which lowers consumption (and welfare) for all generations.

The demographic changes combined with the benefit cut lower welfare for agents in all cohorts with all learning rules (and also in the RE model). The CEV for the final steady state compared to the initial steady state is -7.29% of period consumption. That is, agents in the initial steady state would have to give up 7.29% of period consumption to be indifferent between the initial steady state and being born in the final steady state. The welfare cost is greatest for generations born right after the social security reform. The CEV reaches a minimum of -8.86% for the LCH model agents born in 2030 (the minimum is slightly smaller when comparing the RE model). The welfare cost is decreasing in gain parameter γ; if the gain is decreased to 0.35, the minimum CEV is -9.63% and occurs in the LCH model for the cohort born in 2040. The timing of the welfare cost is also sensitive to the gain parameter, since larger gain parameters producer higher frequency

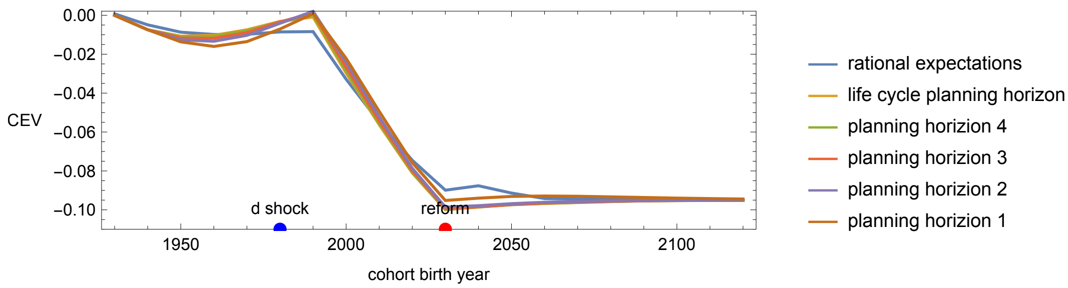


FIGURE 30. Consumption equivalent variation measure of the welfare cost of an announced social security tax increase. he initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. In 2030, the payroll tax rate increases to τ^0=0.1516. γ=0.93 for this example. The cost of the reform is highest (lowest CEV) for agents in the LCH learning model in 2030; agents in the initial steady state would have to give up 9.96% of their period consumption to be indifferent between living in the initial steady and being born in 2030 in the LCH model.

cycles. The faster cycle moves the minimum capital stock a few periods ahead and thus the most harmed cohort is earlier.

Figure 30 depicts the CEV for the tax increase. The demographic changes combined with the tax increase lower welfare for all cohorts in the RE and learning models. The CEV in the final state is -9.49%, indicating agents in the initial steady state would have to give up 9.49% of their consumption to be indifferent between the initial steady state and being born in the final steady state. The welfare cost is largest in the LCH model for the cohort born in 2030. The CEV is -9.96% for this cohort. The magnitude of CEV for the learning models is decreasing in gamma. If γ is decreased to 0.35, the min CEV is -11.91% for the most harmed cohort.

Welfare Cost of Learning

Agents using finite or life-cycle horizon learning don't fully realize the impact of a policy change or other exogenous shock. It take the agents many periods to learn the new steady state values. Converge to the new steady state is slower in the learning models than the RE model. This section measures the welfare cost of the learning models relative to the rational expectations baseline. This welfare comparison will net out the cost of changing demographics and the change in terminal steady state values due to the new social security policy. By comparing consumption in the learning model to consumption in the RE model, this measures the cost of the learning directly.

As in the previous section, welfare is measured using a Consumption Equivalent Variation measure. Here, the CEV measure shows the percent of period income that would have to be added to consumer with rational expectations (in the RE model) in order to be indifferent between living in the RE world or living in the world with learning in the same cohort. A negative CEV indicates that the agent would have higher utility in the world with rational expectations. The CEV is calculated as:

$$\sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j,RE}(1+\Delta_{RE})) = \sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j,L}) \tag{4.31}$$

where $c_{t+j-1}^{j,RE}$ indicates age j consumption in period $t+j-1$ in the model with rational expectations, $c_{t+j-1}^{j,L}$ is consumption in the model with learning, $u(c)$ is the period utility function as defined by (4.5), and Δ_{RE} is the Consumption Equivalent Variation.

Figures 31 and 32 show the CEV for the tax increase and benefit cut examples (parameterized as in the previous sections). In contrast to the previous section, the CEV is not always negative. That is because here the CEV measures learning compared to rational expectations. After many periods, the learning models converge to the RE model, so the CEV converges to zero.

Figure 31 illustrates the welfare cost of the tax increase in the learning models relative to the RE baseline. For the LCH model, the welfare cost is greatest

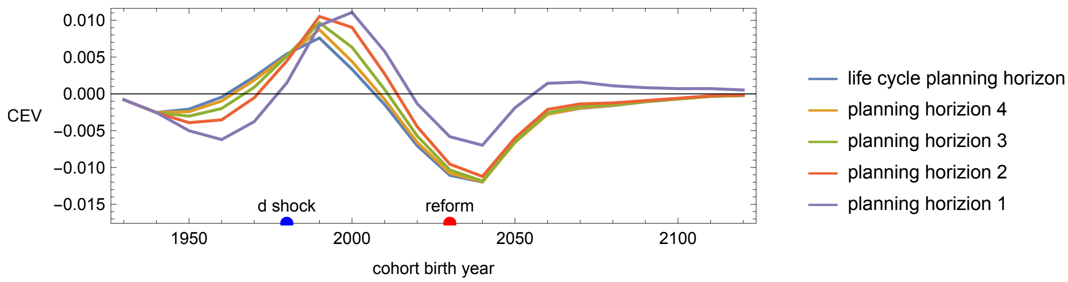


FIGURE 31. Consumption equivalent variation measure of the welfare cost of an announced social security tax increase: learning models compared to RE. he initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. In 2030, the payroll tax rate increases to τ^0=0.1516. The gain parameter γ=0.93. The cost of the reform is highest (lowest CEV) for agents in the LCH learning model born in 2040; agents in the RE model would have to give up 1.20% of their period consumption to be indifferent between being born in the RE model in 2040 and being born in 2040 in the LCH model.

for the tax increase in the year 2040; agents in the RE model born in 2040 would be willing to give up 1.20% of their consumption to avoid being born in 2040 in the LCH model.[12]

Figure 32 illustrates the welfare cost or the benefit cut in the learning models relative to the RE baseline. In the LCH model, the welfare cost is greatest for the cohort born in the year 2030. Agents in the rational model would be willing to give up 0.81% of their consumption to avoid living LCH world in the same time periods.[13]

Many cohorts benefit from the cyclical dynamics introduced via learning. Agents benefit when the capital stock is higher wages are higher. Consider 1-step-ahead FHL learning in the context of the social security benefit cut presented in Figure 32. Cohorts born prior to the reform and many cohorts born after the reform have higher utility in the FHL model than in the RE model. Indeed,

[12]When $\gamma = 0.35$ the min CEV is -3.38% and occurs for the cohort born in 2040.

[13]When $\gamma = 0.35$ the min CEV is -2.43% and occurs for the cohort born in 2040.

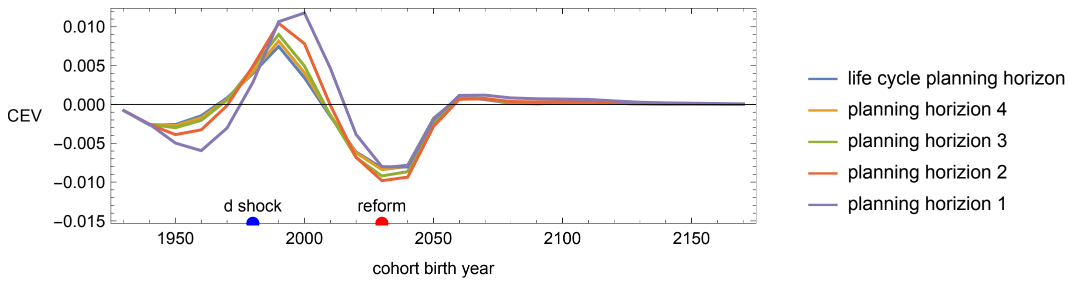


FIGURE 32. Consumption equivalent variation measure of the welfare cost of an announced social security benefit cut: learning models compared to RE. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. In 2030, the benefit replacement rate falls to ϕ=0.332. The gain parameter γ=0.93. The cost of the reform is highest (lowest CEV) for agents in the LCH learning model born in 2030; agents in the RE model would have to give up 0.81% of their period consumption to be indifferent between being born in the RE model in 2030 and being born in 2030 in the LCH model.

the cumulative welfare gain across cohorts can be positive. At first this may seem like a violation of the Welfare Theorems, but it is not. Bounded rationality does not present a Pareto improvement over the RE baseline. The learning dynamics improve the welfare for some agents who enjoy higher wages and higher consumption due to the increased capital stock from higher saving.

Markets exhibit a pecuniary externality in general equilibrium. Higher savings today raises capital and wages tomorrow. Agents can exploit this pecuniary externality in OLG economies, if they can coordinate behavior across generations. Feigenbaum, Caliendo, and Gahramanov (2011) introduce the concept of "optimal irrationality" in OLG economies. They show that an irrational consumption rule *always* exists that can weakly improve upon the lifecycle/permanent-income rule in general equilibrium in a two-period model. They present a calibrated continuous time example with an escalating rule-of-thumb savings rule that increases welfare relative to rational expectations (for all but the first generation). They argue that policies that increase savings in a dynamically efficient economy can improve welfare for many (but not all) generations. The welfare gains to some generations in my model come from the same externality.

The majority of the welfare cost (or welfare gain) of living in a FHL or LCH learning model come from the general equilibrium changes in the capital stock, not from the individual forecast errors made by an agent. One way to measure this cost is to compute the life-cycle consumption for an infinitesimally rational agent who lives in a learning model.

Suppose a single rational agent is born in the LCH model. The rational agent understands that everyone else is using LCH learning, and she is also able to predict future prices with perfect foresight. She is such a small part of the market that her individual choices do not changes prices. I compute the CEV that equates the utility of the infinitesimally rational agent with the utility of the LCH agent. The minimum CEV for this comparison is -0.17% for the benefit cut reform (-0.30% for the tax increase reform). The CEV is less than 0.2% in magnitude for the first six periods (driven mainly by the demographic change), and then quickly falls to zero. The negative sign indicates that LCH agents are worse off than the rational agent living in a LCH world. This CEV is presented below in Figure 33.

The welfare cost of learning can also be explored in the context of a recession. Agents using LCH or FHL learning will fare worse during and after a recession than fully rational agents. This is because the learning agents will not anticipate the general equilibrium effects of the initial decline in production. An example recession is presented in Appendix C.

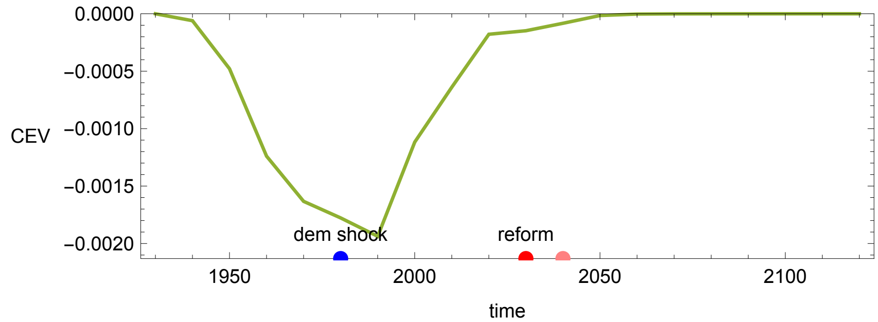


FIGURE 33. Consumption Equivalent Variation (CEV) measure of the welfare cost of using adaptive learning along the transition path of an announced social security benefit cut. The CEV compares an infinitesimally rational agent to a life-cycle horizon learner. The infinitesimally rational agent is effected by the general equilibrium effects of learning, but it able to predict future prices. The initial population growth rate is 0.1802 and falls to 0.01 in 1980. Initial government policy is τ^0, τ^1, ϕ equal to (0.124, 0.045, 0.4)'. In 2030, the benefit replacement rate falls to ϕ=0.332. The gain parameter γ=0.93.

Welfare Costs of Policy Uncertainty

The policy changes of the previous sections are non-stochastic and announced. However, the future of the U.S. social security system seems difficult to anticipate. A growing body of research focuses on the welfare cost of this type of uncertainty.[14] I calculate the welfare cost of policy uncertainty under RE and under LCH learning. To do this, I model government policy as an exogenous, stochastic process.

Let $\omega_t = (\tau_t^0, \tau_t^1, \phi_t)'$ describe the government policy parameters at time t. Suppose that the social security program will be reformed at some future date. The reform date falls within a known, finite set. Suppose that the realization of ω_{t+1} depends on ω_t, and is contained in the finite set Ω. Suppose also that each possible reform converges to a steady state (the path is non-explosive).

[14]I follow Caliendo, Casanova, Gorry, and Slavov (2016) and use the words "uncertainty" and "risk" interchangeably in this paper. All of the examples I will consider have known probabilities, and thus might be called "risky."

Let the probability of realizing a particular value of ω_{t+1} given ω_t be described by $\pi(\omega_{t+1}|\omega_t)$. Using this notation, the expected value in the household first order equations (4.4) can be written as:

$$u'(c^j_{t+j-1}) = \beta \sum_{\omega_{t+j} \in \Omega} \pi(\omega_{t+j}|\omega_{t+j-1}) R_{t+j} u'(c^{j+1}_{t+j}) \quad \text{for } j = 1, \cdots, J-1 \tag{4.32}$$

Introducing aggregate uncertainty to the model raises some concerns; a steady state wealth distribution will not exist in general in an OLG model with aggregate uncertainty (see Krueger and Kubler (2004) for a discussion). I overcome this problem by modeling aggregate policy uncertainty in a simple, stylized way. Policy uncertainty only exists for a small number of periods and policy reform can only be one of a small number of policy options. These simplifications, combined with the calibration of the life-cycle as six periods, allow me to calculate the equilibrium paths for the economy with policy uncertainty.

Suppose that policy uncertainty take the following form: reform is possible in either date S, or in date $S + 1$. Within each period two reforms are possible, either a benefit cut or a tax increase. Thus, there are four possible paths for the economy. The probability of each path is $p = 0.25$. All agents in the economy know the four possible reforms and their relatively probability. I calibrate this example to correspond to the US system. The economy starts in a steady with a ratio of retirees to workers of 0.3 and with government policy: $\tau^0 = 0.124$, $\tau^1 = 0.045$, and $\phi = 0.4$. In 1980, the growth rate of the population falls to 0.01 leading the ratio of retirees to workers to grow, eventually reaching 0.485. The demographic change causes the social security system to run a deficit, which increases government debt.

Reform is calibrated as either a tax increase to $\tau^0 = 0.1516$, or a benefit cut to $\phi = 0.32$. The reform takes place in either date 2030 or 2040. The SSA estimates these reforms would eliminate the funding shortfall in the social security system. In order to have a dynamically efficient economy with positive government debt, the Leeper tax rate is set to $\tau^1 = 0.045$ before and after reform. The gain parameter for the learning model is set to $\gamma = 0.93$.

Before considering the welfare cost of reform, first consider the transition dynamics of the four possible equilibrium paths in the RE model. The four possible paths for the economy are identical up to 2020. Agents in (or before) period five face the same uncertainty about future social security policy. This uncertainty is either fully or partially resolved in 2030. Along the first two paths of the economy, agents observe that policy was reformed in 2030 (either as a tax increase or benefit cut). Uncertainty is resolved for these two paths. Along the other two paths, agents observe that policy was not reformed in 2030, so they know reform will take place in 2040, but they do not which reform until the policy is realized. In 2040, all uncertainty is resolved, and all four paths are (potentially) different. There are two possible steady states for this example, the tax increase steady state and the benefit cut steady state. Both tax increase paths converge to the tax increase steady state; similarly, both benefit cut paths converge to the benefit cut steady state. The asset paths for this example are presented in Figure 34.

This same policy uncertainty experiment is possible in the learning models; I will present results for the LCH model. Agents in the learning model do not know future prices, they make decisions based on their adaptive expectations. In the initial few periods, following the change in the population growth rate, the LCH agents over-save compared to the rational agents because they are overestimating the interest rate. This drives up capital in the LCH model compared to the RE model. The LCH agents benefit from this higher capital stock (since the economy is dynamically efficient) and don't have to save as much in the periods right

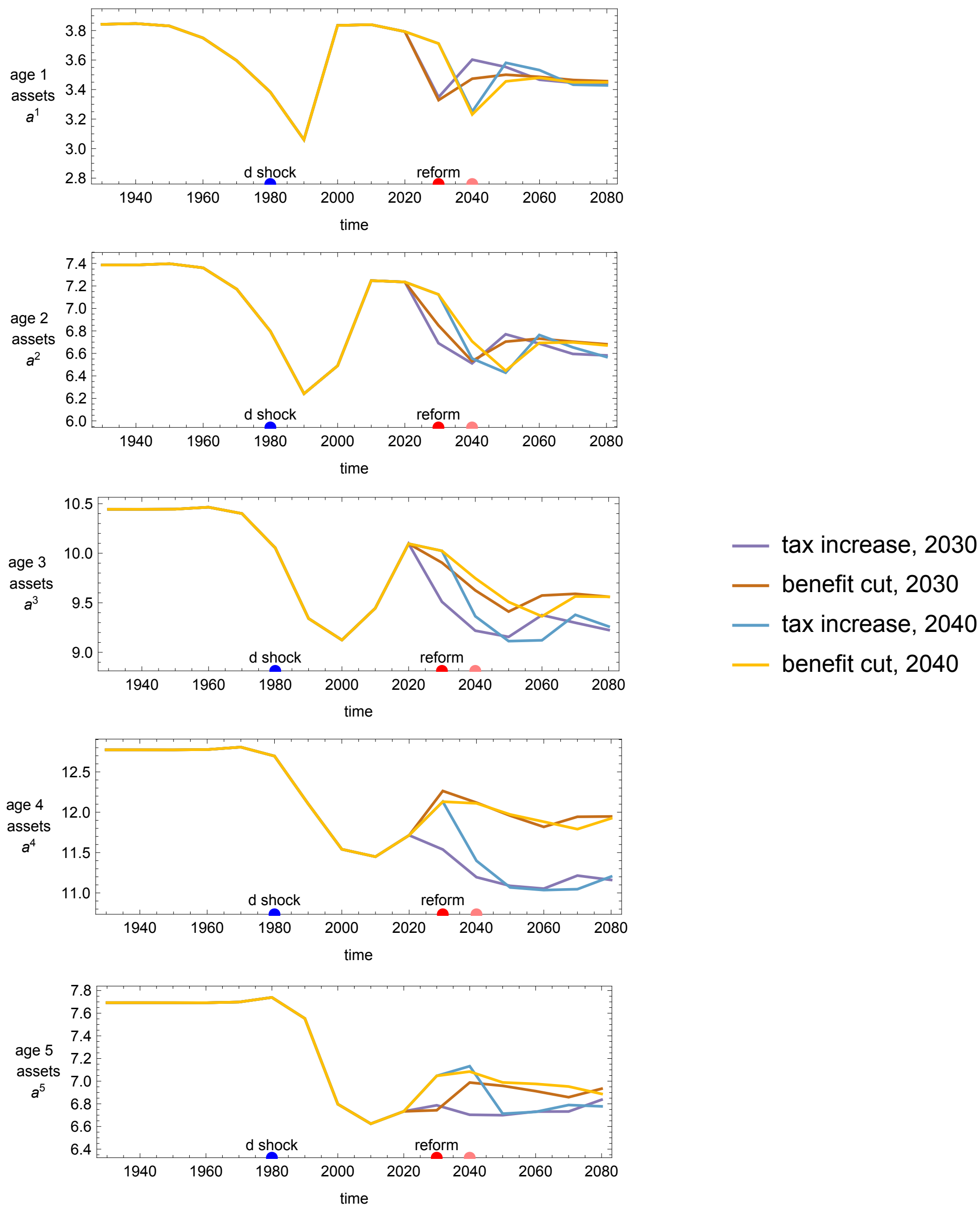


FIGURE 34. Transition paths for asset holdings in the RE model with policy uncertainty. Four reforms are possible. Either taxes are increased in 2030 (purple line), benefits are cut in 2030 (red/brown line), taxes are increased in 2040 (light blue line), or benefits are cut in 2040 (gold line). All four paths are identical until period 2020. This is because all agents face the same policy uncertainty leading up to the first possible reform. The two possible reform dates are indicated with red and pink dots. The beginning of the demographic transition is indicated with a blue dot.

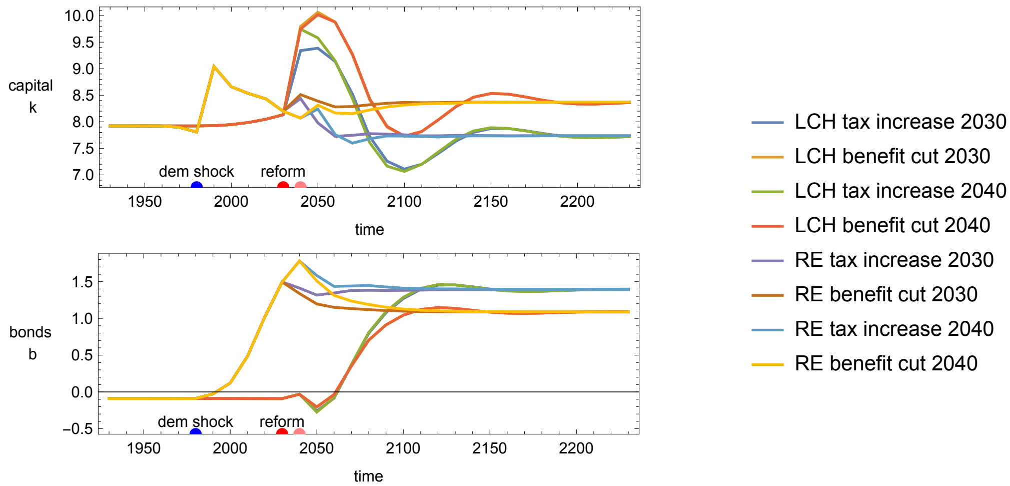


FIGURE 35. Transition paths for capital and bonds in the LCH model and RE model with policy uncertainty. Four reforms are possible in each model. Either taxes are increased in 2030 (LCH: blue, RE: purple line), benefits are cut in 2030 (LCH: yellow, RE: red/brown line), taxes are increased in 2040 (LCH:green, RE: light blue line), or benefits are cut in 2040 (LCH: red, RE: gold line). The LCH paths are cyclical, while the RE paths converge to the new steady states quickly. The dynamics before the reform are driven by demographic change in 1980 and the forward looking behavior of agents anticipating reform. The benefit cut paths converge to a steady state with higher capital.

before the policy reform. Following a policy reform, the paths of savings, capital and bonds are cyclical in the LCH model since agents are slowly updating their forecasts of prices and bonds.

Note, that all of the agents in the RE and LCH model know the policy process. All agents know the possible dates of reform, the new policy parameters, and the probability of each reform. All agents are forward looking (to the end of their life-cycle). The only difference is that the LCH agents forecast future prices and bonds adaptively, while the rational agents make fully rational forecasts. The paths of capital and assets for the LCH and RE models are presented together in Figures 35 and 36.

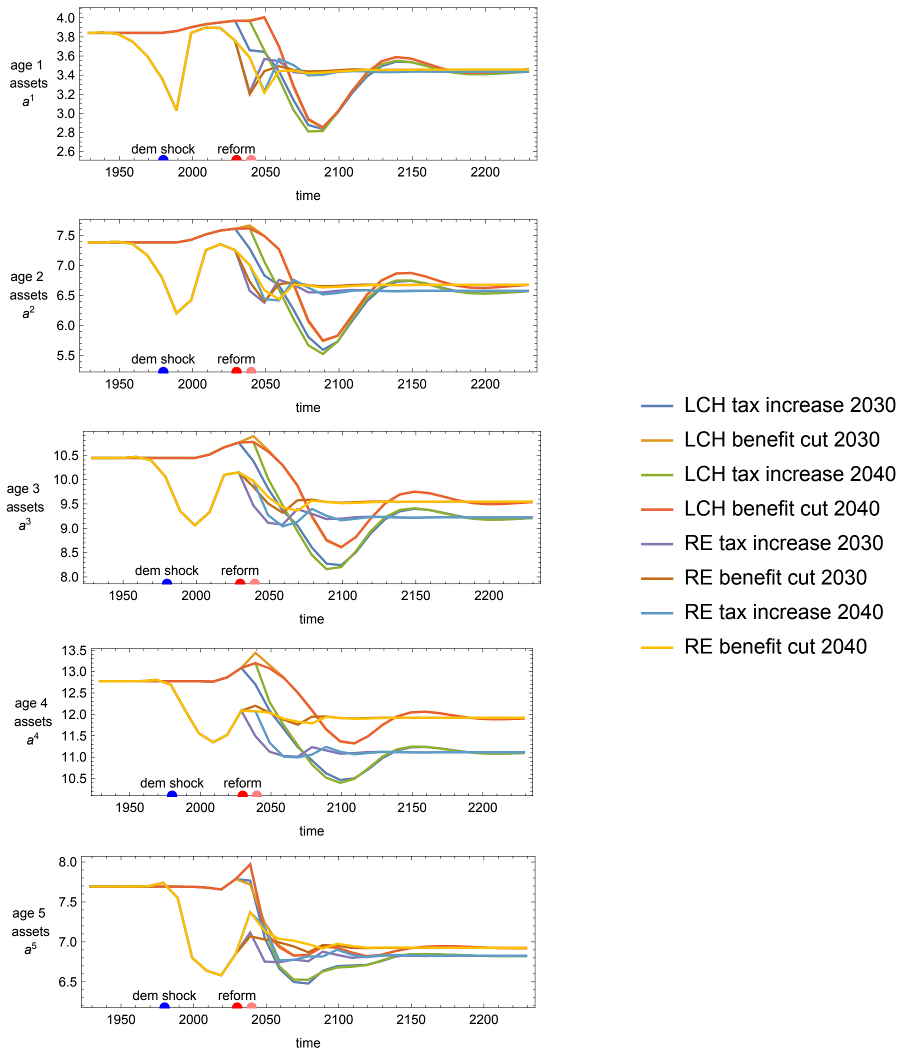


FIGURE 36. Transition paths for asset holdings in the LCH model and RE model with policy uncertainty. Four reforms are possible in each model. Either taxes are increased in 2030 (LCH: blue, RE: purple line), benefits are cut in 2030 (LCH: yellow, RE: red/brown line), taxes are increased in 2040 (LCH:green, RE: light blue line), or benefits are cut in 2040 (LCH: red, RE: gold line). The benefit cut paths converge to a steady state with lower age 1 and 2 savings (a^1 and a^2) and higher age 3, 4 and 5 savings (a^3, a^4, a^5). The LCH paths are cyclical, while the RE paths converge to the new steady states quickly. The dynamics before the reform are driven by demographic change in 1980; the rational agents respond to the demographic changes while the learning agents do not. Both types of agents are forward looking and respond to looming reform.

Welfare Cost

The welfare cost of policy uncertainty is calculated in this section using two ex-post Consumption Equivalent Variation techniques: one for the rational expectations model, and one for the learning model. In each, the utility of an agent who experiences a given realization of the uncertain policy process (that is, they experience a vector of policy parameters over their lifecycle $\omega_t, \ldots, \omega_{t+J}$) is compared to the utility of an agent who experience the same policy realization with certainty (that is, the policy parameters $\omega_t, \ldots, \omega_{t+J}$ are anticipated). Kitao (2016) constructs a similar welfare measure to examine the welfare cost of policy uncertainty (particularly when considering uncertainty over the timing and type of policy reform). Bütler (1999) uses a similar metric (although her model contains agent-level policy misperceptions rather than aggregate policy uncertainty). The ex-post CEV for policy uncertainty in the RE model is calculated as:

$$\sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j,A}(1+\Delta_u)) = \sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j,A,u}) \tag{4.33}$$

where $c_{t+j-1}^{j,A}$ indicates age j consumption in period $t+j-1$ in the model with policy path A and rational expectations and $c_{t+j-1}^{j,A,u}$ is consumption in the model with RE and policy uncertainty where policy path A was realized. As before, $u(c)$ indicates the period utility function, and Δ_u is the CEV for policy uncertainty.

The welfare cost of social security policy uncertainty is illustrated below in Figure 37. The figure depict the CEV for each of the four possible paths in the RE model. The calibration of this example is identical to example 35. Recall, the economy begins in a steady state with a high population growth rate. In 1980, the growth rate falls and the social security system begins to run a deficit. Bonds increase until reform is enacted and the economy converges to a new steady state

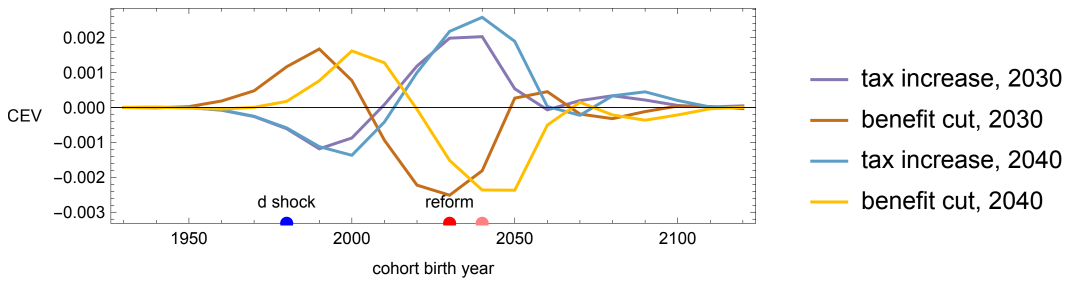


FIGURE 37. Consumption equivalent variation measure of the welfare cost of policy uncertainty in the RE model. The welfare cost is constructed by comparing the utility of realized policy parameters of a given path with the utility of that same policy change in a perfect foresight, RE model. The CEV is show pictured for all four possible paths: taxes are increased in 2030 (purple), benefits are cut in 2030 (red/brown), taxes are increased in 2040 (light blue), or benefits are cut in 2040 (gold). A negative CEV indicates that an agent would have higher utility in a RE model without policy uncertainty. A positive CEV indicates that utility is higher in the uncertain model.

associated with the particular reform. The four possible reforms are a benefit cut in 2030 or 2040 or a tax increase in period 2030 or 2040.

The welfare cost of social security policy uncertainty for the RE model is small. The cost is the less than 0.25% of period consumption for cohorts along all four paths. The initial cohorts benefit if the reform is a benefit cut, since they do not have to pay higher taxes and do not experience the benefit reduction in their lifetime. The agents born a few periods before and after the benefit cut are harmed because they receive lower benefits, but are not alive to experience the feedback effect of higher wages that result in the new steady state. The most harmed cohort are agents born in 2030 if the realized policy is the benefit cut in 2030. These agents would give up 0.25% of consumption to avoid being born in a world with policy uncertainty (even though the uncertainty is resolved in the period in which they are born).

The relatively small welfare cost of policy uncertainty in the rational expectations framework that I find is consistent with work by other authors. Kitao

(2016) finds the welfare cost of pension reform uncertainty in a model calibrated to the Japanese economy to be 0.8%-1.5% of period consumption.[15] Caliendo et al. (2015) find the welfare cost of social security policy uncertainty to be 0.01% of period consumption for an agent with average earnings.[16] Part of the reason Kitao and I find larger welfare costs than Caliendo et al. is because Kitao and I both use a general equilibrium framework. Caliendo et al. use a partial equilibrium model, which misses the feedback effect of savings on future wages and interest rates. Bütler (1999) finds that the welfare cost of misperceived social security benefits is between 0-3.46% of period consumption in a partial equilibrium model. Bütler's results are not easily comparable to other papers, because agent beliefs are model consistent in her paper.[17]

To assess the cost of policy uncertainty in the learning model the ex-post Consumption Equivalent Variation compares the lifetime utility of an agent in a LCH model who experiences a given realization of the uncertain policy process (that is, they experience a vector of policy parameters over their lifecycle $\omega_t, \dots, \omega_{t+J}$) to the utility of an agent in a LCH model who experience the same

[15]Kitao constructs a CEV measure similar to the CEV used in this paper which compares utility of agents who experience a particular reform under uncertain with utility of agents who experience the same reform with certainty. She also compares reforms realized in later periods to the baseline of earlier reform (and finds CEV in the range of 2-3.5% of period consumption). Finally, she also considers a CEV that compares the utility of agents who experience a particular reform with the utility of agents who experience the *expected* reform. Kitao finds similar magnitudes using both CEV measures.

[16] The welfare metric developed in Caliendo et al. (2015) is not exactly the same as the CEV developed in this paper. Caliendo et al. consider a continuous distribution of policy reform dates and options. The CEV metric they construct compares the utility of agent endowed with expected wealth over all possible reforms who experiences the average (or expected) reform to the utility of an agent who faces uncertainty. The also perform heterogeneity analysis and find the welfare cost of social security policy uncertainty to be 2.39 times higher for the lowest earners when the policy is benefit cut, and 1.7 times higher for tax increase. The welfare cost for highest earners is 0.6 times the welfare cost for an average earning when the policy is a benefit cut, and 2.25 times higher for tax increase. These costs are still quite small, since $2.39 \times 0.01 = 0.0239$.

[17]See Phelan (1999) for a discussion of Bütler (1999).

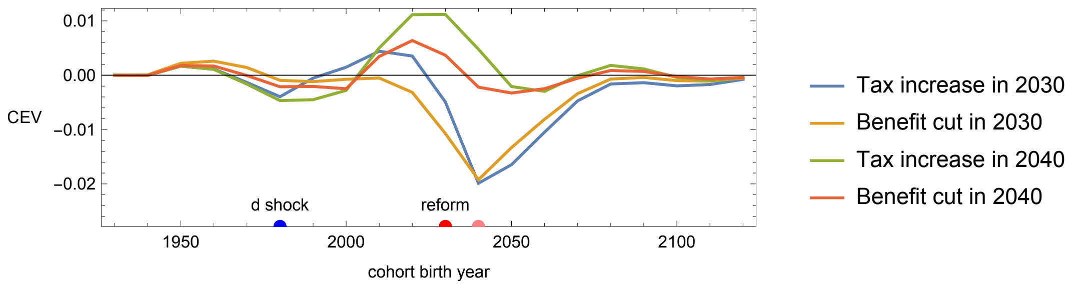


FIGURE 38. Consumption equivalent variation measure of the welfare cost of policy uncertainty in the LCH model. The welfare cost is constructed by comparing the utility of realized policy parameters of a given path with the utility of that same policy change in a LCH model. The CEV is pictured for all four possible paths: tax increased in 2030 (blue), benefit cut in 2030 (yellow), tax increase in 2040 (green), or benefit cut in 2040 (red). A negative CEV indicates that an agent would have higher utility in a LCH model with announced policy (no uncertainty). A positive CEV indicates that utility is higher in the uncertain model. The gain parameter $\gamma = 0.93$ in this example.

announced policy realization (that is, the policy parameters $\omega_t, \ldots, \omega_{t+J}$ are anticipated).

The learning CEV is calculated as:

$$\sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j,A}(1+\Delta_{uL})) = \sum_{j=1}^{J} \beta^{j-1} u(c_{t+j-1}^{j,A,u}) \tag{4.34}$$

where $c_{t+j-1}^{j,A}$ indicates age j consumption in period $t+j-1$ in the model with policy path A and LCH learning, and $c_{t+j-1}^{j,A,u}$ is consumption in the model with learning and policy uncertainty where policy path A was realized. As before, $u(c)$ indicates the period utility function, and Δ_{uL} is the CEV for policy uncertainty comparing learning to learning.

This learning CEV measures the change in consumption that is the result of policy uncertainty in the LCH framework. The learning CEV is presented in below in Figure 38.

The learning uncertainty CEV is positive for several cohorts if taxes are increased in 2040. This is because the possibility of facing reform in 2030 increases agent savings and drives up the capital stock in the model with policy uncertainty, relative to the model with announced tax increases. The accumulation of extra capital raises agent welfare, relative to the baseline. The maximum (positive) CEV is 1.11% and occurs in 2030. The cohort of agents born in 2030 in the LCH model with a tax increase in 2040 would need to have 1.11% added to their period consumption in order to be indifferent between their consumption, and the consumption of agents born in 2030 in a LCH model that had a 25% chance of each reform (where the tax increase in 2040 was realized).

The learning uncertainty CEV is negative for a few early cohorts who are harmed by increased precautionary saving, and for a few cohorts following the reform who experience the bottom of the swing in the capital stock. The swings in state variables are larger in the uncertain model, and thus the welfare cost is larger. The min CEV is -1.98% and occurs in 2040 for agents who are born after the tax increase in 2030. The minimum CEV along the other three paths is between -0.32% and -1.92%. As in the previous examples, the CEV is depends partially on the gain parameter. The min CEV is -1.94% when $\gamma = 0.35$.

My analysis suggests that rational expectations models may understate the welfare cost of social security policy reform *and* the welfare cost of policy uncertainty. The life-cycle horizon learning model I propose is a small deviation from rational expectations in which agents maintain model consistent beliefs Agents in the LCH model are still optimizing and they are still forward looking, yet the welfare cost of policy uncertainty is much larger. The welfare cost is driven mainly by the cyclical changes in capital stock that are introduced into the model by adaptive learning. The most harmed agents in the LCH model would be willing to give up nearly 2% of period consumption to avoid living in a world of policy uncertainty. In the RE model, the most harmed agents would only be willing to give up 0.25% of period consumption to avoid the policy uncertainty. The welfare cost in the learning model is nearly an order of magnitude larger than in the rational case.

Robustness

Forecast tax burden

In main specification, agents fully understand the Leeper-tax and forecast government debt levels in order to estimate their upcoming taxes. I back away from that assumption in this section and assume that agents do not fully understand the tax system. Agents observe the total tax burden (rate) they face in the current period, and they use adaptive learning to forecast their future tax burden. They do not incorporate knowledge about the structure of the tax system. Agents continue to forecast social security benefits as the replacement rate ϕ times the expected wage at the time of retirement.

Agents in this framework understand that a fraction of their wage is taken by the government every period, but they do not distinguish between the Leeper tax and the payroll tax. Agents are not forward looking with regard to tax changes. They are, however, forward looking with regard to social security benefit changes. This assumption will be relaxed in the next section.

Under this specification, agents forecast

$$w^e_{t+1} = \gamma w_t + (1-\gamma) w^e_t \tag{4.35}$$

$$R^e_{t+1} = \gamma R_t + (1-\gamma) R^e_t \tag{4.36}$$

$$\tau^e_{t+1} = \gamma(\tau^0_t + \tau^1_t b_t) + (1-\gamma)\tau^e_t \tag{4.37}$$

with $\gamma \in (0,1)$. Forecasts many periods ahead are equal to the one-step-ahead forecasts: $x^e_{t+j} = x^e_{t,t+j} = x^e_{t+1}$, for $x = R, w, \tau$ and $j > 1$.

Agents do not respond to an announced tax change until after the change has been is implemented. Savings of working-age cohorts increase as the policy is implemented, instead of before. Thus the welfare cost of learning (compared to the RE baseline) is larger in this specification than in the baseline learning model. For example, consider the same set-up as the announced policy change in section IV. The initial ratio of retirees to workers is 0.3. The ratio begins to increase in 1980, which causes the social security system to run deficits; capital falls and bonds increase until reform is enacted. Initial policy is $\tau^0 = 0.124$, $\tau^1 = 0.045$ and $\phi = 0.4$. Reform is calibrated as a payroll tax increase to $\tau^0 = 0.1516$. This policy change is depicted in Figure 39 and the CEV is presented in Figure 40.

Agents in the learning models do not respond to the tax increase until the period of the policy change. The swings in capital stock that follow the policy change are larger than in the baseline learning model when agents anticipate tax changes. The minimum CEV in the tax-forecasting learning model is in 2040. An agent born in 2040 in the RE model would be willing to give up 1.39% of period consumption to avoid being born in the same period in the LCH tax-forecasting model. The minimum CEV in the bond-forecasting model is -1.19%. As with pervious examples, the CEV depends on the learning gain and is smaller for larger gains. The gain for these examples was set to $\gamma = 0.93$.

Forecast Policy Adaptively

As a final robustness check, suppose that agents do not anticipate any policy changes, but rather agents learn all policy adaptively. Agents forecast their tax

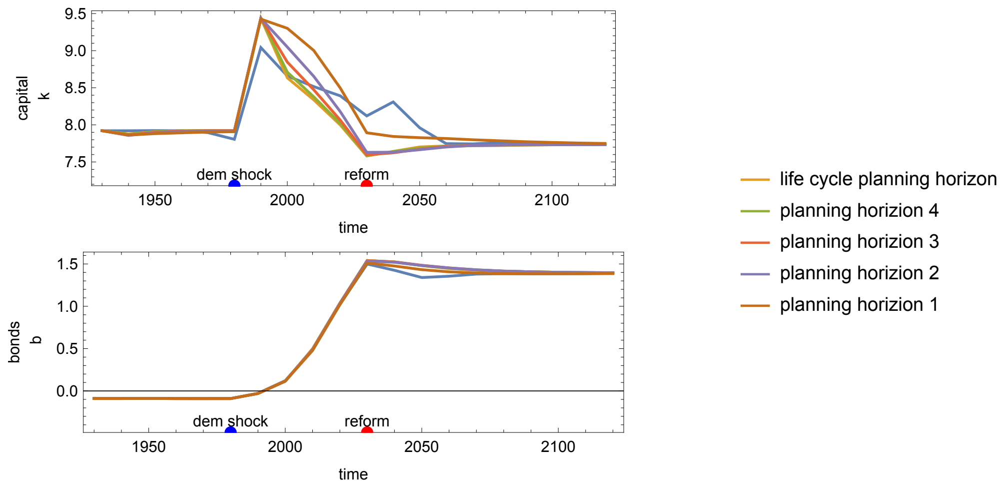


FIGURE 39. Equilibrium paths for the rational expectations economy and learning economies that forecast the tax burden according to equation 4.37. The economy begins with a ratio of retirees to workers of 0.3; this falls to 0.485 over six periods beginning in 1980. Social security deficits lead to an accumulation of government debt until taxes are raised in 2030. The RE path is shown in blue, LCH learning in yellow, finite horizon learning in green (for 4 period planning horizon), red (3), purple (2), and orange/brown (1-step-ahead).

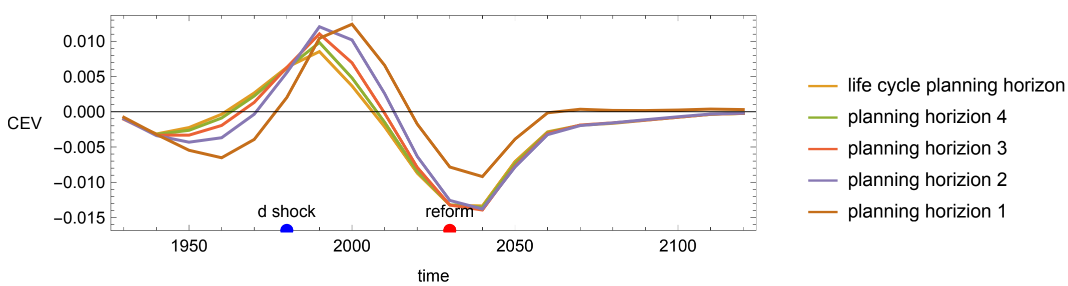


FIGURE 40. Consumption equivalent variation paths learning economies that forecast the tax burden according to equation 4.37. The CEV is calculated according to 4.31. The economy begins with a ratio of retirees to workers of 0.3; this falls to 0.485 over six periods beginning in 1980. Social security deficits lead to an accumulation of government debt until taxes are raised in 2030. The CEV for LCH learning is sown in yellow, finite horizon learning in green (for 4 period planning horizon), red (3), purple (2), and orange/brown (1-step-ahead). The welfare cost is greatest (smallest CEV) for LCH learning.

burden $(1-\tau_t)w_t$ and social security benefit, z_t adaptively. Agents expect to receive the same social security benefit in both periods of retirement (which is consistent with the actual policy process). They forecast social security benefits for the period in which they retire.

Agents forecast

$$w^e_{t+1} = \gamma w_t + (1-\gamma)w^e_t \tag{4.38}$$

$$R^e_{t+1} = \gamma R_t + (1-\gamma)R^e_t \tag{4.39}$$

$$\tau^e_{t+1} = \gamma(\tau^0_t + \tau^1_t b_t) + (1-\gamma)\tau^e_t \tag{4.40}$$

$$z^e_{t+1} = \gamma \phi_t w_t + (1-\gamma)z^e_t \tag{4.41}$$

with $\gamma \in (0,1)$. Forecasts many periods ahead are equal to the one-step-ahead forecasts: $x^e_{t+j} = x^e_{t,t+j} = x^e_{t+1}$, for $x = R, w, \tau, z$ and $j > 1$.

Under this specification, agents in learning models do not respond to announced policy changes. They only respond to tax or benefit changes when they have been implemented. The welfare cost of tax changes is identical to the previous section. However, the welfare cost of social security benefit changes is larger. Using the set-up from the previous section, suppose the population growth rate is initial high, and then falls in 1980, leading to an accumulation of government debt until benefits are cut in 2030. The CEV is lowest in 2030 and equals -1.63%. This means an agent born in the rational model in 2030 would be willing to give up 1.63% of her period consumption in order to avoid being born in the same period in a model where agents adaptively learn policy with a life-cycle planning horizon. In contrast, the minimum CEV in the baseline LCH model (where agents anticipate policy change and adaptively forecast prices and bonds) is -0.81%. The gain parameter for these examples was $\gamma = 0.93$. The welfare costs are larger for a smaller gain.

Sensitivity Analysis

The qualitative results of this paper are sensitive to parameter choice. As a robustness check, I calibrated the model to be dynamically *inefficient* and ran similar experiments (raising taxes or lowering social security benefits). I parameterize a dynamically inefficient model by setting the discount factor β greater than one. This increases agents desire to consume in old age and drives up savings, and thus the capital stock. I use the parameterization of Bullard and Russell (1999): $\sigma = 4.2$, annual $\beta = (\frac{1}{1-0.041})$, $\alpha = 0.25$, and annual $\delta = 0.1$.[18] Under this specification, agents in learning models still suffer relative to RE agents when facing demographic changes, policy changes, and policy uncertainty. However, the welfare cost of learning is much smaller. For example, the welfare cost that compares LCH to RE of an announced benefit cut in response to the demographic shock is less than 1% of period consumption.

Conclusion

Demographic changes in the United States make future social security reform likely, as more beneficiaries are supported by each worker paying taxes. If the program is left unchanged, benefits will exceed tax receipts and the social security trust fund will be depleted by around the year 2034. The welfare cost to agents of social security reform is not limited to agents alive during the policy change. Using two new models of bounded rationality, I show that the welfare cost

[18] My model does not compare directly to Bullard and Russell (1999) in two key ways; first I do not have stochastic survival probability, nor do I have endogenous labor choice. Thus, this calibration should be viewed as a robustness exercise only, and not an attempt to replicate Bullard and Russell (1999).

of announced policy changes can be quite large if there are general equilibrium feedback effects along the transition path.

I relax the rational expectations assumption and model agents who forecast future interest rates and wages adaptively. I model this in two main ways. First, I develop a model of life-cycle horizon learning, in which agents make optimal decisions over their lifecycle, given their (potentially imperfect) forecasts of prices. Second, I model finite horizon life-cycle learning, in which agents plan over a shorter planning horizon and make optimal choices conditional on their forecasts and the assets they plan to hold at the end of their planning horizon.

The maximum welfare cost occurs along the transition path for the tax increase in the LCH learning model; agents born after the reform while the capital stock is depressed are worse off by up to about 1% of period consumption compared to agents in a fully rational model. The welfare costs of learning are not always negative; agents benefit when the capital stock is increased as the result of over-saving by the learning agents.

The uncertainty regarding the future of social security is also economically interesting. Social security policy uncertainty is not terribly costly to agents in a rational expectations framework. Forward looking, rational agents are able to save and partially self-insure against the aggregate risk of unfavorable policy change. The greatest welfare cost of policy uncertainty in the rational model is equivalent to less than 0.25% reduction in period consumption. I find that the maximum welfare cost to agents in a model with social security policy uncertainty is larger in a model with adaptive learning. The maximum welfare cost of policy uncertainty in the LCH model (compared to the LCH model with announced policy) is 1.98%. The CEV are significantly larger in the learning model than the rational expectations model.

Policy makers who use rational expectations models to predict the welfare effects of social security policy change will understate the cost of reform. To the extent that the learning dynamics are a realistic depiction of agent level behavior, the welfare costs of announced or uncertain social security reforms might be quite large. If agents are unable to predict the general equilibrium effects of a change in government policy, they will not be able to respond optimally. The results of this paper suggest that policy makers can help agents by announcing policy and also by explaining how the policy will impact wages and interest rates.

The learning models I develop in the paper include demographic changes and endogenous government debt. Several interesting questions that are beyond the scope of this paper can be addressed in this framework. I plan to explore the relationship between delaying social security reform, growing deficits, and explosive government debt in future work. The learning models developed in this paper provide an excellent framework to examine explosive debt that is not possible in a standard, rational expectations framework. The model could also be used to explore the relationship between recessions, public pensions, and lifecycle savings.

CHAPTER V

CONCLUSION

In the three substantive chapters of this dissertation, I have shown how agents respond to announced and uncertain social security reform in general equilibrium over-lapping generations models.

In chapter II, I examine social security in an analytically tractable model with endogenous deficits and debt. The need for social security reform is motivated by a saddle-node bifurcation. The number and stability of steady states depends on the underlying parameters. I calculate the minimum reform needed to ensure convergence to a steady state in various scenarios. The longer a reform is delayed, the larger the reform needs to be in order to avoid explosive government debt. Comparative static analysis reveals that households increase their saving in response to an increased probability of social security benefit reductions. Comparative static analysis also shows that the savings rate of households is increasing in the social security reform benefit rate; agents save less if they anticipate larger social security benefits. I also conduct welfare analysis using a social welfare function; optimal reform depends critically on the discount factor and planning horizon of the planner.

In chapter III, I extend the analysis of the previous chapter to a model setting with fully forward looking behavior. Households live for multiple periods, and so they respond to announced and potential reforms many periods in the future. I examine the response of households to policy timing uncertainty (policy can take place within one of three periods), and combined timing and option uncertainty (either taxes or benefits are changed). I also examine the long-lasting effects of policy uncertainty and show that the capital stock and can be depressed for generations following a reform. Finally, in this chapter, I study the interaction of policy uncertainty and the sustainability of government debt. I show that if the probability of reform is high enough, that the response from households can be large enough to ensure convergence to a steady state, even if the reform never takes place.

In chapter IV, I introduce two new behavioral models: life-cycle horizon learning, and finite horizon life-cycle learning. I demonstrate that the welfare cost of social security policy uncertainty is much larger when agents use adaptive expectations rather than rational expectations. The welfare cost is not uniform across generations; cohorts born after reform may still be harmed by the cyclical dynamics that result from agents learning with policy uncertainty.

The results presented in this dissertation represent the core of a broader research agenda examining bounded rationality, social security, and retirement. The learning models I developed in chapter IV can be used to explore a variety of policy questions that impact the lifecycle. I am currently working on a project to examine the long-term relationship between unfunded pension liabilities and growing government debt. By using a model of bounded rationality, I am able to explore explosive dynamics that are not possible in a fully rational model. I am also exploring the intergenerational effects of fiscal policy and recessions with boundedly rational agents. Going forward, I plan to explore the relationship between Euler-equation learning and finite-horizon learning in OLG economies, and to explore heterogeneous learning rules in OLG models. I may also model the consumption profile and retirement timing decision of agents under various learning rules.

I am interested in the relationship between stated household beliefs about social security policy and current savings decisions. Many households do not believe they will receive social security benefits (are reported in survey data, such as the American Life Panel), yet aren't increasing their private savings. I plan to test this relationship in a heterogeneous agent life-cycle model with rational expectations and also using a model of bounded rationality. I suspect that bounded rationality will generate savings behavior that is consistent with the data.

The aging of the U.S. population ensures that questions about social security, saving, and retirement will continue to be politically salient. I look forward to contributing research in this policy area.

APPENDIX A

TWO-PERIOD MODEL APPENDIX

Derivation of Golden Rule Level of Capital

The golden rule level of capital is defined as the steady state level of capital which maximizes aggregate consumption (and therefore aggregate utility). Aggregate consumption is given by $C_t = Y_t - I_t$ were I_t indicates aggregate capital investment. $I_t = K_{t+1} - (1-\delta)K_t$, with full depreciation $I_t = K_{t+1}$. Note that principle of the invariance of a maximum implies that the max of C_t will be equal to the max of C_t/L_tA_t. Thus, the golden rule level of capital can be found by maximizing C_t/L_tA_t with the steady state $k_t = k_{t+1} = k$ imposed

$$\frac{C_t}{L_tA_t} = \frac{Y_t}{L_t} - \frac{K_{t+1}}{L_tA_t}\frac{L_{t+1}}{L_{t+1}}\frac{A_{t+1}}{A_{t+1}}$$

$$c_t = f(k_t) - (1+n)(1+g)k_{t+1}$$

$$c = f(k) - (1+n)(1+g)k$$

$$\max_k c = f(k) - (1+n)(1+g)k$$

$$\text{FOC: } f'(k) - (1+n)(1+g) = 0$$

$$f'(k_{gr}) = (1+n)(1+g)$$

$$\alpha(k_{gr})^{\alpha-1} = (1+n)(1+g)$$

$$k_{gr} = \left[\frac{\alpha}{(1+n)(1+g)}\right]^{\frac{1}{1-\alpha}}$$

Balanced Budget

It is possible for the government to run a balanced budget by setting social security benefits equal to social security payroll taxes in each period. If the government runs a balanced budget (zero deficit) each period, two special cases are possible. Either the government has an initial stock of debt (which asymptotes to a steady state value), or the government government doesn't have any debt (and the model collapses to the univariate Diamond model with capital and social security). I will begin by discussing the planar case, when $b_0 \geq 0$. I will discuss the simple univariate case in section A.

When the government runs a balanced social security budget, the payroll taxes exactly cover social security benefits in each period:

$$\hat{\phi} = (1+n)\hat{\tau} \tag{A.1}$$

where $\hat{\phi}$ and $\hat{\tau}$ notation is used to emphasize that the government is running a true, balanced-budget, pay-as-you-go social security program. When this is the case, the government only has one choice variable (either $\hat{\tau}$ or $\hat{\phi}$) which determines the size of the social security program. Without loss of generality, I will assume the government chooses $\hat{\tau}$ and sets $\hat{\phi} = (1+n)\hat{\tau}$ so that the budget is balanced in each period and there are no deficits.

Plugging the balanced budget identity (A.1) into the transition equations (2.8) and (2.9) gives the dynamics for the balanced budget economy:

$$\begin{aligned} b_{t+1} &= \frac{1}{(1+n)(1+g)}\left[\alpha k_t^{\alpha-1} b_t + \left(\frac{(1+n)\hat{\tau}}{1+n} - \hat{\tau}\right)(1-\alpha)k_t^{\alpha}\right] \\ &= \frac{\alpha k_t^{\alpha-1} b_t}{(1+n)(1+g)} \end{aligned} \tag{A.2}$$

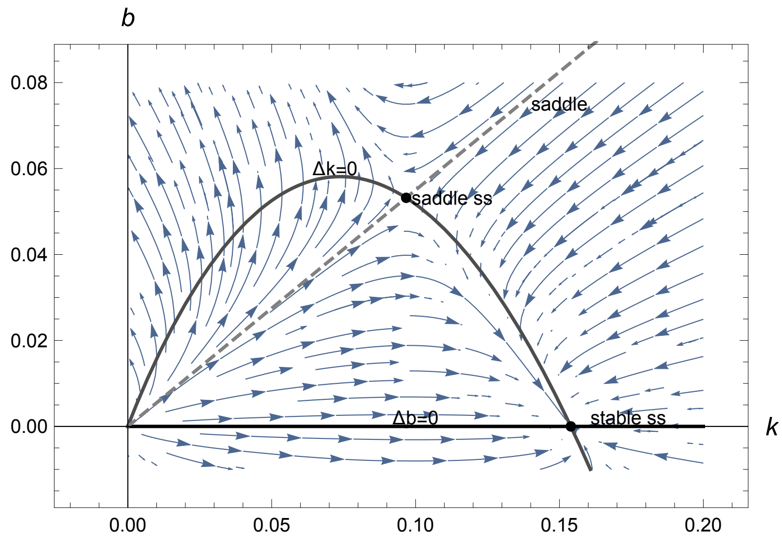


FIGURE A.41. Phase diagram for balanced budget $\hat{\phi}$=(1+n)$\hat{\tau}$. In the case of a balanced budget, the constant bond loci collapses to horizontal line at b=0. The model still has two steady states: the stable sink where the constant capital and constant bond loci intersect, and a steady state governed by the saddle path. In the balanced budget case, the saddle steady state is at the golden rule level of capital.

$$
\begin{aligned}
k_{t+1} = & \left[(1+n)(1+g) + \frac{(1+n)\hat{\tau}(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1} \\
& \left[\frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)k_t^{\alpha} - \alpha k_t^{\alpha-1}b_t - \left(\frac{(1+n)\hat{\tau}}{1+n} - \hat{\tau}\right)(1-\alpha)k_t^{\alpha}\right] \\
= & \left[(1+n)(1+g) + \frac{(1+n)\hat{\tau}(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1} \\
& \left[\frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)k_t^{\alpha} - \alpha k_t^{\alpha-1}b_t\right]
\end{aligned}
\tag{A.3}
$$

In the planar model with balanced budgets, the model has two steady states: one at the golden rule level of capital and positive bonds $(k_{gr}, b(k_{gr}))$, and a second steady state with zero bonds. Under the baseline parameterization of the model, the zero bond steady state is dynamically inefficient $k > k_{gr}$. These steady states are depicted in the phase diagrams in Figure A.41.

It is easy to show that $(k_{gr}, b(k_{gr}))$ is a steady-state of the model, by examining equations (A.2) and (A.3). Setting $k_{t+1} = k_t = k$ and $b_{t+1} = b_t = b$ in (A.2), we find:

$$
\begin{aligned}
(1+n)(1+g)b &= \alpha k^{\alpha-1} b + \left(\frac{(1+n)\hat{\tau}}{1+n} - \hat{\tau} \right)(1-\alpha)k^{\alpha} \\
(1+n)(1+g)b &= \alpha k^{\alpha-1} b \\
\alpha k^{\alpha-1} &= \frac{(1+n)(1+g)}{\alpha} \\
k &= \left[\frac{\alpha}{(1+n)(1+g)} \right]^{\frac{1}{1-\alpha}} \\
k &= k_{gr}
\end{aligned}
$$

Setting $k_{t+1} = k_t = k_{gr}$ and $b_{t+1} = b_t = b$ in (A.3), we find $b = b(k_{gr})$.

$$
\begin{aligned}
\psi k_{gr} &= \frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)k_{gr}^{\alpha} - \alpha k_{gr}^{\alpha-1} b \\
b &= \frac{\frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)k_{gr}^{\alpha} - \psi k_{gr}}{\alpha k_{gr}^{\alpha-1}} \\
b &= b(k_{gr})
\end{aligned}
$$

Where $\psi = (1+n)(1+g) + \frac{(1+n)\hat{\tau}(1-\alpha)(1+g)}{(1+\beta)\alpha}$.

It is also easy to show that $b = 0$ and $k = k(0)$ is a steady state of the model. Plugging $b = 0$ into (A.2) we get $0 = 0 + 0$. Plugging $b = 0$ into (A.3) we find the steady state value of k associated with zero bonds and zero deficits:

$$
k(0) = \left(\frac{\alpha\beta(1-\hat{\tau})(1-\alpha)}{(1+n)(1+g)(\alpha(1+\beta) + \hat{\tau}(1-\alpha))} \right)^{\frac{1}{1-\alpha}} \tag{A.4}
$$

Transcritical Bifurcation of the Balanced Budget Model

With a balanced budget, the model has two steady states: the golden rule steady state $(k_{gr}, b(k_{gr}))$, and a zero bond steady state (with capital given by equation A.4). The stability properties of the two steady states depend on the parameters of the model. This is called a transciritical bifurcation. For illustrative purposes, let $\hat{\tau}$ be the bifurcation parameter. Holding all other parameters constant, the number and stability properties of the steady states depends on the value of the parameter $\hat{\tau}$. At a critical value of the parameter $\hat{\tau} = \hat{\tau}_c$, a single steady state exists (at that steady state $k = k_{gr}$ and $b = 0$). At values of $\hat{\tau}$ below the critical value, two steady states exist and the zero bond steady state is stable while the golden rule steady state is a saddle (this is true under the baseline parameterization of the model as illustrated in Figure A.41). At values of $\hat{\tau}$ above the critical value, the stability properties are reserved; the zero bond steady state is a saddle and the golden rule steady state is stable.

The transcritical bifurcation is illustrated in figures A.42 and A.43. Figure A.42 plots the steady state values of capital associated with different levels of social security $\hat{\tau}$. For low values of $\hat{\tau}$, the zero bond steady state is stable (and has capital above the golden rule level). At the critical value $\hat{\tau}_c$, only one steady state exists. At values of $\hat{\tau}$ above the critical value, there are two steady states and the golden rule level of capital steady state is stable.

Phase diagrams of the balanced budget model are presented in Figure A.43. In the top panel, the size of the social security system is relatively small ($\hat{\tau} < \hat{\tau}_c$) and the zero bond steady state is stable. As the size of the system increases, capital is crowded out, and the capital stock at the stable steady declines. At the critical value $\hat{\tau}_c$, only one steady state exists (at the golden rule level of capital, with zero

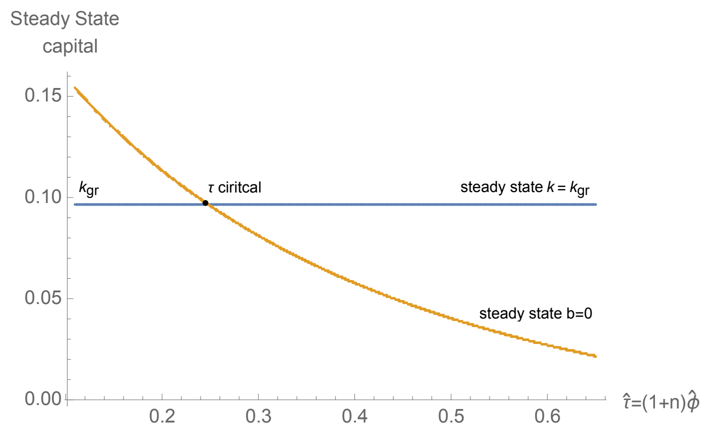


FIGURE A.42. Phase diagrams demonstrating the transcritical bifurcation with a balanced budget. As the size of the social security system $\hat{\tau}$ is increased, capital is crowded out and the stable steady state (with zero bonds) shifts to the left. At a critical value of $\hat{\tau}$, only one steady state exists, $(k_{gr}, 0)$. As the tax is further increased, the zero bond steady state moves further to the left to a lower value of capital. The stability properties of the steady states switch following the transcirtical bifurcation, and now the golden rule steady state is stable.

bonds). This is illustrated in the middle panel. As the size of the system grows and $\hat{\tau} > \hat{\tau}_c$, the stability properties of the steady states switch, as shown in the bottom panel. The golden rule steady state is now stable and the zero bond steady state is a saddle.

Analytical Transcritical Bifurcation

The transcritical bifurcation can be demonstrated analytically by showing that the stability properties of the steady states change as the bifurcation parameter passes through a critical value.[1] I will demonstrate the bifurcation using the parameter $\hat{\tau}$. As discussed in the main text of the paper (II), the stability of a steady state depends on the eigenvalues of the Jacobian matrix evaluated at the

[1]See Sander (1990) for a discussion of transcirtical bifurcations. De La Croix and Michael (2002) point out that transcritical bifurcations are uncommon in economics, often because they involve a trivial fixed-point.

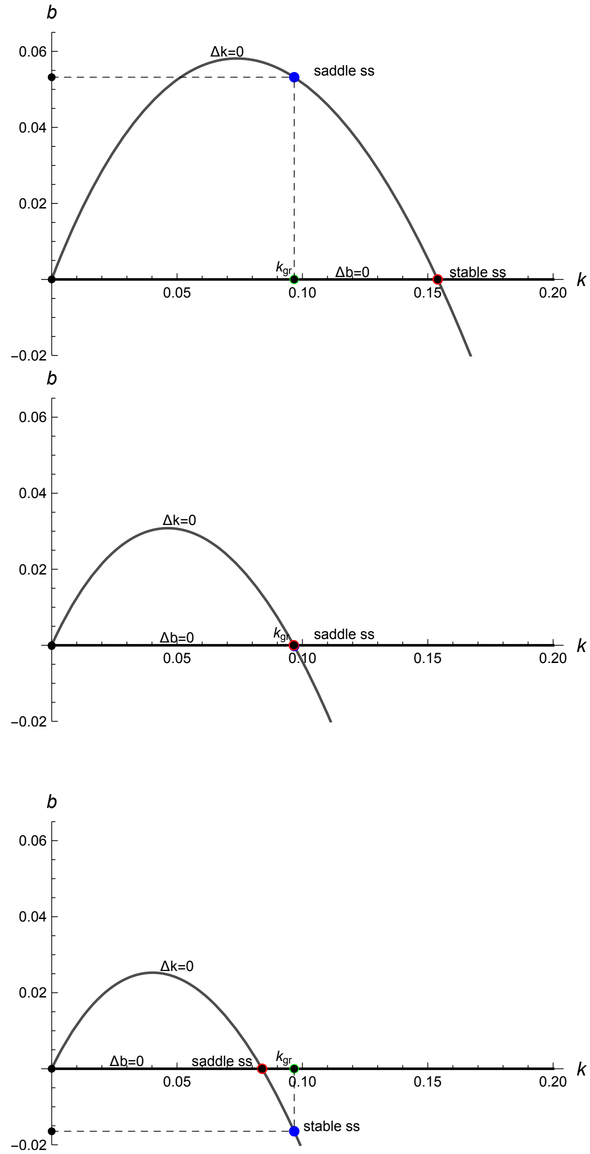


FIGURE A.43. Phase diagrams demonstrating the transcritical bifurcation with a balanced budget. As the size of the social security system $\hat{\tau}$ is increased, capital is crowded out and the stable steady state (with zero bonds) shifts to the left. At a critical value of $\hat{\tau}$, only one steady state exists, $(k_{gr}, 0)$. As the tax is further increased, the zero bond steady state moves further to the left to a lower value of capital. The stability properties of the steady states switch following the transcritical bifurcation, and now the golden rule steady state is stable.

steady state. If both eigenvalues are less than one in modulus, the steady state is stable; if one eigenvalue is greater than one in modulus and the other is less than one in modulus, the steady state is a saddle.

Letting $\gamma = (1+n)(1+g)$ and $\psi = \left(\gamma + \frac{\hat{\tau}(1-\alpha)}{(1+\beta)\alpha}\right)$, the Jacobian of the balanced budget model is given by:

$$J = \begin{pmatrix} \psi^{-1}\left(\frac{\alpha\beta(1-\hat{\tau})(1-\alpha)}{1+\beta}k_t^{\alpha-1} - (\alpha-1)\alpha k_t^{\alpha-2}b_t\right) & \psi^{-1}\alpha k_t^{\alpha-1} \\ \\ \frac{(\alpha-1)\alpha}{\gamma}k_t^{\alpha-2}b_t & \frac{\alpha}{\gamma}k_t^{\alpha-1} \end{pmatrix} \tag{A.5}$$

The critical value of $\hat{\tau}$ can be found by noting that when only one steady exists, $k = k_{gr}$ and $b = 0 = b(k_{gr})$. The second identity can be used to solve for

$$\hat{\tau}_c = \frac{\alpha[\beta(1-\alpha)k^\alpha - \gamma(1+\beta)k]}{(1-\alpha)(\alpha k^\alpha + \gamma k)}. \tag{A.6}$$

For values of $\hat{\tau} < \hat{\tau}_c$, the golden rule steady state is a saddle and the zero bond steady state is stable; for values of $\hat{\tau} > \hat{\tau}_c$, the reserve is true. This can be shown by finding the eigenvalues of the Jacobian evaluated at each of the steady states near the critical value $\hat{\tau}$.

Evaluated at the golden rule steady state, the Jacobian A.5 simplifies to

$$J|_{k_t=k_{gr},b_t=b(k_{gr})} = \begin{pmatrix} \frac{-(1-\alpha)((\alpha+\beta)(1-\hat{\tau})+\beta\alpha+\hat{\tau})}{\alpha(1+\beta)+(1-\alpha)\hat{\tau}} & \frac{-\alpha(1+\beta)}{\alpha(1+\beta)+(1-\alpha)\hat{\tau}} \\ \\ \frac{(1-\alpha)((1+\beta)(\alpha+(1-\alpha)\hat{\tau})-(1-\alpha)\beta)}{\alpha(1+\beta)} & 1 \end{pmatrix} \tag{A.7}$$

The eigenvalues of this matrix are

$$\left(\frac{(1-\alpha)\beta(1-\hat{\tau})}{\alpha(1+\beta)+(1-\alpha)\hat{\tau}}, \alpha\right) \tag{A.8}$$

The eigenvalues simplify to $(1,\alpha)$ evaluated at the critical value $\hat{\tau}_c$. The second eigenvalue is always less than one and does not depend on the bifurcation parameter. The magnitude of the first eigenvalue depends on the bifurcation parameter, $\hat{\tau}$. To demonstrate this, first note that the critical value evaluated at the golden rule level of capital is given by:

$$\hat{\tau}_c(k_{gr}) = \frac{\beta(1-\alpha) - \alpha(1+\beta)}{(1-\alpha)(1+\beta)} \tag{A.9}$$

Next, evaluate the first eigenvalue (A.8) at $\hat{\tau}_c(k_{gr}) + \epsilon$ where ϵ is very small. This simplifies to:

$$\lambda_1 = \frac{\beta(1+\beta\alpha) - \beta\epsilon(1+\beta)(1-\alpha)}{\beta(1+\beta\alpha) + \epsilon(1+\beta)(1-\alpha)} \tag{A.10}$$

When $\epsilon = 0$, the $\lambda_1 = 1$, as expected. When $\epsilon > 0$, the numerator of the fraction is smaller than the denominator, and $\lambda_1 < 1$. When $\epsilon < 0$, the reverse is true, and $\lambda_1 > 1$. Thus, for tax rates above the critical value, the golden rule steady state is stable, and for taxes below the critical value, the golden rule steady state is a saddle.

Evaluated at the zero bond steady state, the Jacobian A.5 simplifies to

$$J|_{k_t=k, b_t=0} = \begin{pmatrix} \alpha & -\frac{\alpha(1+\beta)}{(1-\alpha)\beta(1-\hat{\tau})} \\ & \\ 0 & \frac{\alpha(1+\beta)+(1-\alpha)\hat{\tau}}{(1-\alpha)\beta(1-\hat{\tau})} \end{pmatrix} \tag{A.11}$$

The eigenvalues of this matrix are $\left(\frac{\alpha(1+\beta)+(1-\alpha)\hat{\tau}}{(1-\alpha)\beta(1-\hat{\tau})}, \alpha\right)$. Note that the second eigenvalue is equal to α and does not depend on the bifurcation variable. Additionally, the first eigenvalue is the inverse of the eigenvalue evaluated at the golden rule steady state. Thus, when $\hat{\tau} = \hat{\tau}_c$, the eigenvalue is equal to one; for values of $\hat{\tau}$ above the critical value, the eigenvalue will be great than one; and for $\hat{\tau}$ below the critical value, the eigenvalue will be less than one. Thus the stability properties of the zero bond steady state are the mirror opposite of the golden rule steady state.

Special Case: Univariate Model

If the government runs a balanced budget and the initial stock of government debt is zero, then the model collapses into the univariate Diamond model. This can be shown by plugging $b_{t+1} = b_t = 0$ into equations (A.2) and (A.3). The dynamic system is then characterized by the single equation:

$$k_{t+1} = \left[(1+n)(1+g) + \frac{(1+n)\hat{\tau}(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1} \left[\frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)k_t^{\alpha}\right] \tag{A.12}$$

Equation (A.12) can also be derived from the household problem and the market clearing conditions, noting that if the government does not issue debt, that the capital market clearing condition (2.6) simplifies to

$$(1+n)(1+g)k_{t+1} = a_t^*/A_t$$

The steady state of the univariate model is given by equation (A.4), just as in the previous section

$$k = \left(\frac{\alpha\beta(1-\hat{\tau})((1-\alpha)}{(1+n)(1+g)(\alpha(1+\beta)+\hat{\tau}(1-\alpha))}\right)^{\frac{1}{1-\alpha}}$$

As is standard in the Diamond model, a benevolent planner would only choose a social security system (that is, choose a value of $\hat{\tau} > 0$) if the economy is dynamically inefficient, and social security can be used to crowd out private savings and move the economy closer to the golden rule level of capital. It can be shown that $k = k_{gr}$ in the univariate model with social security if $\hat{\tau} = \hat{\tau}_c$ from equation (A.9). This is the optimal tax level for the univariate model.

In the univariate case of the model (when the government runs a balanced budget and never borrows), the unique steady state is always stable. This can be shown by noting that the univariate model is stable if the derivative of the transition equation (A.12) evaluated at the steady state is less than one in absolute value. This will be true as long as the production parameter α is less than one. The derivate of the univariate transition equation is given by:

$$\frac{dk_{t+1}}{dk_t} = \alpha \left[(1+n)(1+g) + \frac{(1+n)\hat{\tau}(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1} \left[\frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)k_t^{\alpha-1}\right] \tag{A.13}$$

Evaluating equation (A.13) at the steady state given by equation (A.4) and simplifying:

$$\frac{dk_{t+1}}{dk_t}|_{k_t=k_{ss}} = \alpha \left[(1+n)(1+g) + \frac{(1+n)\hat{\tau}(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1} \left[\frac{\beta}{1+\beta}(1-\hat{\tau})(1-\alpha)\left[\left(\frac{\alpha\beta(1-\hat{\tau})((1-\alpha)}{(1+n)(1+g)(\alpha(1+\beta)+\hat{\tau}(1-\alpha))}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha-1}\right]$$

$$= \alpha \left[\frac{(1+n)(1+g)(\alpha(1+\beta)+\hat{\tau}(1-\alpha))}{(1+\beta)\alpha}\right]^{-1} \left[\frac{\beta(1-\hat{\tau})(1-\alpha)}{(1+\beta)}\left(\frac{\alpha\beta(1-\hat{\tau})(1-\alpha)}{(1+n)(1+g)(\alpha(1+\beta)+\hat{\tau}(1-\alpha))}\right)^{\frac{\alpha-1}{1-\alpha}}\right]$$

letting $\epsilon = (1+n)(1+g)(\alpha(1+\beta)+\hat{\tau}(1-\alpha))$

$$\frac{dk_{t+1}}{dk_t}|_{k_t=k_{ss}} = \alpha \left[\frac{\epsilon}{(1+\beta)\alpha}\right]^{-1} \left[\frac{\beta(1-\hat{\tau})(1-\alpha)}{(1+\beta)}\left(\frac{\alpha\beta(1-\hat{\tau})(1-\alpha)}{\epsilon}\right)^{-1}\right]$$

$$= \alpha \left(\frac{(1+\beta)\alpha}{\epsilon}\right)\left(\frac{\beta(1-\hat{\tau})(1-\alpha)}{(1+\beta)}\right)\left(\frac{\epsilon}{\alpha\beta(1-\hat{\tau})(1-\alpha)}\right)$$

$$= \alpha \left(\frac{(1+\beta)\alpha}{\epsilon}\right)\left(\frac{\epsilon}{(1+\beta)\alpha}\right)$$

$$= \alpha$$

The stability results are not unique to this model. See Azariadis Chapter 7.4 for a discussion of stability in scalar Diamond models.

Endogenous Government Surpluses with Dynamic Inefficiency

In the baseline calibration of the two-period model, government runs an endogenous deficit in each period. It is also possible for the government to run an endogenous surplus.[2] Recall from equation (2.5) that the time t deficit is defined as

$$d_t = \left(\frac{\phi}{1+n} - \tau\right) w_t$$

If the tax rate is sufficiently high (if $\tau > \phi/(1+n)$), the government will run a surplus each period. The government can be a net lender in this scenario, if the initial stock of debt is low enough. Conversely, if the initial stock of debt is high enough, the government will be a borrower at steady state, even if it runs a surplus each period. These two possible steady states are depicted graphically in figures A.44. Note that if the model is calibrated to be dynamically inefficient, then there are always two steady states when the government runs a surplus. This is because the feasible parameter combinations that lead to the government running a surplus at steady state are such that the bifurcation variable (for example, τ) is always on the side of the threshold where two steady states exist.

Finally, it is possible for the government to run a surplus that is too large and approaches infinity. For example, if the government *taxes* the elderly (that is, $\phi < 0$), and the young ($\tau > 0$), then the government accumulates large asset holdings, and the capital stock increases. If the government taxes both generations enough, the economy fails to converge and the governments assets approach infinity (bonds approach negative infinity). Recall in this model the government doesn't

[2]The United States Social Security system ran a surplus most years since 1983.

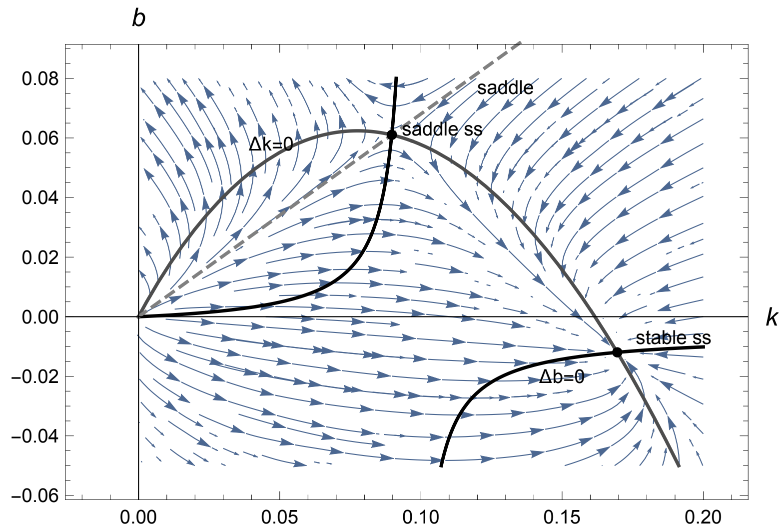


FIGURE A.44. If the government runs a surplus each period, it is possible that there are two non-trivial steady states: a saddle with positive bonds (government borrows) and a stable sink with negative bonds (the government lends).

spend on consumption or goods; the only function of the government is to run a social security system. So, if the government taxes both generations, it adds a pile of tax revenue to its balance sheet each period. The government's assets grow by receiving interest each period and by adding the additional revenue (surplus) each period. This leads government assets to approach infinity, which prevents the economy from reaching the steady state. In practice, this only occurs if both generations are taxed a high rate. In the analysis that follows, I rule out government policies that lead to explosively large surpluses.

Special Case with Dynamic Efficiency

Leeper Tax

The baseline 2-period model is calibrated to be dynamically inefficient. A special case that can give rise to a dynamically efficient level of steady state capital is a Leeper tax. A Leeper tax responds automatically to increases in government debt. Although actual social security taxes do not change based on government debt, a Leeper tax can be thought of as a way to capture legislative unease with increasing debt. I consider a Leeper tax of the following form:

$$\tau_t = \tau_0 + \tau_1 b_t \tag{A.14}$$

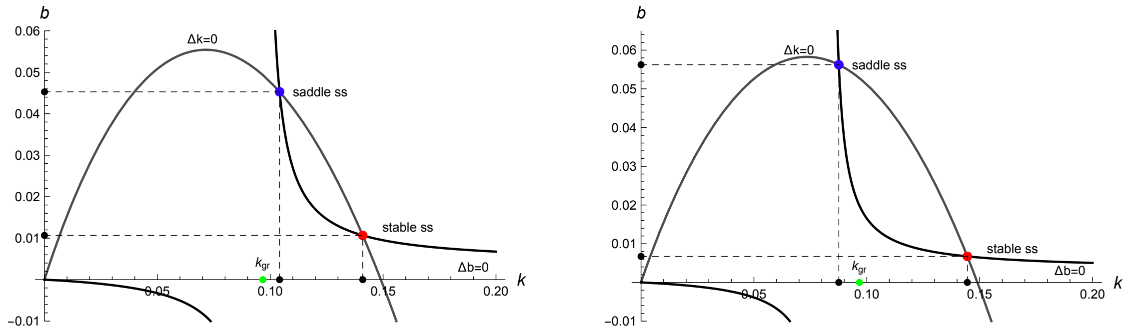
Where $\tau_0 \in [0,1]$ is the baseline tax rate when government debt is zero, and $\tau_1 \in [0,1]$ is the incremental tax change if the government borrows. When $\tau_1 = 0$ the Leeper tax corresponds to the simple, exogenous tax rate of the baseline model. When $\tau_1 > 0$, taxes increase when government debt increases. This slows the accumulation of debt and increases the basin of attraction of the stable (high capital) steady state. The responsive tax rate also allows for a dynamically efficient saddle steady state. The phase diagram for the model with a Leeper tax is depicted in Figure A.45.

Leeper taxes are generally modeled as responding to the previous period's debt $\tau_t = f(b_{t-1})$ (see Davig and Leeper 2011 as an example). However, in this simple model, government debt (b_t) and capital (k_t) are both predetermined. Thus, a tax rate in time t that responds to bond in time t is reasonable, since agents with perfect foresight know b_t at the beginning of time t. Following the process outlined in sections II, II and II, and substituting in equation (A.14) yields the following (modified) transition equations:

$$b_{t+1} = \frac{1}{(1+n)(1+g)}\left[\alpha k_t^{\alpha-1} b_t + \left(\frac{\phi}{1+n} - \tau_0 - \tau_1 b_t\right)(1-\alpha)k_t^{\alpha}\right]$$

$$k_{t+1} = \left[(1+n)(1+g) + \frac{\phi(1-\alpha)(1+g)}{(1+\beta)\alpha}\right]^{-1}$$

$$\left[\frac{\beta}{1+\beta}(1-\tau_0-\tau_1 b_t)(1-\alpha)k_t^{\alpha} - \alpha k_t^{\alpha-1} b_t - \left(\frac{\phi}{1+n} - \tau_0 - \tau_1 b_t\right)(1-\alpha)k_t^{\alpha}\right]$$



Leeper tax with $\tau_1 = 0.1$ Leeper tax with $\tau_1 = 0.336$

FIGURE A.45. The Leeper tax becomes more responsive to government debt as the τ_1 parameter increases. When τ_1 is larger, the basin of attraction for the stable steady state is larger, and it is possible that the saddle steady state is dynamically efficient, as depicted in panel (b).

The dynamics implied by these modified transitions equations are depicted graphically in Figure A.45. Note finally that under a Leeper tax, the parameters τ_0 and τ_1 can both be bifurcation variables. Holding the other parameters of the model fixed, there are tax response rates τ_1 or baseline tax rate τ_0 such that two steady states exist, one (non-unique) steady state exists, or no steady states exist.

Replacement Rate Uncertainty: Time Varying Probability of Reform

The policy experiments of section II highlight the importance of agents' expectations about future benefit rates. If agents believe the probability of reform (benefit cut) is high enough, that belief can keep the economy on a stable trajectory. This captures the intuitive tension between unstable policy and agent expectations. A simple extension that ties agents beliefs with the current state of the economy is to assume that the probability of reform is an increasing function of the government's outstanding debt. If government debt is very low, agents are less worried about debt accumulation and do not place a high likelihood on their social

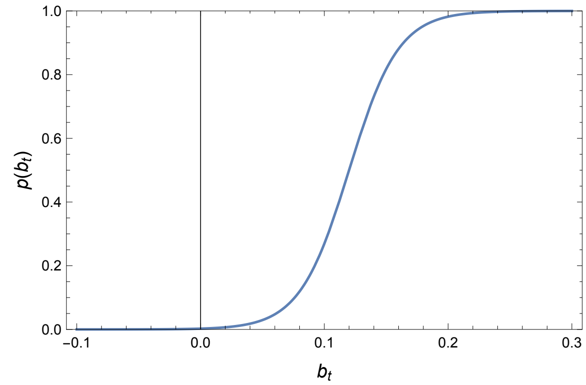


FIGURE A.46. Graph of $p(b_t)$ equation (A.15). The perceived probability of reform is increasing in debt. As the stock of government debt grows, agents believe reform (benefit cuts) are more likely.

security benefits being cut. If government debt is high, agents are more concerned about benefit reductions.

Suppose agents believe the probability of benefit cuts is given by the following logistic equation:

$$p_t = \frac{1}{1 + \exp(-\eta(b_t - \bar{b}))} = p(b_t) \tag{A.15}$$

where $\eta > 0$, and $\bar{b}$ is a target debt level. (Note, if $\bar{b} = 0$, then agents believe there is a 0.5 probability of reform when debt is zero, $p(0) = 0.5$. If $\bar{b} > 0$, then $p(0) < 0.5$. The parameter $\bar{b}$ can be chosen to ensure that the probability of reform when debt is zero is very small.)

Preliminary results using point expectations suggest the effects of agent's believing there is a time-varying probability of reform are qualitatively similar to the simple case of a fixed probability. In both cases, if agents believe reform is likely and large (enough), then their beliefs can keep the economy on a stable trajectory. This time-varying probability $p(b_t)$ could be incorporated fully in a computational model in my future work.

Derivation of Social Welfare Function

The social welfare function is the (discounted) sum of consumption for all generations into the future until time $\overline{T}$ (which is the planning horizon).

$$\begin{aligned}
W = &(\text{young}_0 * c_{1,0} + \text{old}_0 * c_{2,0}) \\
&+ \rho(\text{young}_1 * c_{1,1} + \text{old}_1 * c_{2,1}) \\
&+ \rho^2(\text{young}_2 * c_{1,2} + \text{old}_2 * c_{2,2}) \\
&+ \cdots \\
&+ \rho^{\overline{T}}(\text{young}_{\overline{T}} * c_{1,\overline{T}} + \text{old}_{\overline{T}} * c_{2,\overline{T}})
\end{aligned}$$

which can be rewritten noting that the young in period i are equal to $(1+n)$ times the old:

$$\begin{aligned}
W = &((1+n)\text{old}_0 * c_{1,0} + \text{old}_0 * c_{2,0}) \\
&+ \rho((1+n)\text{old}_1 * c_{1,1} + \text{old}_1 * c_{2,1}) \\
&+ \cdots \\
&+ \rho^{\overline{T}}((1+n)\text{old}_{\overline{T}} * c_{1,\overline{T}} + \text{old}_{\overline{T}} * c_{2,\overline{T}})
\end{aligned}$$

The constant population growth rate implies: $\text{old}_{t+1} = (1+n)\text{old}_t$

$$\begin{aligned}
W = &((1+n)\text{old}_0 * c_{1,0} + \text{old}_0 * c_{2,0}) \\
&+ \rho((1+n)^2\text{old}_0 * c_{1,1} + (1+n)\text{old}_0 * c_{2,1}) \\
&+ \cdots \\
&+ \rho^{\overline{T}}((1+n)^{\overline{T}+1}\text{old}_0 * c_{1,\overline{T}} + (1+n)^{\overline{T}}\text{old}_0 * c_{2,\overline{T}})
\end{aligned}$$

Normalizing the initial number of old to one, $\text{old}_0 = 1$, welfare can be written as:

$$\sum_{t=0}^{\overline{T}} \rho^t (1+n)^t \left[(1+n)c_{1,t} + c_{2,t}\right]$$

Finally, noting that the consumption values depend on the policy parameters, we arrive at the specification given in equation (2.27).

APPENDIX B

THREE PERIOD MODEL APPENDIX

Solution Method

In the prefect foresight case, the model can be solved by first finding a steady state, and then using a Gauss-Sidel algorithm to solve for the equilibrium path of the variables. The steady state can be found numerically by searching for the values $\{k, b, a^1, a^2\}$ that solve the steady state equations (3.24), (3.25), (3.26), (3.27). The steady state is then used as an endpoint to simulate the transition path (or equilibrium dynamics) of the economy using equations (3.6), (3.7), (3.17), and (3.18).

Time Path Iteration

The model can be solved computationally using a process called Time Path Iteration (TPI). This approach was pioneered by Auerbach and Kotlikoff in their 1987 book *Dynamic Fiscal Policy*, chapter 4. R. W. Evans and Phillips (2014) give detailed instructions on using TPI to solve OLG models.

The key assumption of TPI is that the economy will reach the steady state within a finite number of periods $T < \infty$, as long as the initial conditions lie within the basin of attraction of the steady state. Using the steady state, the TPI method can be outlined as follows:

– Solve for the (non-stochastic) steady state given by equations (3.24), (3.25), (3.26), (3.27). Assume that the economy reaches the steady state in period T.

– Assume a path for the state variables $\{k_t\}_{t=0}^T$ and $\{b_t\}_{t=0}^T$. The path for the state variables must include the initial conditions k_0 and b_0 and the steady state condition k_T and b_T.

– Solve for the path of choice variables $\{a_t^1\}_{t=0}^T$ and $\{a_t^2\}_{t=0}^T$ given the path of k and b assumed in the previous step.

– Use the values of $\{a_t^1\}_0^T$ and $\{a_t^2\}_0^T$ from the previous step and the initial conditions $\{k_0, b_0\}$ to find the implied values of $\{k_t\}_1^T$ and $\{b_t\}_1^T$.

– If $\{k_t\}_1^T$ and $\{b_t\}_1^T$ from the previous step differs from the initial guesses (by more than a given small threshold), update the guess (to a convex combination of the implied value of capital and bonds, and the values from the previous iteration) and repeat the cycle. If the distance between the initial guess and the implied paths is arbitrarily close to zero, stop.

Examples of Steady States in the Three-Period Model

I begin with an economy that is dynamically inefficient in the absence of social security (Figure B.1). That is, the steady state capital stock is above the golden rule level of capital. Introducing a balanced budget social security system crowds out capital (since private savings is depressed), and results in a dynamically efficient stable steady state (Figure B.2). The steady states and the eigenvalues of the linearized system evaluated at the steady state are presented in a summary Table B.1.

Figure	τ^0	ϕ	deficit	Stable Steady State k	Stable b	Stable a^1	Stable a^2	Unstable Steady State k	Unstable b	Unstable a^1	Unstable a^2
B.1	0.00	0.00	0.00	0.13	0.00	0.09	0.23	0.13	0.01	0.09	0.23
				3×10^8	*0.98*	*0.25*	*0.24*	*3×10^8*	*1.02*	*0.25*	*0.24*
B.2	0.0193	0.05	0.00	0.13	-0.01	0.08	0.22	0.12	0.00	0.08	0.22
				65.11	*0.98*	*0.25*	*0.24*	*66.76*	*1.03*	*0.25*	*0.24*
11	0.00	0.01	0.0024	0.13	0.00	0.07	0.26	0.11	0.04	0.10	0.27
				172.40	*0.79*	*0.20*	*0.15*	*219.68*	*1.27*	*0.25*	*0.15*
	0.00	0.025	0.0059	0.12	0.01	0.05	0.28	0.09	0.08	0.11	0.30
				76.64	*0.92*	*0.21*	*0.15*	*83.54*	*1.09*	*0.23*	*0.15*
12	0.077	0.20	-0.0003	0.10	0.002	0.078	0.18	0.09	0.02	0.087	0.18
	0.8			*20.63*	*0.89*	*0.27*	*0.25*	*22.25*	*1.12*	*0.28*	*0.23*

TABLE B.1. Steady States for figures in section III and B. Unless otherwise specified, the Leeper tax was set to zero for these examples (τ^1=0). Figures B.1 and B.2 are based on the same parameter values. The golden rule level of capital in for these first simulations is k=0.126. The eigenvalue from matrix M_1 evaluated at a particular steady state are presented below each steady state. If three eigenvalues are less than one in absolute value, the steady state is saddle stable. If more than one eigenvalue lies outside the unit circle, the steady state is explosive. For all of the examples considered, the higher capital steady state is saddle stable and the lower capital steady state is explosive. The bottom section of the table shows the steady states and eigenvalues for Figure 11. The underlying parameters here are different, such that the golden rule level of capital is k=0.0881. Note that as the deficit increases, the steady states move closer together. As this occurs, the second eigenvalue for each steady state gets closer to one. By definition, an eigenvalue of the system evaluated at a steady state will *equal* one at the critical value where the saddle-node bifurcation switches between two steady states and one steady state. The parameters for Figure 12 are τ^0=0.077 and τ^1=0.8.

FIGURE B.1. No social security

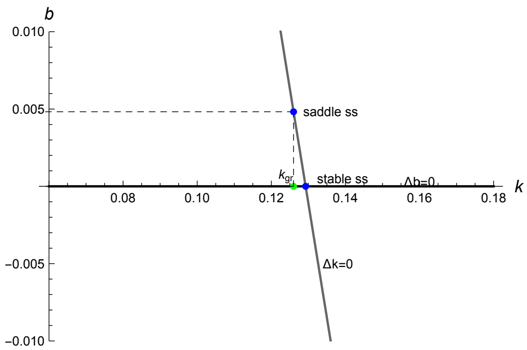
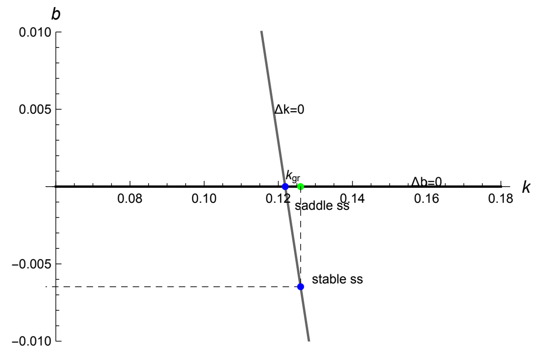


FIGURE B.2. Social security, $\phi = 0.05$



Figures B.1 - B.2: Steady state contour graphs of equations (3.32) and (3.33) in (k, b) space (capital is plotted on the horizontal axis, bonds on the vertical). This shows dynamically inefficient economy with no social security. As a balanced budget social security is introduced, it crowds out capital. The size of the social security system is indicated by the replacement rate ϕ. When the social security system is large enough, the steady state level of capital falls below the golden rule. The higher capital steady state is saddle stable in each case. The lower capital steady state is explosive.

Additional Types of Uncertainty

Simple three-period timing uncertainty

Building off the simple case of section III, consider policy that will be reformed (to know parameters) in period S with probability p, in period $S + 1$ with probability q. If policy change is not implemented in S or $S + 1$, it is implemented with certainty in period $S+2$. Assume that $p+q < 0$. Policy parameters are certain for time periods $t < S$ and $t > S + 1$; policy is now uncertain in periods $t = S$ and $t = S + 1$.

Under this simple type of policy uncertainty, there are three possible paths for the economy: either reform occurs in period S, in period $S + 1$, or in period $S + 2$. For notational ease, I will distinguish the three paths using tildes, hats, and bars: $\tilde{x}$ for the path of the economy when reform occurs in period S, and $\hat{x}$ for the path of the economy when reform happens in $S+1$, and $\bar{x}$ for the path of the economy when reform happens in $S+2$.

This uncertainty impacts up to three generations: the generation who is young at time $S-2$ (they form expectations for ϕ_S for their retirement), and the generation who is young at time $S-1$ (expectations for τ_S in middle age and ϕ_{S+1} in retirement), and then generation who is young at time S, if policy is not changed in S. This generation potentially faces uncertainty about the taxes ϕ_{S+1} they will face in middle age. They do not face uncertainty regarding retirement benefits, since policy reform is enacted with certainty by period $S+2$.

The generation who is young at time $S-2$ chooses a^1_{S-2} using the non-stochastic Euler equation (B.1) and a^2_{S-1} using a stochastic Euler equation. There will be a distinct stochastic Euler equation for each of the three paths; only the tilde path is represented below as (B.2).

$$\begin{aligned}&((1-\tau_{S-2})w(k_{S-2})-a^1_{S-2})^{-\sigma}\\&=\beta(R(k_{S-1})(R(k_{S-1})a^1_{S-2}+(1-\tau_{S-1})w(k_{S-1})-a^2_{S-1})^{-\sigma})\end{aligned}\tag{B.1}$$

For path $\tilde{x}$:

$$\begin{aligned}&(R(\tilde{k}_{S-1})\tilde{a}^1_{S-2}+(1-\tilde{\tau}_{S-1})w(\tilde{k}_{S-1})-\tilde{a}^2_{S-1})^{-\sigma}\\&=\beta[p(R(\tilde{k}_S)(R(k_S)\tilde{a}^2_{S-1}+\tilde{\phi}_{new}w(\tilde{k}_S))^{-\sigma})\\&+q(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^2_{S-1}+\hat{\phi}_{old}w(\hat{k}_S))^{-\sigma})\\&+(1-p-q)(R(\bar{k}_S)(R(\bar{k}_S)\bar{a}^2_{S-1}+\bar{\phi}_{old}w(\bar{k}_S))^{-\sigma})]\end{aligned}\tag{B.2}$$

The generation who is young at time $S-1$ chooses a^1_{S-1} and a^2_S using stochastic Euler equations. There will be distinct stochastic Euler equations for each of the three paths; only the tilde path is represented below as (B.3) and (B.4).

$$
\begin{aligned}
&((1-\tilde{\tau}_{S-1})w(\tilde{k}_{S-1})-\tilde{a}^1_{S-1})^{-\sigma}\\
&=\beta[p(R(k_{\tilde{S}})(R(\tilde{k}_S)\tilde{a}^1_{S-1}+(1-\tilde{\tau}_S)w(\tilde{k}_S)-\tilde{a}^2_S)^{-\sigma})\\
&+q(R(\hat{k}_S)(R(\hat{k}_S)\hat{a}^1_{S-1}+(1-\hat{\tau}_S)w(\hat{k}_S)-\hat{a}^2_S)^{-\sigma})\\
&+(1-p-q)(R(\bar{k}_S)(R(\bar{k}_S)\bar{a}^1_{S-1}+(1-\bar{\tau}_S)w(\bar{k}_S)-\bar{a}^2_S)^{-\sigma})]
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
&p(R(\tilde{k}_S)\tilde{a}^1_{S-1}+(1-\tilde{\tau}_S)w(\tilde{k}_S)-\tilde{a}^2_S)^{-\sigma})\\
&+q(R(\hat{k}_S)\hat{a}^1_{S-1}+(1-\hat{\tau}_S)w(\hat{k}_S)-\hat{a}^2_S)^{-\sigma})\\
&+(1-p-q)(R(\bar{k}_S)\bar{a}^1_{S-1}+(1-\bar{\tau}_S)w(\bar{k}_S)-\bar{a}^2_S)^{-\sigma})]\\
&=\beta[p(R(\tilde{k}_{S+1})(R(\tilde{k}_{S+1})\tilde{a}^2_S+\tilde{\phi}_{new}w(\tilde{k}_{S+1}))^{-\sigma})\\
&+q(R(\hat{k}_{S+1})(R(\hat{k}_{S+1})\hat{a}^2_S+\hat{\phi}_{new}w(\hat{k}_{S+1}))^{-\sigma})\\
&+(1-p-q)(R(\bar{k}_{S+1})(R(\bar{k}_{S+1})\bar{a}^2_S+\bar{\phi}_{old}w(\bar{k}_{S+1}))^{-\sigma})]
\end{aligned}
\tag{B.4}
$$

The generation who is young at time S chooses a^1_S and a^2_{S+1} using non-stochastic Euler equations along the tilde path (reform in period S). Along the other two paths, a^1_S is chosen using a stochastic Euler equation.

Path $\tilde{x}$, non-stochastic Euler equations (3.6) and (3.7).

Path $\hat{x}$ (reform in $S+1$), stochastic Euler equation:

$$
\begin{aligned}
&((1-\hat{\tau}_{old,S})w(\hat{k}_S)-\hat{a}^1_S)^{-\sigma}\\
&=\beta[q(R(\hat{k}_{S+1})(R(\hat{k}_{S+1})\hat{a}^1_S+(1-\hat{\tau}_{new,S+1})w(\hat{k}_{S+1})-\hat{a}^2_{S+1})^{-\sigma})\\
&+(1-q)(R(\bar{k}_{S+1})(R(\bar{k}_{S+1})\bar{a}^1_S+(1-\bar{\tau}_{old,S+1})w(\bar{k}_{S+1})-\bar{a}^2_{S+1})^{-\sigma})]
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
&q((R(\hat{k}_{S+1})\hat{a}^1_S+(1-\hat{\tau}_{new,S+1})w(\hat{k}_{S+1})-\hat{a}^2_{S+1})^{-\sigma})\\
&+(1-q)((R(\bar{k}_{S+1})\bar{a}^1_S+(1-\bar{\tau}_{old,S+1})w(\bar{k}_{S+1})-\bar{a}^2_{S+1})^{-\sigma})\\
&=\beta R(\hat{k}_{S+2})(R(\hat{k}_{S+2})\hat{a}^2_{S+1}+\hat{\phi}_{new}w(\hat{k}_{S+2}))^{-\sigma}
\end{aligned}
\tag{B.6}
$$

I illustrate the impact of three-period timing uncertainty by using the same example as Figure 16 extended to three-periods. The economy begins in a steady state with social security, and taxes are increased in either period $t = 3$ (with probability $p = 0.5$), in period $t = 4$ (with probability $q = 0.25$), or taxes are increased in period $t = 5$. This example is displayed below in Figure B.3. As was the case with two-period timing uncertainty, young agents who expect that they may have to pay higher taxes next period increase their savings relative to the previous generation. Agents facing a possible tax hike also save more than agents who know taxes won't be increased in their life-cycle. Agents facing uncertainty do not increase their savings as much as agents in the counterfactual world who know taxes will be raised next period. This is evident by young saving in period $t = 2$, a_2^1. Agents who know taxes will be raised in period $t = 3$ save the most (depicted in the graph as the open triangle). Agents who know taxes won't be raised until period $t = 4$ or $t = 5$ save the least (in graph, open square and open circle). Agents who face uncertainty save an amount in between. Similarly, the savings of the young in period $t = 3$, a_3^1 is highest for the agents who face a certain tax increase in $t = 4$; savings is lowest for agents who know the tax increase isn't coming until $t = 5$; agents who face uncertainty save in between the two certain amounts. The initial values, policy parameters, and final steady state for this experiment are listed in Table B.2.

I also present the time paths for an economy in a steady state that experiences a large social security benefit reduction in either period $t = 3$, $t = 4$, or $t = 5$. This policy experiment is presented in Figure B.4. The probability of policy change in period $t = 3$ is $p = 0.5$, probability of change in period $t = 4$ is $q = 0.25$, and if benefits are not decreased in period 3 or 4, benefits are cut in period 5 with

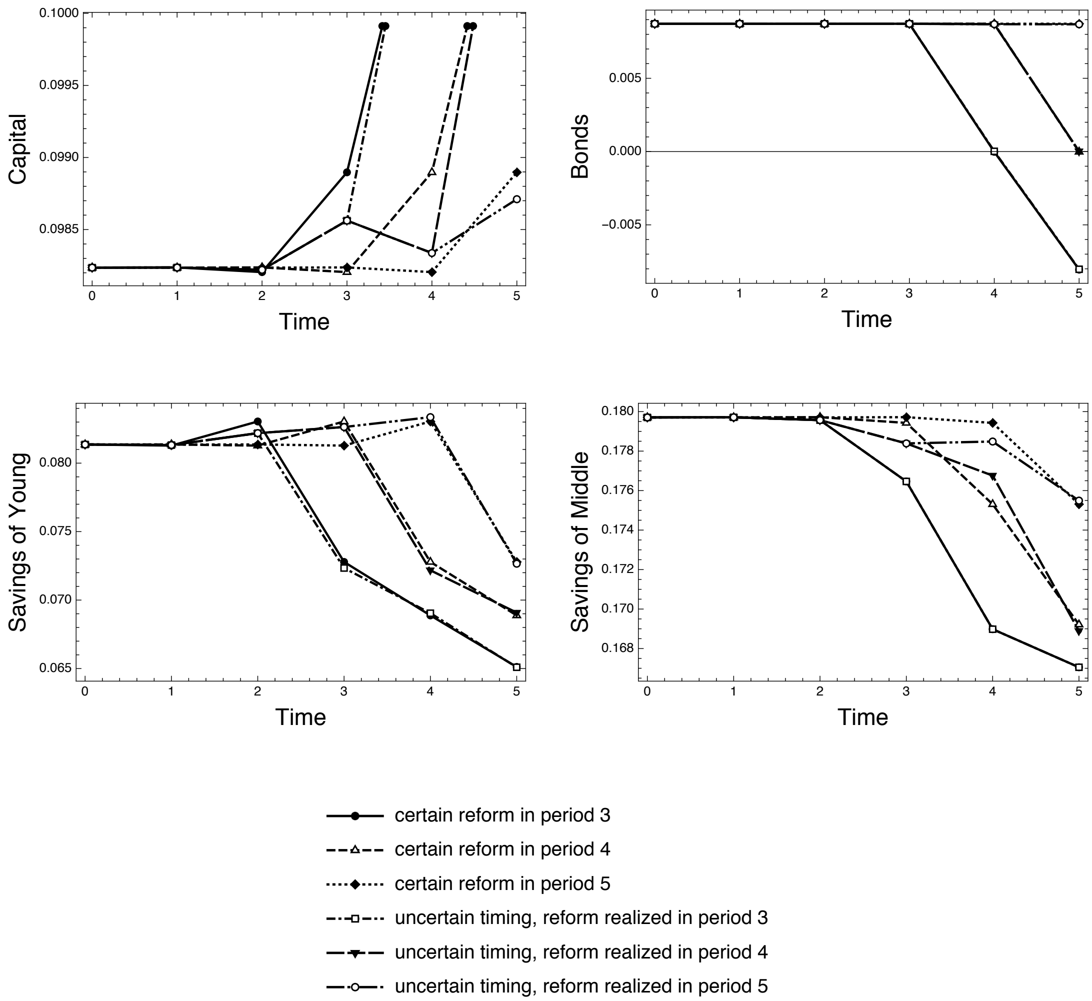


FIGURE B.3. Time paths for an economy is in a steady state and experiences a large tax increase in period 3, 4, or 5. Time paths are plotted for reform that happens with certainty in any of the three period. Time paths are plotted in addition for an economy that faces probability p=0.5 of reform in period 3, probability q=0.25 that reform takes place in period 4. If reform doesn't occur in period 3 or 4, it occurs with certainty in period 5. There are three possible paths for the economy under this type of stochastic policy, either the reform takes place in period 3 (these paths are labeled uncertain reform in 3), period 4 (these paths are labeled uncertain reform in 4), or in period 5 (labeled uncertain reform in 5).

Fig	Initial policy			New policy			Initial values				Final steady state(s)			
	τ^0	τ^1	ϕ	τ^0	τ^1	ϕ	k	b	a^1	a^2	k	b	a^1	a^2
B.3	0.076	0.8	0.2	0.1	0.8	0.2	0.098	0.009	0.081	0.18	0.121	-0.031	0.058	0.163
B.4	0.076	0.8	0.2	0.077	0.8	0.18	0.098	0.009	0.081	0.18	0.113	-0.012	0.071	0.176

TABLE B.2. Policy Experiments with three-period timing uncertainty. Experiment B.3: Economy in steady state with social security, taxes raised in either period $t = 3$, $t = 4$, or $t = 5$. Experiment B.4: Economy in steady state with social security, benefits cut in either period $t = 3$, $t = 4$, or $t = 5$.

certainty. The results of this experiment are qualitatively similar to the two-period example presented in Figure 19. The initial values, policy parameters, and final steady state for this experiment are listed in Table B.2.

Two Special Cases with Balanced Social Security Budget

In simplified version of the model without a Leeper tax ($\tau^1 = 0$, and $\tau^1 = \tau$), the social security budget is balanced in every time period if the government does not run a primary deficit or surplus ($\tau_t = \phi_t/(1+n)(2+n)$). The government's initial stock of debt b_0 is continually rolled over according to (3.17), which simplifies to $b_{t+1}(1+n) = R_t b_t$. In this framework, two steady states are possible. In the first, the initial government stock is zero and the government never borrows. That is, $b_t = 0$ for all t. When that is true, the model collapses to a three variable, three equation system (which will be discussed below). In the second case, steady state capital occurs at the golden rule level, and steady state bonds are at a corresponding level implied by equation (3.22).

Balanced Budget, No Government Debt

In the simplest case with balanced budgets, the government simply never issues debt. The model can be written as a system of three equations by replacing τ_t with $\phi_t/(1+n)(2+n)$ and noting that $b_t = 0$ for all t. This simplified model

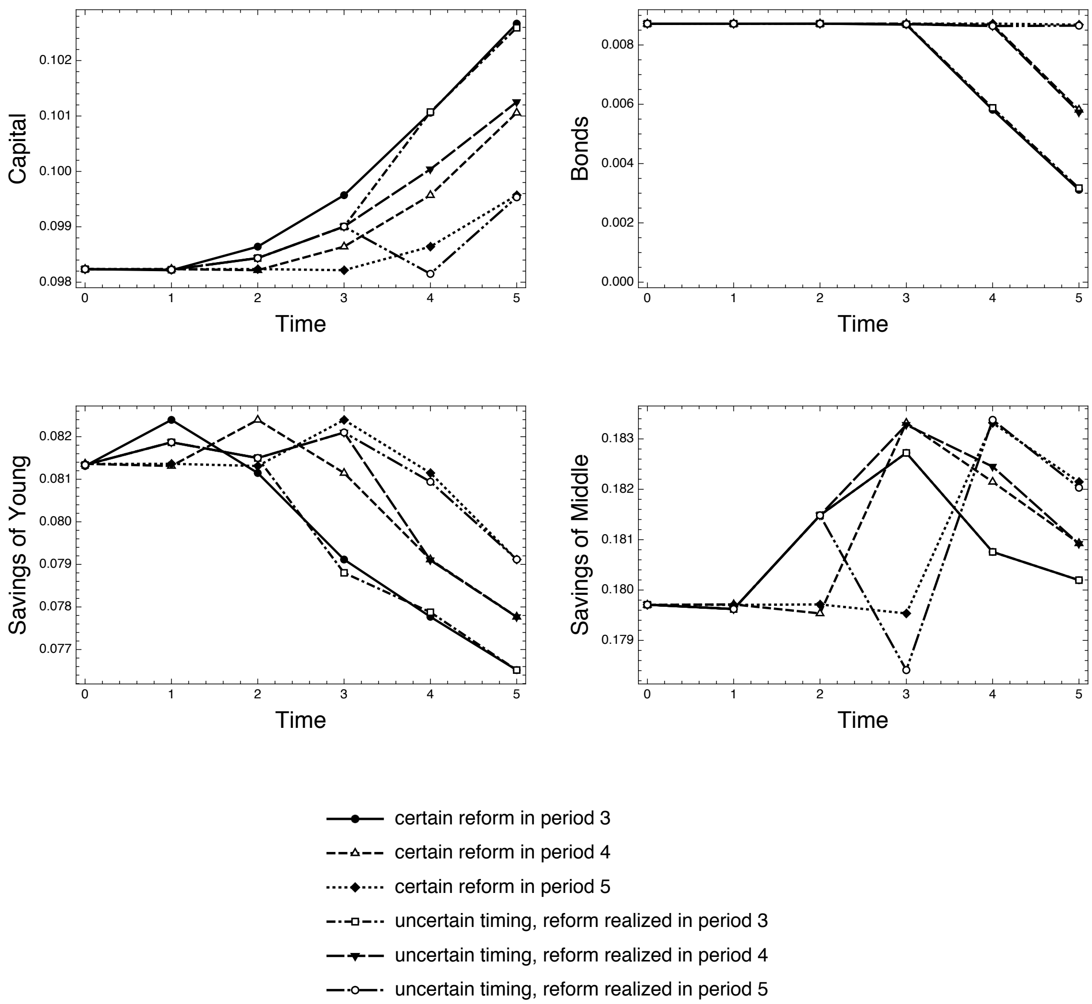


FIGURE B.4. Time paths for an economy is in a steady state and experiences a large benefit reduction in period 3, 4, or 5. Time paths are plotted for reform that happens with certainty in any of the three period. Time paths are plotted in addition for an economy that faces probability p=0.5 of reform in period 3, probability q=0.25 that reform takes place in period 4. If reform doesn't occur in period 3 or 4, it occurs with certainty in period 5. There are three possible paths for the economy under this type of stochastic policy, either the reform takes place in period 3 (these paths are labeled uncertain reform in 3), period 4 (these paths are labeled uncertain reform in 4), or in period 5 (labeled uncertain reform in 5).

has two predetermined variables (capital and savings of the young), and one free variable (savings of the middle aged). The model has a unique, stable steady.

The unique steady state of the model is the collection $\{k, a^1, a^2\}$ that solves:

$$\left(\left(1 - \frac{\phi}{(1+n)(2+n)}\right) w(k) - a^1\right)^{-\sigma} - \beta R(k) \left(R(k)a^1 + \left(1 - \frac{\phi}{(1+n)(2+n)}\right) w(k) - a^2\right)^{-\sigma} = 0 \tag{B.7}$$

$$\left(R(k)a^1 + \left(1 - \frac{\phi}{(1+n)(2+n)}\right) w(k) - a^2\right)^{-\sigma} - \beta R(k)(R(k)a^2 + \phi w(k))^{-\sigma} = 0 \tag{B.8}$$

$$(1+n)k - (1-\delta)k - a^1 \frac{1+n}{2+n} - a^2 \frac{1}{2+n} = 0 \tag{B.9}$$

Equation (B.9) reveals the unique steady state value of capital $k = \frac{2+n}{n+\delta}\left((1+n)a^1 + a^2\right)$. The steady state values of a^1 and a^2 are given jointly by equation (B.7) and (B.8).

Analysis of the linearized model reveals the steady state is stable. Define $\hat{X}_t = (a^1_{t-1}, a^2_t, k_t)'$ and $\hat{X}_{t+1} = (a^1_t, a^2_{t+1,} k_{t+1})'$. Then the balanced budget, no debt model can be written as

$$\hat{G}(\hat{X}_t \hat{X}_{t+1}) = 0 \tag{B.10}$$

where $\hat{G}(\hat{X}_t \hat{X}_{t+1})$ is the following:

$$\left(\left(1 - \frac{\phi_t}{(1+n)(2+n)}\right) w(k_t) - a^1_t\right)^{-\sigma} - \beta R(k_{t+1}) \left(R(k_{t+1})a^1_t + \left(1 - \frac{\phi_{t+1}}{(1+n)(2+n)}\right) w(k_{t+1}) - a^2_{t+1}\right)^{-\sigma} = 0 \tag{B.11}$$

$$\left(R(k_{t+1})a^1_t + \left(1 - \frac{\phi_{t+1}}{(1+n)(2+n)}\right) w(k_{t+1}) - a^2_{t+1}\right)^{-\sigma} - \beta R(k_{t+2})(R(k_{t+2})a^2_{t+1} + \phi_{t+2} w(k_{t+2}))^{-\sigma} = 0 \tag{B.12}$$

$$(k_{t+1})(1+n) - (1-\delta)k_t - a^1_t \frac{1+n}{2+n} - a^2_t \frac{1}{2+n} = 0 \tag{B.13}$$

Total differentiation of $\hat{G}$ yields $\hat{H}_1 d\hat{X}_{t+1} + \hat{H}_2 d\hat{X}_t + \hat{H}_3 dZ_t = 0$, where $\hat{H}$ are matrices of partial derivatives. Multiplying through by $-(\hat{H}_2^{-1})$, the linearized model can be written as:

$$d\hat{X}_{t+1} = \hat{M}_1 d\hat{X}_t + \hat{M}_3 dZ_t$$

where $\hat{M}_1 = -(\hat{H}_2^{-1})\hat{H}_1$. The stability of the system is governed by the eigenvalues of $\hat{M}_1$ evaluated at a steady state $\hat{X}^*$. I found a unique steady state for each parameter combination I tested. Additionally, two of the eigenvalues of $\hat{M}_1$ evaluated at a steady state were always within in the unit circle, and one was outside. Thus, for a given set of parameters, the model has a unique, stable steady state (See Azariadis (1993) page 59).

Balanced Budget, Constant Debt

The second special case that is possible when the government balances the social security budget each period, is a steady state at the golden rule level of capital. Recall the government budget equation:

$$b_{t+1}(1+n) = R(k_t)b_t + \frac{1}{(2+n)(1+n)}\phi_t w(k_t) - \tau_t w(k_t) \tag{3.17}$$

When the government balances the social security budget, this simplifies to:

$$b_{t+1}(1+n) = R(k_t)b_t \tag{B.14}$$

The steady state version of equation (B.14) is only true at the golden rule level of capital, when $R(k) = 1+n$.

$$b(1+n) = R(k)b \tag{B.15}$$

When capital is at the golden rule level and $R(k) = 1+n$, any level of bonds solves equation (B.15). The unique level of bonds associated with the steady state is jointly determined by the remaining steady state equations, reprinted below:

$$((1-\tau)w(k) - a^1)^{-\sigma} = \beta R(k)(R(k)a^1 + (1-\tau)w(k) - a^2)^{-\sigma}$$

$$\left(R(k)a^1 + (1-\tau)w(k) - a^2\right)^{-\sigma} = \beta R(k)\left(R(k)a^2 + \phi w(k)\right)^{-\sigma}$$

$$b(1+n) = R(k)b$$

$$(k+b)(1+n) = (1-\delta)k + a^1\frac{1+n}{2+n} + a^2\frac{1}{2+n}$$

The stability of the golden rule steady state is governed by the eigenvalues of matrix M_1, as described in Section III and equation (3.30).

APPENDIX C

LIFE-CYCLE HORIZON LEARNING APPENDIX

Stability of REE under Learning

Determinacy is often used as an equilibrium selection device in rational expectations models with multiple equilibria. A complementary approach to selecting equilibria is to conduct stability analysis under learning. If agents' (non-rational) expectations and forecasts converge to a rational expectations equilibrium (REE) in a model with learning, the REE is said to be stable under learning (see G. W. Evans and Honkapohja (2001)). W. Branch et al. (2013) show that the unique REE in an infinite-horizon Ramsey model is stable under N-step optimal learning. Similarly, I numerically verify that determinate REE in my model are stable under LCH learning and FHL Learning.

Given constant (potentially incorrect) expectations $p^e = (R^e, w^e, b^e, a^{j,e}_{terminal})'$, the learning dynamics of the FHL model asymptotically converge to $p = (R, w, b, a^j)'$. Similarly, the dynamics of the LCH model converge from beliefs $p^e = (R^e, w^e, b^e)'$ to $p = (R, w, b)'$. This converge from beliefs to actual prices is called a T-map.

$$T : \mathbb{R}^{J-H+3} \rightarrow \mathbb{R}^{J-H+3} \tag{C.1}$$

A fixed point of the T map is E-stable if it locally stable under the ordinary differential equation$\frac{dp}{d\tau} = T(p) - p$. E-stability requires the real parts the eigenvalues of the derivative matrix $dT < 1$. I have numerically verified all determinate steady states in the paper are E-stable under LCH learning and FHL learning (at all horizons).

The stability under learning of the determinate REE in this model can be illustrated graphically. The dynamics of the learning model converge to the REE given arbitrary initial conditions near the steady state. Figure C.5 illustrates this convergence in the LCH and FHL learning models, calibrated with $J = 6$ (six period lives) and parameters calibrated as detailed in section IV. The example begins with capital, bonds, assets, and expectations of all of these things below the steady state values. Agents update their expectations as they receive more information and eventually learn the steady state values, which are indicated with a dashed red line. Agents overshoot the steady state initially. This pattern is observed in G. W. Evans et al. (2009).

W. Branch et al. (2013) find that shorter planning horizons converge to the REE more quickly; and that agents make larger errors (and there are larger aggregate fluctuations) when the planning horizon is longer. I find a similar result; the fluctuations in the economy are greatest for life-cycle horizon learning and decrease as the planning horizon decreases.

Recessions and Life-cycle Learning

The examples in the main text illustrate the welfare cost of life-cycle and finite-horizon learning in the face of social security policy changes. An alternative framework to view the welfare cost of learning is to simulate a recession and compare the lifecycle utility of agents in the RE model with the utility of agents in a learning model. I do this using the same CEV as defined in equation 4.31.

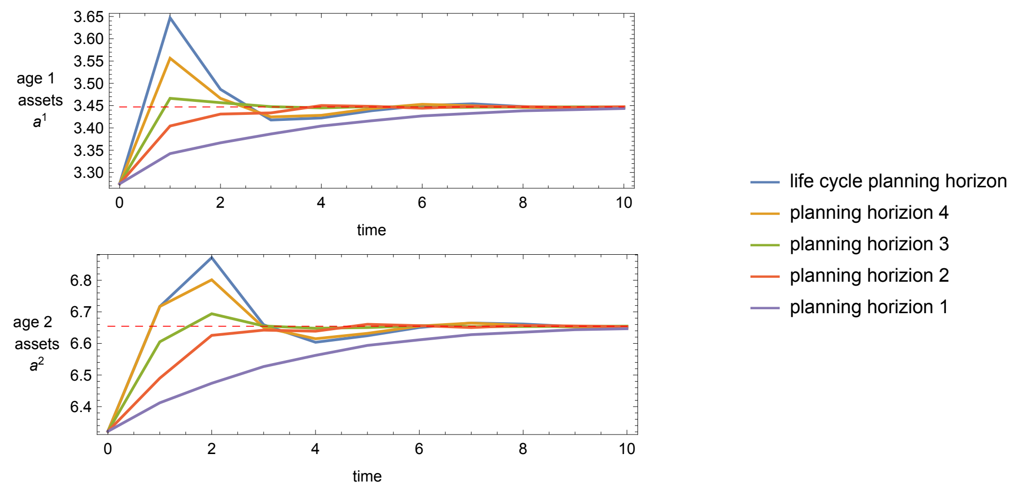


FIGURE C.5. This graphs shows convergence to a steady state for the Finite Horizon Lifecycle Learning model (FHL) and Life-cycle Horizon Learning model (LCH) with different planning horizons. The red dashed line indicates the steady-state. The initial values were chosen to be near the steady-state. The initial expectations were all set equal to the initial values. The top graph shows the path of age 1 assets; the bottom graph shows the path of age 2 assets. The converge of the other asset choices $(a^3, \ldots, a^5)$, capital, and bonds, follow a similar oscillating pattern.

I simulate as recession as a surprise decrease in the TFP factor A. The recession is depicted in Figure C.6 below. I model the recession as a one period reduction in the TFP factor from $A = 10$ to $A = 9.5$. The recession reduces output per worker by 2.9% and aggregate consumption by 7.53%.[1] Agents in the RE model know that the recession is only one period. Agents in the learning models do not know the recession will end; they continue to forecast prices and bonds adaptively. The recession generates cycles in capital and bonds, as agents overshoot the steady state before converging. Note that converge back to the steady state is non-monotonic in the rational and learning models (see Azariadis, Bullard, and Ohanian (2004) for a discussion of cycles in rational, multi-period OLG economies). The oscillatory dynamics are larger and last longer in the learning dynamics. The decrease in output and consumption are also larger in the learning models relative to the RE baseline.[2] The welfare cost of experiencing the recession in a model with learning compared to the same recession in the RE model is depicted in Figure C.7. The welfare cost varies by cohort. The cost is greatest (smallest CEV) for agents born the period after recession in the LCH model. Agents born in period $t = 11$, following a recession in period $t = 10$, are worse off in the learning model by 1.10% of period consumption. The largest welfare cost occurs after the recession due to the depressed capital stock in the learning model relative to the RE baseline.

The welfare cost is greatest to the cohorts born the period after the recession in the LCH model since they experience lower consumption relative to

[1]During the Great Recession (2007-2009) output fell by 7.2% and consumption fell by 5.4%. The average post-war recession saw a decline in output and consumption of 4.4% and 2.1%, respectively. See Christiano (2017).

[2]Eusepi and Preston (2011) show that adaptive learning in an RBC model creates dynamics that more closely match that data than a standard, RE model. They also show that the shocks required to generate a realistic recession are smaller in the model with learning.

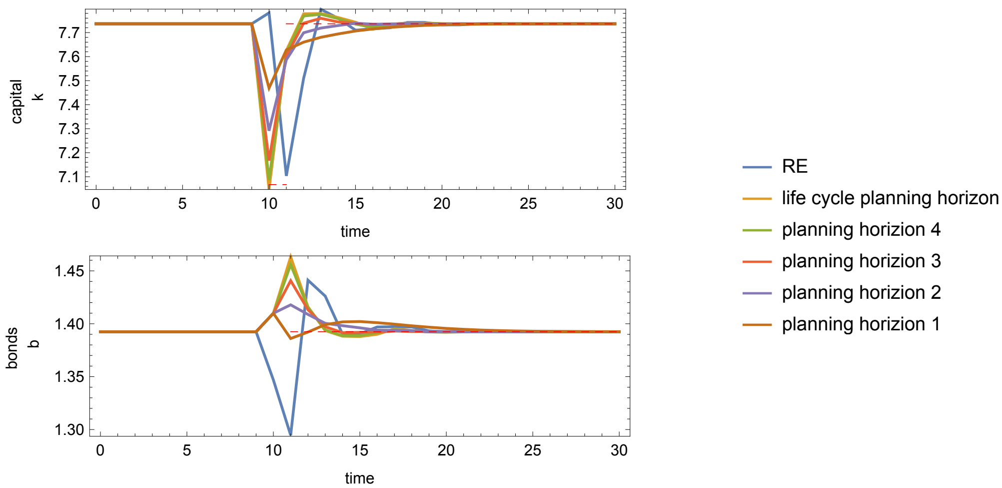


FIGURE C.6. Equilibrium paths for capital and bonds in an economy with a recession in period t=10 that lasts for 1 period. The population growth rate is constant at n=0.01, and social security policy is constant τ^0=0.1516, τ^1=0.045, ϕ=0.4. The recession is a surprise reduction in the TFP factor from A=10 to A=9.5 in period t=10. The TPF factor is depressed for a single period, and the returns to the pre-recession level. In the learning models, the gain parameter γ=0.93.

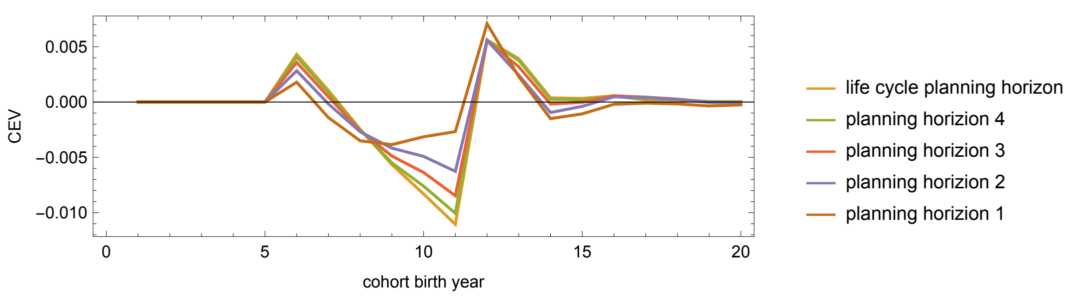


FIGURE C.7. Consumption equivalent variation measure of the welfare cost of a recession. The population growth rate is constant at n=0.01, and social security policy is constant τ^0=0.1516, τ^1=0.045, ϕ=0.4. The recession is a surprise reduction in the TFP factor from A=10 to A=9.5 in period t=10. The TPF factor is depressed for a single period, and the returns to the pre-recession level. In the learning models, the gain parameter γ=0.93.

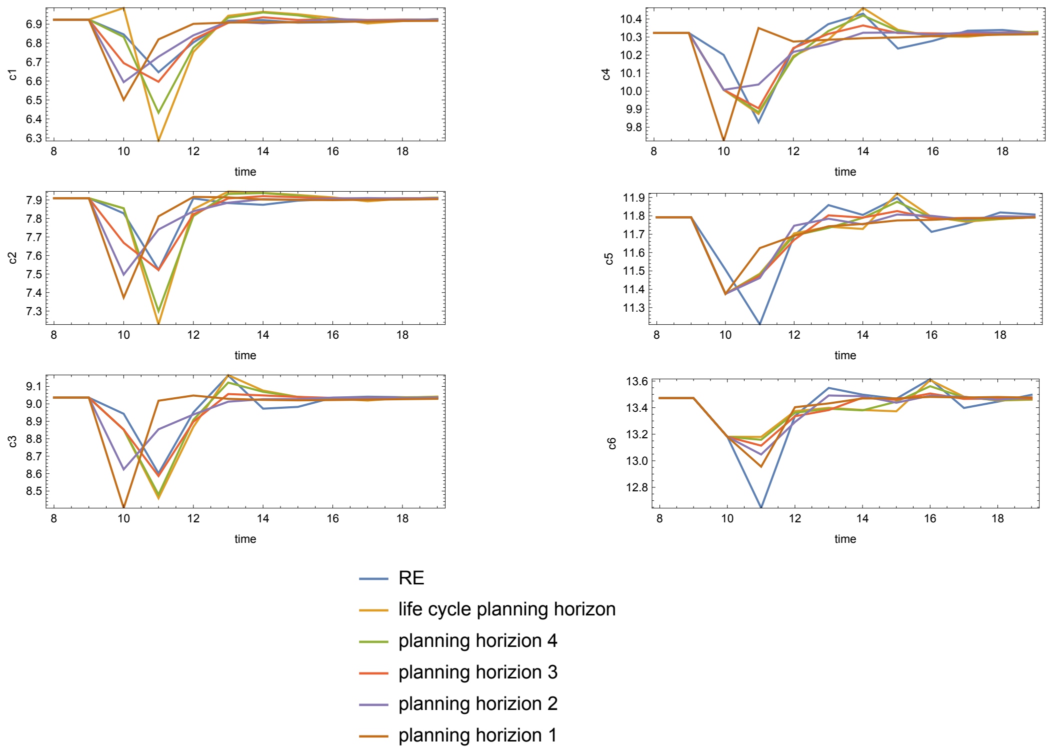


FIGURE C.8. Path of consumption for the rational and learning models for a recession in period t=10. Consumption falls the most during the recession for the 1-period ahead finite horizon learners (dark orange line). Consumption is depressed for many periods in the LCH learning model (yellow line, often very close to the green line). The population growth rate is constant at n=0.01, and social security policy is constant τ^0=0.1516, τ^1=0.045, ϕ=0.4. The recession is a surprise reduction in the TFP factor from A=10 to A=9.5 in period t=10. The TPF factor is depressed for a single period, and the returns to the pre-recession level. In the learning models, the gain parameter γ=0.93.

the other generations. Consumption falls during the recession (period $t = 10$) and during the following for rational agents. The dip in consumption after the recession reflect the decision of agents to save as the economy recovers. Consumption also falls for learning agents during the recession. Since the learning agents adaptively forecast prices, they anticipate depressed wages and interest rates the following period, so they continue to consume less for several periods. This is depicted in Figure C.8.

The welfare cost of the learning depends on both the length and depth of the recession, as well as on the gain parameter γ. If the recession were three periods rather than one, the greatest welfare cost would be 1.98% of period consumption (compared to 1.10% for a one period recession). Similarly, if the recession is deeper and the TFP factor falls from $A = 10$ to $A = 9.1$ instead of $A = 9.5$, the largest welfare cost of a one period recession is 2.26% of period consumption. In all three cases, the largest welfare cost occurs under LCH learning for the cohort of agents born after the recession. The welfare cost also depends on the gain parameter γ. When the gain parameter is larger, the welfare cost is also larger. This is because agents place more weight on recent data when the gain parameter is high, and so they expect prices to be more depressed following a one period recession than agents using a smaller gain. The minimum CEV for LCH agents experiencing a one period recession is 0.7% of period consumption for a gain parameter of 0.01, and is equal to 1.10% of period consumption for a gain of 1.

Note on Golden Rule

The golden rule level of capital maximizes consumption of all generations and occurs when $R = (1 + n)$ in this model.

$$k_{gr} = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

The consumption profile is smooth over the lifecycle if $R = \beta^{-1}$. This is not the case in general. $R = \beta^{-1}$ if

$$k_{smooth} = \left(\frac{\alpha}{\beta^{-1}+\delta-1}\right)^{\frac{1}{1-\alpha}}$$

Thus, the golden rule level of capital corresponds to the level of capital such that $R = \beta^{-1}$ (and the lifecycle consumption profile is constant) when $n = \frac{1-\beta}{\beta}$.

The social security program crowds out capital and can be set at a level to ensure $k = k_{gr}$, or $k = k_{smooth}$; it's less clear why a planner would want the latter.

REFERENCES CITED

Adam, K. (2007). Experimental evidence on the persistence of output and inflation. *The Economic Journal*, *117*(520), 603–636. Retrieved from http://dx.doi.org/10.1111/j.1468-0297.2007.02043.x doi: 10.1111/j.1468-0297.2007.02043.x

Adam, K., Marcet, A., & Nicolini, J. P. (2016, February). Stock market volatility and learning. *Journal of Finance*, *71*(1), 33–82.

Auerbach, A. J. (2014). Fiscal uncertainty and how to deal with it [Hutchins Center on Fiscal and Monetary Policy at Brookings Working Paper]. Retrieved from http://www.brookings.edu/~/media/research/files/papers/2014/12/15-fiscal-uncertainty-auerbach/15_fiscal_uncertainty_auerbach.pdf

Auerbach, A. J., & Hassett, K. (2007). Optimal long-run fiscal policy: Constraints, preferences and the resolution of uncertainty [Journal Article]. *Journal of Economic Dynamics and Control*, *31*(5), 1451-1472. Retrieved from <GotoISI>://WOS:000246538000001 doi: 10.1016/j.jedc.2006.05.003

Auerbach, A. J., & Kotlikoff, L. J. (1987). *Dynamic fiscal policy* [Book]. Cambridge Cambridgeshire ; New York: Cambridge University Press.

Azariadis, C. (1993). *Intertemporal macroeconomics* [Book]. Oxford, UK ; Cambridge, USA: Blackwell.

Azariadis, C., Bullard, J., & Ohanian, L. (2004). Trend-reverting fluctuations in the life-cycle model. *Journal of Economic Theory*, *119*(2), 334 - 356. Retrieved from http://www.sciencedirect.com/science/article/pii/S0022053103000322 doi: https://doi.org/10.1016/S0022-0531(03)00032-2

Bohn, H. (2009). Intergenerational risk sharing and fiscal policy [Journal Article]. *Journal of Monetary Economics*, *56*(6), 805-816. Retrieved from <GotoISI>://WOS:000271335900005 doi: 10.1016/j.jmoneco.2009.06.010

Bouzahzah, M., De la Croix, D., & Docquier, F. (2002). Policy reforms and growth in computable olg economies [Journal Article]. *Journal of Economic Dynamics and Control*, *26*(12), 2093-2113. Retrieved from <GotoISI>://WOS:000176469400005 doi: 10.1016/s0165-1889(01)00023-9

Branch, W., Evans, G. W., & McGough, B. (2013). Finite horizon learning. In T. Sargent & J. Vilmunen (Eds.), *Macroeconomics at the service of public policy* (p. 141-163). Oxford: Oxford University Press.

Branch, W. A. (2004, July). The theory of rationally heterogeneous expectations: Evidence from survey data on inflation expectations*. *Economic Journal*, *114*(497), 592–621.

Branch, W. A., & Mcgough, B. (2011, June). Business cycle amplification with heterogeneous expectations. *Economic Theory*, *47*(2/3), 395–421.

Bullard, J., & Duffy, J. (2001, April). Learning and excess volatility. *Macroeconomic Dynamics*, *5*(2), 272–302.

Bullard, J., & Feigenbaum, J. (2007). A leisurely reading of the life-cycle consumption data. *Journal of Monetary Economics*, *54*(8), 2305 - 2320. Retrieved from http://search.ebscohost.com.libproxy.uoregon.edu/login.aspx?direct=true&db=eoh&AN=0953573&login.asp&site=ehost-live&scope=site

Bullard, J., & Russell, S. (1999). An empirically plausible model of low real interest rates and unbacked government debt [Journal Article]. *Journal of Monetary Economics*, *44*(3), 477-508. Retrieved from <GotoISI>://WOS:000084148300006 doi: 10.1016/s0304-3932(99)00031-8

Bütler, M. (1999). Anticipation effects of looming public-pension reforms. *Carnegie-Rochester Conference Series on Public Policy*, *50*, 119 - 159. Retrieved from http://www.sciencedirect.com/science/article/pii/S016722319900024X doi: http://dx.doi.org/10.1016/S0167-2231(99)00024-X

Caliendo, F., & Aadland, D. (2007). Short-term planning and the life-cycle consumption puzzle. *Journal of Economic Dynamics and Control*, *31*(4), 1392-1415. Retrieved from http://EconPapers.repec.org/RePEc:eee:dyncon:v:31:y:2007:i:4:p:1392-1415

Caliendo, F., Casanova, M., Gorry, A., & Slavov, S. (2016). *The welfare cost of retirement uncertainty* (Working Paper). Retrieved from http://www.fcaliendo.com/application/files/9714/6430/1336/uncertain_retirement_with_disability_figures_included.pdf

Caliendo, F., Gorry, A., & Slavov, S. (2015). *The cost of uncertainty about the timing of social security reform* (NBER Working Papers No. 21585). National Bureau of Economic Research, Inc. Retrieved from http://EconPapers.repec.org/RePEc:nbr:nberwo:21585

Caprioli, F. (2015). Optimal fiscal policy under learning. *Journal of Economic Dynamics and Control*, *58*(Supplement C), 101 - 124. Retrieved from http://www.sciencedirect.com/science/article/pii/S0165188915000871 doi: https://doi.org/10.1016/j.jedc.2015.05.008

CBO, C. B. O. (2017). *The 2017 long-term budget outlook* (Reports No. 52480). Congressional Budget Office. Retrieved from `https://www.cbo.gov/system/files/115th-congress-2017-2018/reports/52480-ltbo.pdf`

Chalk, N. A. (2000). The sustainability of bond-financed deficits: An overlapping generations approach [Journal Article]. *Journal of Monetary Economics*, *45*(2), 293-328. Retrieved from `<GotoISI>://WOS:000086203400003` doi: 10.1016/s0304-3932(99)00052-5

Christiano, L. J. (2017, 02). *The Great Recession: A Macroeconomic Earthquake* (Tech. Rep.). Federal Reserve Bank of Minneapolis.

Davig, T., & Foerster, A. (2014). *Uncertainty and fiscal cliffs* (Research Working Paper No. RWP 14-4). Federal Reserve Bank of Kansas City. Retrieved from `http://EconPapers.repec.org/RePEc:fip:fedkrw:rwp14-04`

Davig, T., & Leeper, E. M. (2011a). Monetary-fiscal policy interactions and fiscal stimulus [Journal Article]. *European Economic Review*, *55*(2), 211-227. Retrieved from `<GotoISI>://WOS:000288309800004` doi: 10.1016/j.euroecorev.2010.04.004

Davig, T., & Leeper, E. M. (2011b). Temporarily unstable government debt and inflation [Journal Article]. *Imf Economic Review*, *59*(2), 233-270. Retrieved from `<GotoISI>://WOS:000292526600005` doi: 10.1057/imfer.2011.7

Davig, T., Leeper, E. M., & Walker, T. B. (2010). "unfunded liabilities" and uncertain fiscal financing [Journal Article]. *Journal of Monetary Economics*, *57*(5), 600-619. Retrieved from `<GotoISI>://WOS:000280969900009` doi: 10.1016/j.jmoneco.2010.05.007

Davig, T., Leeper, E. M., & Walker, T. B. (2011). Inflation and the fiscal limit [Journal Article]. *European Economic Review*, *55*(1), 31-47. Retrieved from `<GotoISI>://WOS:000287069700003` doi: 10.1016/j.euroecorev.2010.11.005

De La Croix, D., & Michael, P. (2002). *A theory of economic growth: Dynamics and policy in overlapping generations* [Book]. The Edinburgh Building, Cambridge CB2 2RU, UK: Cambridge University Press.

Dominitz, J., & Manski, C. F. (2006). Measuring pension-benefit expectations probabilistically. *LABOUR*, *20*(2), 201–236. Retrieved from `http://dx.doi.org/10.1111/j.1467-9914.2006.00343.x` doi: 10.1111/j.1467-9914.2006.00343.x

Eusepi, S., & Preston, B. (2011, October). Expectations, learning, and business cycle fluctuations. *The American Economic Review*, *101*(6), 2844–2872.

Evans, G., & Ramey, G. (2006). Adaptive expectations, underparameterization and the lucas critique. *Journal of Monetary Economics*, *53*(2), 249-264. Retrieved from https://EconPapers.repec.org/RePEc:eee:moneco:v:53:y:2006:i:2:p:249-264

Evans, G. W., & Honkapohja, S. (2001). *Learning and expectations in macroeconomics*. Princeton: Princeton University Press.

Evans, G. W., & Honkapohja, S. (2009). Learning and macroeconomics. *Annual Review of Economics*, *1*(1), 421-449. Retrieved from https://doi.org/10.1146/annurev.economics.050708.142927 doi: 10.1146/annurev.economics.050708.142927

Evans, G. W., Honkapohja, S., & Mitra, K. (2009). Anticipated fiscal policy and adaptive learning. *Journal of Monetary Economics*, *56*(7), 930 - 953. Retrieved from http://www.sciencedirect.com/science/article/pii/S0304393209001251 doi: https://doi.org/10.1016/j.jmoneco.2009.09.007

Evans, R., Kotlikoff, L., & Phillips, K. (2012). Game over: Simulating unsustainable fiscal policy. In *Fiscal policy after the financial crisis* (p. 177-202). National Bureau of Economic Research, Inc. Retrieved from http://EconPapers.repec.org/RePEc:nbr:nberch:12642

Evans, R. W., & Phillips, K. L. (2014). Olg life cycle model transition paths: Alternate model forecast method. *Computational Economics*, *43*(1), 105–131. Retrieved from http://dx.doi.org/10.1007/s10614-012-9359-2 doi: 10.1007/s10614-012-9359-2

Feigenbaum, J. (2008). Can mortality risk explain the consumption hump?. *Journal of Macroeconomics*, *30*(3), 844 - 872. Retrieved from http://search.ebscohost.com.libproxy.uoregon.edu/login.aspx?direct=true&db=eoh&AN=1000346&site=ehost-live&scope=site

Feigenbaum, J., Caliendo, F. N., & Gahramanov, E. (2011). Optimal irrational behavior. *Journal of Economic Behavior and Organization*, *77*(3), 285 - 303. Retrieved from http://www.sciencedirect.com/science/article/pii/S0167268110002192 doi: https://doi.org/10.1016/j.jebo.2010.11.002

Feldstein, M. (1985). The optimal level of social-security benefits [Journal Article]. *Quarterly Journal of Economics*, *100*(2), 303-320. Retrieved from <GotoISI>://WOS:A1985AJZ2700001 doi: 10.2307/1885383

Gali, J. (2014). Monetary policy and rational asset price bubbles [Journal Article]. *American Economic Review*, *104*(3), 721-752. Retrieved from <GotoISI>://WOS:000334497800001 doi: 10.1257/aer.104.3.721

Gasteiger, E., & Zhang, S. (2014). Anticipation, learning and welfare: the case of distortionary taxation. *Journal of Economic Dynamics and Control*, *39*(Supplement C), 113 - 126. Retrieved from http://www.sciencedirect.com/science/article/pii/S0165188913002364 doi: https://doi.org/10.1016/j.jedc.2013.11.012

Gomes, F. J., Kotlikoff, L. J., & Viceira, L. M. (2012). The Excess Burden of Government Indecision. *Tax Policy and the Economy*, *26*(1), 125-164. Retrieved from https://ideas.repec.org/a/ucp/tpolec/doi10.1086-665505.html

Gourinchas, P.-O., & Parker, J. A. (2002). Consumption over the life cycle. *Econometrica*, *70*(1), 47 - 89. Retrieved from http://search.ebscohost.com.libproxy.uoregon.edu/login.aspx?direct=true&db=eoh&AN=0599259&login.asp&site=ehost-live&scope=site

Imrohoroglu, A., Imrohoroglu, S., & Joines, D. H. (1995). A life-cycle analysis of social-security [Journal Article]. *Economic Theory*, *6*(1), 83-114. Retrieved from <GotoISI>://WOS:A1995RF36800005 doi: 10.1007/bf01213942

Kitao, S. (2016). *Policy uncertainty and the cost of delaying reform: A case of aging japan* (RIETI Working Papers No. 16-E-013). Research Institute of Economy, Trade and Industry. Retrieved from http://www.rieti.go.jp/en/publications/summary/16020014.html

Krueger, D., & Kubler, F. (2002). Intergenerational risk-sharing via social security when financial markets are incomplete [Journal Article]. *American Economic Review*, *92*(2), 407-410. Retrieved from <GotoISI>://WOS:000176395100076 doi: 10.1257/000282802320191705

Krueger, D., & Kubler, F. (2004). Computing equilibrium in olg models with stochastic production [Journal Article]. *Journal of Economic Dynamics and Control*, *28*(7), 1411-1436. Retrieved from <GotoISI>://WOS:000188201800010 doi: 10.1016/s0165-1889(03)00111-8

Krueger, D., & Kubler, F. (2006). Pareto-improving social security reform when financial markets are incomplete!? [Journal Article]. *American Economic Review*, *96*(3), 737-755. Retrieved from <GotoISI>://WOS:000238567400013 doi: 10.1257/aer.96.3.737

Laitner, J. (1990). Tax changes and phase-diagrams for an overlapping generations model [Journal Article]. *Journal of Political Economy*, *98*(1), 193-220. Retrieved from <GotoISI>://WOS:A1990CN69500010 doi: 10.1086/261675

Marcet, A., & Sargent, T. (1989). Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory*, *48*(2), 337-368. Retrieved from http://EconPapers.repec.org/RePEc:eee:jetheo:v:48:y:1989:i:2:p:337-368

Martin, A., & Ventura, J. (2012). Economic growth with bubbles [Journal Article]. *American Economic Review*, *102*(6), 3033-3058. Retrieved from <GotoISI>://WOS:000309822200022 doi: 10.1257/aer.102.6.3033

Mitra, K., Evans, G. W., & Honkapohja, S. (2013). Policy change and learning in the rbc model. *Journal of Economic Dynamics and Control*, *37*(10), 1947 - 1971. Retrieved from http://www.sciencedirect.com/science/article/pii/S0165188913001176 doi: https://doi.org/10.1016/j.jedc.2013.05.011

Muth, J. F. (1960). Optimal properties of exponentially weighted forecasts. *Journal of the American Statistical Association*, *55*(290), 299-306. Retrieved from http://www.tandfonline.com/doi/abs/10.1080/01621459.1960.10482064 doi: 10.1080/01621459.1960.10482064

Nelson, J. (2016). *Welfare implications of uncertain social security reform* (Indiana University Department of Economics' Jordan River Economics Conference). Retrieved from https://economics.indiana.edu/home/conferences/2016-jordan-river-economics-conference/files/2016-04-29-14.pdf

Olovsson, C. (2010). Quantifying the risk-sharing welfare gains of social security [Journal Article]. *Journal of Monetary Economics*, *57*(3), 364-375. Retrieved from <GotoISI>://WOS:000277528300009 doi: 10.1016/j.jmoneco.2010.02.009

Park, H., & Feigenbaum, J. (2017). *Bounded rationality, lifecycle consumption, and social security* (Working Papers). Retrieved from http://www.platonicadventures.com/uploads/3/8/7/0/38707599/boundedrationalityhumpjebo.pdf

Pfajfar, D., & Santoro, E. (2010). Heterogeneity, learning and information stickiness in inflation expectations. *Journal of Economic Behavior and Organization*, *75*(3), 426–444.

Phelan, C. (1999). Anticipation effects of looming public-pension reforms. *Carnegie-Rochester Conference Series on Public Policy*, *50*, 161 - 164. Retrieved from http://www.sciencedirect.com/science/article/pii/S0167223199000251 doi: http://dx.doi.org/10.1016/S0167-2231(99)00025-1

Preston, B. (2005). *Learning about monetary policy rules when long-horizon expectations matter* (MPRA Paper). University Library of Munich, Germany. Retrieved from http://EconPapers.repec.org/RePEc:pra:mprapa:830

Preston, B. (2006, April). Adaptive learning, forecast-based instrument rules and monetary policy. *Journal of Monetary Economics*, *53*(3), 507-535. Retrieved from https://ideas.repec.org/a/eee/moneco/v53y2006i3p507-535.html

Sanchez-Marcos, V., & Sanchez-Martin, A. R. (2006). Can social security be welfare improving when there is demographic uncertainty? [Journal Article]. *Journal of Economic Dynamics and Control*, *30*(9-10), 1615-1646. Retrieved from <GotoISI>://WOS:000240574100007 doi: 10.1016/j.jedc.2005.09.017

Sander, J. T. (1990). *Discrete dynamical systems: Theory and applications* [Book]. Oxford: Claredon Press.

Sargent, T. (1993). *Bounded rationality in macroeconomics*. Oxford University Press. Retrieved from https://EconPapers.repec.org/RePEc:oxp:obooks:9780198288695

SSA, B. o. T. (2017). *The 2017 annual report of the board of trustees of the federal old-age and survivors insurance and federal disability insurance trust funds* (House Documents No. 114-145). Social Security Administration. Retrieved from https://www.ssa.gov/OACT/TR/2017/